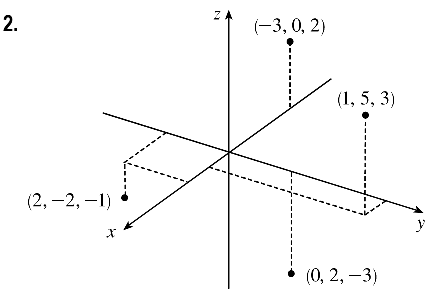
Solutions Manual *for* Multivariable Calculus III with Applications in the Sciences

12 □ VECTORS AND THE GEOMETRY OF SPACE

12.1 Three-Dimensional Coordinate Systems

1. We start at the origin, which has coordinates $(0, 0, 0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4, 0, -3)$.

2.

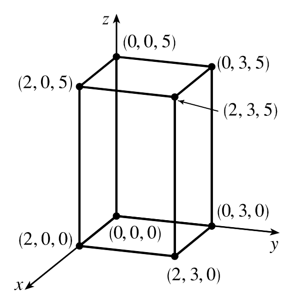


3. The distance from a point to the yz-plane is the absolute value of the x-coordinate of the point. $C(2, 4, 6)$ has the x-coordinate with the smallest absolute value, so C is the point closest to the yz-plane. $A(-4, 0, -1)$ must lie in the xz-plane since the distance from A to the xz-plane, given by the y-coordinate of A, is 0.

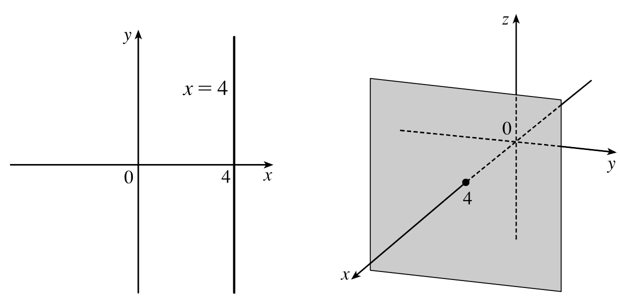
4. The projection of $(2, 3, 5)$ onto the xy-plane is $(2, 3, 0)$; onto the yz-plane, $(0, 3, 5)$; onto the xz-plane, $(2, 0, 5)$.

The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16$$



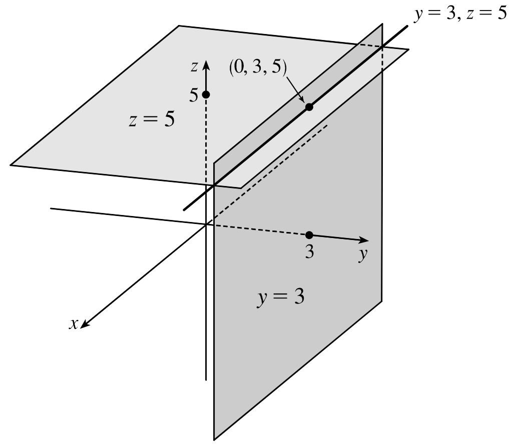
5. In $\mathbb{R}^2$, the equation $x = 4$ represents a line parallel to the y-axis and 4 units to the right of it. In $\mathbb{R}^3$, the equation $x = 4$ represents the set $\{(x, y, z) \mid x = 4\}$, the set of all points whose x-coordinate is 4. This is the vertical plane that is parallel to the yz-plane and 4 units in front of it.



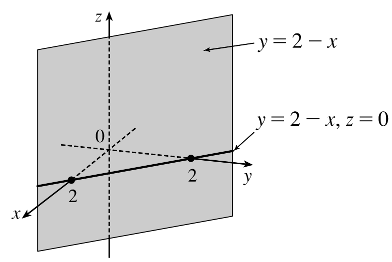
6. In $\mathbb{R}^3$, the equation $y = 3$ represents a vertical plane that is parallel to the xz-plane and 3 units to the right of it. The equation $z = 5$ represents a horizontal plane parallel to the xy-plane and 5 units above it. The pair of equations $y = 3$, $z = 5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 3$, $z = 5$.

[continued]

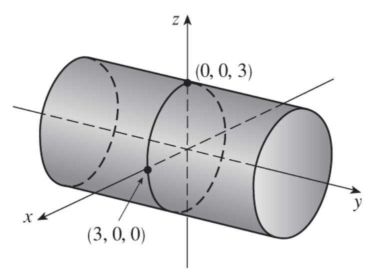
This line can also be described as the set $\{(x, 3, 5) \mid x \in \mathbb{R}\}$, which is the set of all points in $\mathbb{R}^3$ whose x-coordinate may vary but whose y- and z-coordinates are fixed at 3 and 5, respectively. Thus the line is parallel to the x-axis and intersects the yz-plane in the point $(0, 3, 5)$.



7. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$.



8. The equation $x^2 + z^2 = 9$ has no restrictions on y, and the x- and z-coordinates satisfy the equation for a circle of radius 3 with center the origin. Thus the surface $x^2 + z^2 = 9$ in $\mathbb{R}^3$ consists of all possible vertical circles (parallel to the xz-plane) $x^2 + z^2 = 9$, $y = k$, and is therefore a circular cylinder with radius 3 whose axis is the y-axis.



9. The distance between the points $P_1(3, 5, -2)$ and $P_2(-1, 1, -4)$ is

$$|P_1P_2| = \sqrt{(-1-3)^2 + (1-5)^2 + [-4-(-2)]^2} = \sqrt{16+16+4} = 6$$

10. The distance between the points $P_1(-6, -3, 0)$ and $P_2(2, 4, 5)$ is

$$|P_1P_2| = \sqrt{[2-(-6)]^2 + [4-(-3)]^2 + (5-0)^2} = \sqrt{64+49+25} = \sqrt{138}$$

11. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{(7-3)^2 + [0-(-2)]^2 + [1-(-3)]^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$
$$|QR| = \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{36+4+0} = \sqrt{40} = 2\sqrt{10}$$
$$|RP| = \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

The longest side is QR, but the Pythagorean Theorem is not satisfied: $|PQ|^2 + |RP|^2 \neq |QR|^2$. Thus PQR is not a right triangle. PQR is isosceles, as two sides have the same length.

12. Compute the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{(4-2)^2 + [1-(-1)]^2 + (1-0)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$
$$|QR| = \sqrt{(4-4)^2 + (-5-1)^2 + (4-1)^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$
$$|RP| = \sqrt{(2-4)^2 + [-1-(-5)]^2 + (0-4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

Since the Pythagorean Theorem is satisfied by $|PQ|^2 + |RP|^2 = |QR|^2$, PQR is a right triangle. PQR is not isosceles, as no two sides have the same length.

13. (a) First we find the distances between points:

$$|AB| = \sqrt{(3-2)^2 + (7-4)^2 + (-2-2)^2} = \sqrt{26}$$
$$|BC| = \sqrt{(1-3)^2 + (3-7)^2 + [3-(-2)]^2} = \sqrt{45}$$
$$|AC| = \sqrt{(1-2)^2 + (3-4)^2 + (3-2)^2} = \sqrt{3}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $\sqrt{26} + \sqrt{3} \neq \sqrt{45}$, the three points do not lie on a straight line.

(b) First we find the distances between points:

$$|DE| = \sqrt{(1-0)^2 + [-2-(-5)]^2 + (4-5)^2} = \sqrt{11}$$
$$|EF| = \sqrt{(3-1)^2 + [4-(-2)]^2 + (2-4)^2} = \sqrt{44} = 2\sqrt{11}$$
$$|DF| = \sqrt{(3-0)^2 + [4-(-5)]^2 + (2-5)^2} = \sqrt{99} = 3\sqrt{11}$$

Since $\sqrt{11} + 2\sqrt{11} = 3\sqrt{11}$, the three points lie on a straight line.

14. (a) The distance from a point to the xy-plane is the absolute value of the z-coordinate of the point. Thus, the distance from $(4,-2,6)$ to the xy-plane is $|6| = 6$.

(b) Similarly, the distance to the yz-plane is the absolute value of the x-coordinate of the point: $|4| = 4$.

(c) The distance to the xz-plane is the absolute value of the y-coordinate of the point: $|-2| = 2$.

(d) The point on the x-axis closest to $(4,-2,6)$ is the point $(4,0,0)$. (Approach the x-axis perpendicularly.) The distance from $(4,-2,6)$ to the x-axis is the distance between these two points:
$\sqrt{(4-4)^2 + (-2-0)^2 + (6-0)^2} = \sqrt{40} = 2\sqrt{10} \approx 6.32$.

(e) The point on the y-axis closest to $(4,-2,6)$ is $(0,-2,0)$. The distance between these points is
$\sqrt{(4-0)^2 + [-2-(-2)]^2 + (6-0)^2} = \sqrt{52} = 2\sqrt{13} \approx 7.21$.

(f) The point on the z-axis closest to $(4,-2,6)$ is $(0,0,6)$. The distance between these points is
$\sqrt{(4-0)^2 + (-2-0)^2 + (6-6)^2} = \sqrt{20} = 2\sqrt{5} \approx 4.47$.

15. An equation of the sphere with center $(-3,2,5)$ and radius 4 is $[x-(-3)]^2 + (y-2)^2 + (z-5)^2 = 4^2$, or $(x+3)^2 + (y-2)^2 + (z-5)^2 = 16$. The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $9 + (y-2)^2 + (z-5)^2 = 16$, $x = 0$ or $(y-2)^2 + (z-5)^2 = 7$, $x = 0$, which represents a circle in the yz-plane with center $(0,2,5)$ and radius $\sqrt{7}$.

16. An equation of the sphere with center $(2,-6,4)$ and radius 5 is $(x-2)^2+[y-(-6)]^2+(z-4)^2=5^2$, or $(x-2)^2+(y+6)^2+(z-4)^2=25$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z=0$ into the equation, we have $(x-2)^2+(y+6)^2=9$, $z=0$ which represents a circle in the xy-plane with center $(2,-6,0)$ and radius 3. To find the intersection with the xz-plane, we set $y=0$: $(x-2)^2+(z-4)^2=-11$. Since no points satisfy this equation, the sphere does not intersect the xz-plane. (Also note that the distance from the center of the sphere to the xz-plane is greater than the radius of the sphere.) To find the intersection with the yz-plane, we set $x=0$: $(y+6)^2+(z-4)^2=21$, $x=0$, a circle in the yz-plane with center $(0,-6,4)$ and radius $\sqrt{21}$.

17. The radius of the sphere is the distance between $(4,3,-1)$ and $(3,8,1)$: $r=\sqrt{(3-4)^2+(8-3)^2+[1-(-1)]^2}=\sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2+(y-8)^2+(z-1)^2=30$.

18. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1,2,3)$: $r=\sqrt{(1-0)^2+(2-0)^2+(3-0)^2}=\sqrt{14}$. Then an equation of the sphere is $(x-1)^2+(y-2)^2+(z-3)^2=14$.

19. Completing squares in the equation $x^2+y^2+z^2+8x-2z=8$ gives $(x^2+8x+16)+y^2+(z^2-2z+1)=8+16+1 \quad\Rightarrow\quad (x+4)^2+y^2+(z-1)^2=25$, which we recognize as an equation of a sphere with center $(-4,0,1)$ and radius $\sqrt{25}=5$.

20. Completing squares in the equation $x^2-6x+y^2+4y+z^2+10z=0$ gives $(x^2-6x+9)+(y^2+4y+4)+(z^2+10z+25)=9+4+25 \quad\Rightarrow\quad (x-3)^2+(y+2)^2+(z+5)^2=38$, which we recognize as an equation of a sphere with center $(3,-2,-5)$ and radius $\sqrt{38}$.

21. Completing squares in the equation $2x^2-2x+2y^2+4y+2z^2=-1$ gives $2\left(x^2-x+\frac{1}{4}\right)+2(y^2+2y+1)+2z^2=-1+\frac{1}{2}+2 \quad\Rightarrow\quad 2\left(x-\frac{1}{2}\right)^2+2(y+1)^2+2z^2=\frac{3}{2} \quad\Rightarrow$ $\left(x-\frac{1}{2}\right)^2+(y+1)^2+z^2=\frac{3}{4}$, which we recognize as an equation of a sphere with center $\left(\frac{1}{2},-1,0\right)$ and radius $\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$.

22. Completing the squares in the equation $4x^2-16x+4y^2+6y+4z^2=-12$ gives $4(x^2-4x+4)+4\left(y^2+\frac{3}{2}y+\frac{9}{16}\right)+4z^2=-12+16+\frac{9}{4} \quad\Rightarrow\quad 4(x-2)^2+4\left(y+\frac{3}{4}\right)^2+4z^2=\frac{25}{4} \quad\Rightarrow$ $(x-2)^2+\left(y+\frac{3}{4}\right)^2+z^2=\frac{25}{16}$, which we recognize as the equation of a sphere with center $\left(2,-\frac{3}{4},0\right)$ and radius $\sqrt{\frac{25}{16}}=\frac{5}{4}$.

23. If the midpoint of the line segment from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$ is $Q=\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2},\dfrac{z_1+z_2}{2}\right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

[continued]

$$
\begin{aligned}
|P_1Q| &= \sqrt{\left[\tfrac{1}{2}(x_1+x_2)-x_1\right]^2+\left[\tfrac{1}{2}(y_1+y_2)-y_1\right]^2+\left[\tfrac{1}{2}(z_1+z_2)-z_1\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2-\tfrac{1}{2}x_1\right)^2+\left(\tfrac{1}{2}y_2-\tfrac{1}{2}y_1\right)^2+\left(\tfrac{1}{2}z_2-\tfrac{1}{2}z_1\right)^2} = \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \tfrac{1}{2}\,|P_1P_2|
\end{aligned}
$$

$$
\begin{aligned}
|QP_2| &= \sqrt{\left[x_2-\tfrac{1}{2}(x_1+x_2)\right]^2+\left[y_2-\tfrac{1}{2}(y_1+y_2)\right]^2+\left[z_2-\tfrac{1}{2}(z_1+z_2)\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2-\tfrac{1}{2}x_1\right)^2+\left(\tfrac{1}{2}y_2-\tfrac{1}{2}y_1\right)^2+\left(\tfrac{1}{2}z_2-\tfrac{1}{2}z_1\right)^2} = \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \tfrac{1}{2}\,|P_1P_2|
\end{aligned}
$$

So Q is indeed the midpoint of P_1P_2.

24. By Exercise 23(a), the midpoint of the diameter that has endpoints $(5,4,3)$ and $(1,6,-9)$ (and thus the center of the sphere) is $\left(\dfrac{5+1}{2}, \dfrac{4+6}{2}, \dfrac{3+(-9)}{2}\right) = (3,5,-3)$. The radius is half the diameter, so $r = \frac{1}{2}\sqrt{(1-5)^2+(6-4)^2+(-9-3)^2} = \frac{1}{2}\sqrt{164} = \sqrt{41}$. Therefore, an equation of the sphere is $(x-3)^2+(y-5)^2+(z+3)^2 = 41$.

25. (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(-1,4,5)$, to the xy-plane, which is 5. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 25$.

(b) Since the sphere touches the yz-plane, its radius is the distance from its center, $(-1,4,5)$, to the yz-plane, which is 1. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 1$.

(c) Since the sphere touches the xz-plane, its radius is the distance from its center, $(-1,4,5)$, to the xz-plane, which is 4. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 16$.

26. The shortest distance from the center, $(7,3,8)$, to any of the three coordinate planes is 3, which is the distance to the xz-plane. Therefore, an equation of the sphere is $(x-7)^2+(y-3)^2+(z-8)^2 = 9$.

27. The equation $z=-2$ represents a plane, parallel to the xy-plane and 2 units below it.

28. The equation $x=3$ represents a plane, parallel to the yz-plane and 3 units in front of it.

29. The inequality $y \geq 1$ represents a half-space consisting of all the points on or to the right of the plane $y=1$.

30. The inequality $x < 4$ represents a half-space consisting of all the points behind the plane $x=4$.

31. The inequality $-1 \leq x \leq 2$ represents all points on or between the vertical planes $x=-1$ and $x=2$.

32. The equation $z=y$ represents a plane, perpendicular to the yz-plane, and intersecting the yz-plane in the line $z=y$, $x=0$.

33. Because $z=-1$, all points in the region must lie in the horizontal plane $z=-1$. In addition, $x^2+y^2=4$, so the region consists of all points that lie on a circle with radius 2 and center on the z-axis that is contained in the plane $z=-1$.

34. Here $x^2 + y^2 = 4$ with no restrictions on z, so a point in the region must lie on a circle of radius 2, center on the z-axis, but it could be in any horizontal plane $z = k$ (parallel to the xy-plane). Thus the region consists of all possible circles $x^2 + y^2 = 4$, $z = k$ and is therefore a circular cylinder with radius 2 whose axis is the z-axis.

35. The inequality $y^2 + z^2 \leq 25$ is equivalent to $\sqrt{y^2 + z^2} \leq 5$, which describes the set of all points in $\mathbb{R}^3$ whose distance from the x-axis is at most 5. Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 5 with axis the x-axis.

36. The inequality $x^2 + z^2 \leq 25$ is equivalent to $\sqrt{x^2 + z^2} \leq 5$, which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 5. Further, $0 \leq y \leq 2$ consists of the points on or between the planes $y = 0$ and $y = 2$. Thus, the inequalities represent the region consisting of all points on or inside a circular cylinder of radius 5 with axis the y-axis from $y = 0$ to $y = 2$.

37. The equation $x^2 + y^2 + z^2 = 4$ is equivalent to $\sqrt{x^2 + y^2 + z^2} = 2$, so the region consists of those points whose distance from the origin is 2. This is the set of all points on a sphere with radius 2 and center $(0, 0, 0)$.

38. The inequality $x^2 + y^2 + z^2 \leq 4$ is equivalent to $\sqrt{x^2 + y^2 + z^2} \leq 2$, so the region consists of those points whose distance from the origin is at most 2. This is the set of all points on or inside a sphere with radius 2 and center $(0, 0, 0)$.

39. The inequalities $1 \leq x^2 + y^2 + z^2 \leq 5$ are equivalent to $1 \leq \sqrt{x^2 + y^2 + z^2} \leq \sqrt{5}$, so the region consists of those points whose distance from the origin is at least 1 and at most $\sqrt{5}$. This is the set of all points on or between spheres with radii 1 and $\sqrt{5}$ and centers $(0, 0, 0)$.

40. The inequalities $1 \leq x^2 + y^2 \leq 5$ are equivalent to $1 \leq \sqrt{x^2 + y^2} \leq \sqrt{5}$, which represents the set of all points in $\mathbb{R}^3$ whose distance is at least 1 and at most $\sqrt{5}$ from the z-axis. Thus, the region consists of all points on or between a circular cylinder of radius 1 and a circular cylinder of radius $\sqrt{5}$ with axis the z-axis.

41. The inequalities $0 \leq x \leq 3, 0 \leq y \leq 3, 0 \leq z \leq 3$ represent the set of all points in $\mathbb{R}^3$ that lie on or between the planes $x = 3$, $y = 3$, $z = 3$ in the first octant. Thus, the region is a cube with dimensions $3 \times 3 \times 3$.

42. The inequality $x^2 + y^2 + z^2 > 2z \quad \Leftrightarrow \quad x^2 + y^2 + (z - 1)^2 > 1$ is equivalent to $\sqrt{x^2 + y^2 + (z - 1)^2} > 1$, so the region consists of those points whose distance from the point $(0, 0, 1)$ is greater than 1. This is the set of all points outside the sphere with radius 1 and center $(0, 0, 1)$.

43. This describes all points whose x-coordinate is between 0 and 5, that is, $0 < x < 5$.

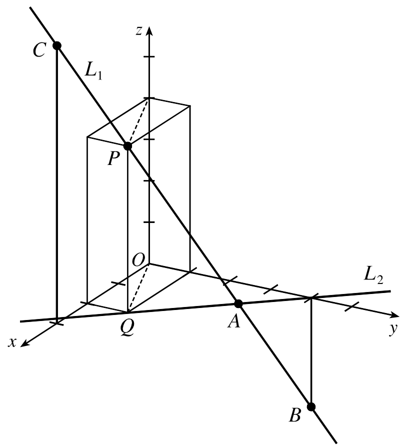
44. For any point on or above the disk in the xy-plane with center the origin and radius 2 we have $x^2 + y^2 \leq 4$. Also each point lies on or between the planes $z = 0$ and $z = 8$, so the region is described by $x^2 + y^2 \leq 4,\ \ 0 \leq z \leq 8$.

45. This describes a region all of whose points have a distance to the origin which is greater than r, but smaller than R. So inequalities describing the region are $r < \sqrt{x^2 + y^2 + z^2} < R$, or $r^2 < x^2 + y^2 + z^2 < R^2$.

46. The solid sphere itself is represented by $\sqrt{x^2+y^2+z^2} \le 2$. Since we want only the upper hemisphere, we restrict the z-coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2+y^2+z^2} \le 2$, $z \ge 0$, or $x^2+y^2+z^2 \le 4$, $z \ge 0$.

47. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2, 1, 4)$.

(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.



48. Let $P = (x, y, z)$. Then $2\,|PB| = |PA| \quad\Leftrightarrow\quad 4\,|PB|^2 = |PA|^2 \quad\Leftrightarrow$

$4\big[(x-6)^2+(y-2)^2+(z+2)^2\big] = (x+1)^2+(y-5)^2+(z-3)^2 \quad\Leftrightarrow$

$4\big(x^2-12x+36\big) - x^2 - 2x + 4\big(y^2-4y+4\big) - y^2 + 10y + 4\big(z^2+4z+4\big) - z^2 + 6z = 35 \quad\Leftrightarrow$

$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \quad\Leftrightarrow\quad x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$.

By completing the square three times we get $\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{-423\,+\,625\,+\,9\,+\,121}{9} = \frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

49. We need to find a set of points $\big\{P(x,y,z) \;\big|\; |AP| = |BP|\big\}$.

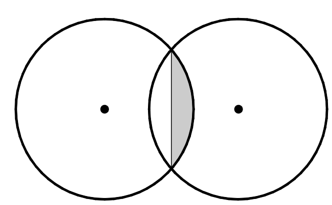
$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} = \sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \quad\Rightarrow$

$(x+1)^2+(y-5)^2+(z-3)^2 = (x-6)^2+(y-2)^2+(z+2)^2 \quad\Rightarrow$

$x^2+2x+1+y^2-10y+25+z^2-6z+9 = x^2-12x+36+y^2-4y+4+z^2+4z+4 \quad\Rightarrow\quad 14x-6y-10z=9$.

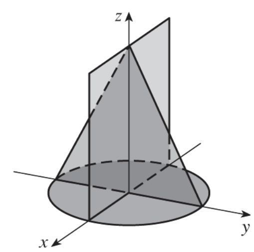
Thus, the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

50. Completing the square three times in the first equation gives $(x+2)^2+(y-1)^2+(z+2)^2 = 2^2$, a sphere with center $(-2, 1, -2)$ and radius 2. The second equation is that of a sphere with center $(0, 0, 0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2+(1-0)^2+(-2-0)^2} = \sqrt{4+1+4} = 3$. Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 6.2.61, the volume of a cap of a sphere is $V = \pi h^2\left(r - \frac{1}{3}h\right) = \pi\left(\frac{1}{2}\right)^2\left(2 - \frac{1}{3}\cdot\frac{1}{2}\right) = \frac{11\pi}{24}$. So the total volume is $2\cdot\frac{11\pi}{24} = \frac{11\pi}{12}$.

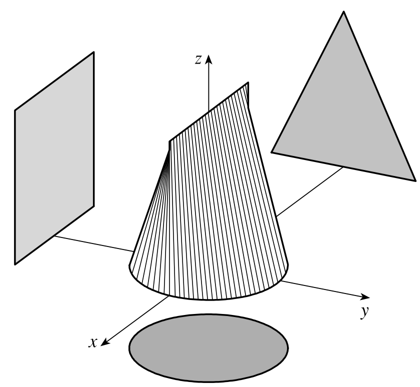


51. The sphere $x^2 + y^2 + z^2 = 4$ has center $(0, 0, 0)$ and radius 2. Completing squares in $x^2 - 4x + y^2 - 4y + z^2 - 4z = -11$ gives $(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = -11 + 4 + 4 + 4 \quad \Rightarrow \quad (x-2)^2 + (y-2)^2 + (z-2)^2 = 1$, so this is the sphere with center $(2, 2, 2)$ and radius 1. The (shortest) distance between the spheres is measured along the line segment connecting their centers. The distance between $(0, 0, 0)$ and $(2, 2, 2)$ is $\sqrt{(2-0)^2 + (2-0)^2 + (2-0)^2} = \sqrt{12} = 2\sqrt{3}$, and subtracting the radius of each circle, the distance between the spheres is $2\sqrt{3} - 2 - 1 = 2\sqrt{3} - 3$.

52. There are many different solids that fit the given description. However, any possible solid must have a circular horizontal cross-section at its top or at its base. Here we illustrate a solid with a circular base in the xy-plane. (A circular cross-section at the top results in an inverted version of the solid described below.) The vertical cross-section through the center of the base that is parallel to the xz-plane must be a square, and the vertical cross-section parallel to the yz-plane (perpendicular to the square) through the center of the base must be a triangle with two vertices on the circle and the third vertex at the center of the top side of the square. (See the figure.)



The solid can include any additional points that do not extend beyond these three "silhouettes" when viewed from directions parallel to the coordinate axes. One possibility shown here is to draw the circular base and the vertical square first. Then draw a surface formed by line segments parallel to the yz-plane that connect the top of the square to the circle.



Problem 8 in the Problems Plus section at the end of the chapter illustrates another possible solid.

12.2 Vectors

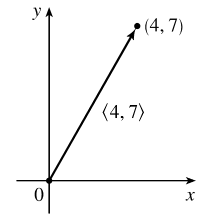
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

(d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7\rangle$ is placed at the origin, then $\langle 4, 7\rangle$ is the position vector of the point $(4, 7)$.



3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

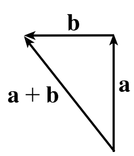
4. (a) The initial point of $\overrightarrow{BC}$ is positioned at the terminal point of $\overrightarrow{AB}$, so by the Triangle Law the sum $\overrightarrow{AB} + \overrightarrow{BC}$ is the vector with initial point A and terminal point C, namely $\overrightarrow{AC}$.

(b) By the Triangle Law, $\overrightarrow{CD} + \overrightarrow{DB}$ is the vector with initial point C and terminal point B, namely $\overrightarrow{CB}$.

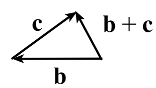
(c) First we consider $\overrightarrow{DB} - \overrightarrow{AB}$ as $\overrightarrow{DB} + \left(-\overrightarrow{AB}\right)$. Then since $-\overrightarrow{AB}$ has the same length as $\overrightarrow{AB}$ but points in the opposite direction, we have $-\overrightarrow{AB} = \overrightarrow{BA}$ and so $\overrightarrow{DB} - \overrightarrow{AB} = \overrightarrow{DB} + \overrightarrow{BA} = \overrightarrow{DA}$.

(d) We use the Triangle Law twice: $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB} = \left(\overrightarrow{DC} + \overrightarrow{CA}\right) + \overrightarrow{AB} = \overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB}$.

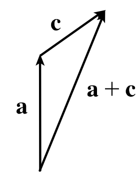
5. (a)



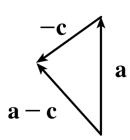
(b)



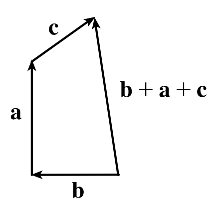
(c)



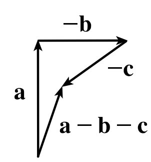
(d)



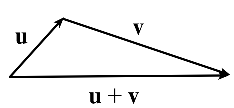
(e)



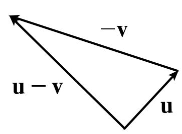
(f)



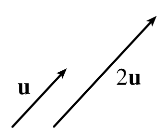
6. (a)



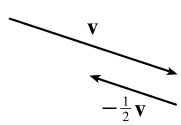
(b)



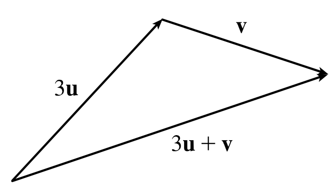
(c)



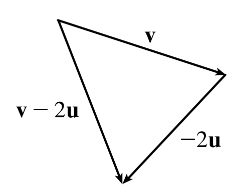
(d)



(e)

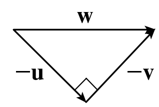


(f)



7. Because the tail of $\mathbf{d}$ is the midpoint of QR we have $\overrightarrow{QR} = 2\mathbf{d}$, and by the Triangle Law, $\mathbf{a} + 2\mathbf{d} = \mathbf{b} \;\Rightarrow\; 2\mathbf{d} = \mathbf{b} - \mathbf{a} \;\Rightarrow\; \mathbf{d} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$. Again by the Triangle Law, we have $\mathbf{c} + \mathbf{d} = \mathbf{b}$ so $\mathbf{c} = \mathbf{b} - \mathbf{d} = \mathbf{b} - \left(\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}\right) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

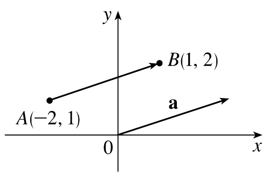
8. We are given $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, so $\mathbf{w} = (-\mathbf{u}) + (-\mathbf{v})$. (See the figure.)



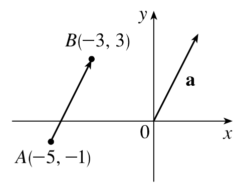
Vectors $-\mathbf{u}$, $-\mathbf{v}$, and $\mathbf{w}$ form a right triangle, so from the Pythagorean Theorem

we have $|-\mathbf{u}|^2 + |-\mathbf{v}|^2 = |\mathbf{w}|^2$. But $|-\mathbf{u}| = |\mathbf{u}| = 1$ and $|-\mathbf{v}| = |\mathbf{v}| = 1$, so $|\mathbf{w}| = \sqrt{|-\mathbf{u}|^2 + |-\mathbf{v}|^2} = \sqrt{2}$.

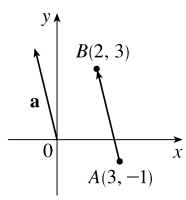
9. $\mathbf{a} = \langle 1 - (-2), 2 - 1 \rangle = \langle 3, 1 \rangle$



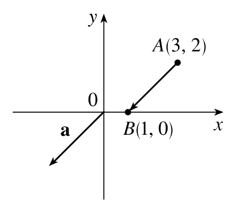
10. $\mathbf{a} = \langle -3 - (-5), 3 - (-1) \rangle = \langle 2, 4 \rangle$



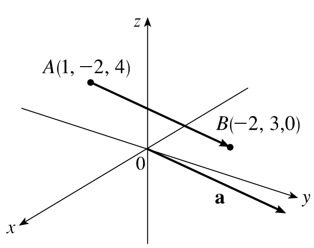
11. $\mathbf{a} = \langle 2 - 3, 3 - (-1) \rangle = \langle -1, 4 \rangle$



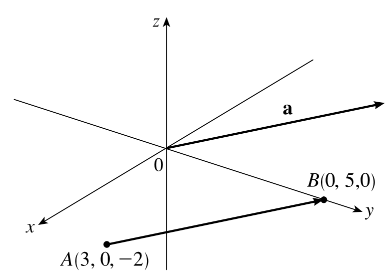
12. $\mathbf{a} = \langle 1 - 3, 0 - 2 \rangle = \langle -2, -2 \rangle$



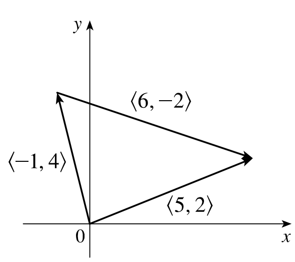
13. $\mathbf{a} = \langle -2 - 1, 3 - (-2), 0 - 4 \rangle = \langle -3, 5, -4 \rangle$



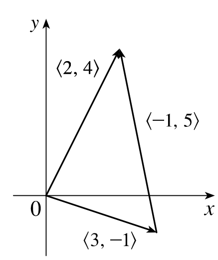
14. $\mathbf{a} = \langle 0 - 3, 5 - 0, 0 - (-2) \rangle = \langle -3, 5, 2 \rangle$



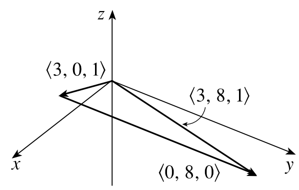
15. $\langle -1, 4 \rangle + \langle 6, -2 \rangle = \langle -1 + 6, 4 + (-2) \rangle = \langle 5, 2 \rangle$



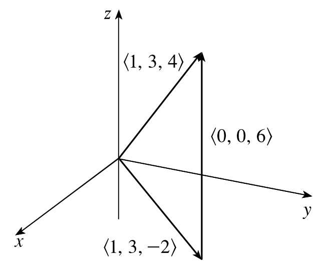
16. $\langle 3, -1 \rangle + \langle -1, 5 \rangle = \langle 3 + (-1), -1 + 5 \rangle = \langle 2, 4 \rangle$



17. $\langle 3,0,1\rangle + \langle 0,8,0\rangle = \langle 3+0, 0+8, 1+0\rangle$
$= \langle 3,8,1\rangle$



18. $\langle 1,3,-2\rangle + \langle 0,0,6\rangle = \langle 1+0, 3+0, -2+6\rangle$
$= \langle 1,3,4\rangle$



19. $\mathbf{a}+\mathbf{b} = \langle -3,4\rangle + \langle 9,-1\rangle = \langle -3+9, 4+(-1)\rangle = \langle 6,3\rangle$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,\langle -3,4\rangle + 2\,\langle 9,-1\rangle = \langle -12,16\rangle + \langle 18,-2\rangle = \langle 6,14\rangle$

$|\mathbf{a}| = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

$|\mathbf{a}-\mathbf{b}| = |\langle -3-9, 4-(-1)\rangle| = |\langle -12,5\rangle| = \sqrt{(-12)^2+5^2} = \sqrt{169} = 13$

20. $\mathbf{a}+\mathbf{b} = (5\,\mathbf{i}+3\,\mathbf{j}) + (-\mathbf{i}-2\,\mathbf{j}) = 4\,\mathbf{i}+\mathbf{j}$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,(5\,\mathbf{i}+3\,\mathbf{j}) + 2\,(-\mathbf{i}-2\,\mathbf{j}) = 20\,\mathbf{i}+12\,\mathbf{j}-2\,\mathbf{i}-4\,\mathbf{j} = 18\,\mathbf{i}+8\,\mathbf{j}$

$|\mathbf{a}| = \sqrt{5^2+3^2} = \sqrt{34}$

$|\mathbf{a}-\mathbf{b}| = |(5\,\mathbf{i}+3\,\mathbf{j})-(-\mathbf{i}-2\,\mathbf{j})| = |6\,\mathbf{i}+5\,\mathbf{j}| = \sqrt{6^2+5^2} = \sqrt{61}$

21. $\mathbf{a}+\mathbf{b} = (4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}) + (2\,\mathbf{i}-4\,\mathbf{k}) = 6\,\mathbf{i}-3\,\mathbf{j}-2\mathbf{k}$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,(4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}) + 2\,(2\,\mathbf{i}-4\,\mathbf{k}) = 16\,\mathbf{i}-12\,\mathbf{j}+8\,\mathbf{k}+4\,\mathbf{i}-8\,\mathbf{k} = 20\,\mathbf{i}-12\,\mathbf{j}$

$|\mathbf{a}| = \sqrt{4^2+(-3)^2+2^2} = \sqrt{29}$

$|\mathbf{a}-\mathbf{b}| = |(4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k})-(2\,\mathbf{i}-4\,\mathbf{k})| = |2\,\mathbf{i}-3\,\mathbf{j}+6\,\mathbf{k}| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{49} = 7$

22. $\mathbf{a}+\mathbf{b} = \langle 8,1,-4\rangle + \langle 5,-2,1\rangle = \langle 8+5, 1+(-2), -4+1\rangle = \langle 13,-1,-3\rangle$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,\langle 8,1,-4\rangle + 2\,\langle 5,-2,1\rangle = \langle 32,4,-16\rangle + \langle 10,-4,2\rangle = \langle 42,0,-14\rangle$

$|\mathbf{a}| = \sqrt{8^2+1^2+(-4)^2} = \sqrt{81} = 9$

$|\mathbf{a}-\mathbf{b}| = |\langle 8-5, 1-(-2), -4-1\rangle| = |\langle 3,3,-5\rangle| = \sqrt{3^2+3^2+(-5)^2} = \sqrt{43}$

23. The vector $\langle 6,-2\rangle$ has length $|\langle 6,-2\rangle| = \sqrt{6^2+(-2)^2} = \sqrt{40} = 2\sqrt{10}$, so by Equation 4 the unit vector with the same direction is $\dfrac{1}{2\sqrt{10}}\,\langle 6,-2\rangle = \left\langle \dfrac{3}{\sqrt{10}}, -\dfrac{1}{\sqrt{10}}\right\rangle$.

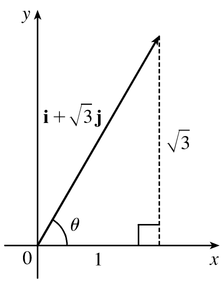
24. The vector $-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}$ has length $|-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}| = \sqrt{(-5)^2+3^2+(-1)^2} = \sqrt{35}$, so by Equation 4 the unit vector with the same direction is $\dfrac{1}{\sqrt{35}}(-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}) = -\dfrac{5}{\sqrt{35}}\,\mathbf{i} + \dfrac{3}{\sqrt{35}}\,\mathbf{j} - \dfrac{1}{\sqrt{35}}\,\mathbf{k}$.

25. The vector $8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ has length $|8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}| = \sqrt{8^2+(-1)^2+4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}) = \frac{8}{9}\,\mathbf{i} - \frac{1}{9}\,\mathbf{j} + \frac{4}{9}\,\mathbf{k}$.

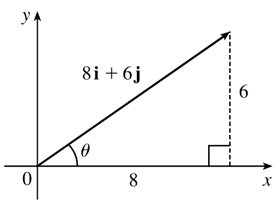
26. $|\langle 6,2,-3\rangle| = \sqrt{6^2+2^2+(-3)^2} = \sqrt{49} = 7$, so a unit vector in the direction of $\langle 6,2,-3\rangle$ is $\mathbf{u} = \frac{1}{7}\langle 6,2,-3\rangle$.

A vector in the same direction but with length 4 is $4\mathbf{u} = 4\cdot\frac{1}{7}\langle 6,2,-3\rangle = \left\langle \frac{24}{7}, \frac{8}{7}, -\frac{12}{7}\right\rangle$.

27. From the figure, we see that $\tan\theta = \dfrac{\sqrt{3}}{1} = \sqrt{3} \quad\Rightarrow\quad \theta = 60^\circ$.



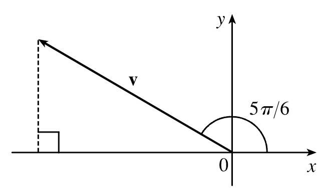
28. From the figure, we see that $\tan\theta = \frac{6}{8} = \frac{3}{4}$, so $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9^\circ$.



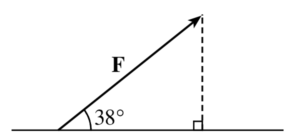
29. From the figure, we see that the x-component of $\mathbf{v}$ is

$v_1 = |\mathbf{v}|\cos(5\pi/6) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ and the y-component is

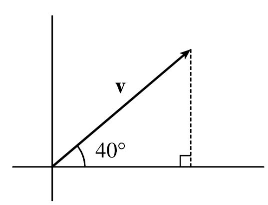
$v_2 = |\mathbf{v}|\sin(5\pi/6) = 4\left(\frac{1}{2}\right) = 2$. Thus, $\mathbf{v} = \left\langle -2\sqrt{3}, 2\right\rangle$.



30. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}|\cos 38^\circ = 50\cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}|\sin 38^\circ = 50\sin 38^\circ \approx 30.8$ N.



31. The velocity vector $\mathbf{v}$ makes an angle of 40° with the horizontal and has magnitude equal to the speed at which the football was thrown. From the figure, we see that the horizontal component of $\mathbf{v}$ is $|\mathbf{v}|\cos 40^\circ = 60\cos 40^\circ \approx 45.96$ ft/s and the vertical component is $|\mathbf{v}|\sin 40^\circ = 60\sin 40^\circ \approx 38.57$ ft/s.



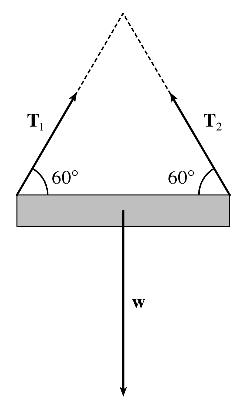
32. The given force vectors can be expressed in terms of their horizontal and vertical components as

$20\cos 45^\circ\,\mathbf{i} + 20\sin 45^\circ\,\mathbf{j} = 10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j}$ and $16\cos 30^\circ\,\mathbf{i} - 16\sin 30^\circ\,\mathbf{j} = 8\sqrt{3}\,\mathbf{i} - 8\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = \left(10\sqrt{2} + 8\sqrt{3}\right)\mathbf{i} + \left(10\sqrt{2} - 8\right)\mathbf{j} \approx 28.00\,\mathbf{i} + 6.14\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(28.00)^2 + (6.14)^2} \approx 28.7$ lb and, letting θ be the angle $\mathbf{F}$ makes with the positive x-axis,

$$\tan\theta = \frac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}} \quad\Rightarrow\quad \theta = \tan^{-1}\left(\frac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}}\right) \approx 12.4^\circ.$$

33. The given force vectors can be expressed in terms of their horizontal and vertical components as $-300\,\mathbf{i}$ and $200\cos 60^\circ\,\mathbf{i} + 200\sin 60^\circ\,\mathbf{j} = 200\left(\frac{1}{2}\right)\,\mathbf{i} + 200\left(\frac{\sqrt{3}}{2}\right)\,\mathbf{j} = 100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = (-300 + 100)\,\mathbf{i} + \left(0 + 100\sqrt{3}\right)\mathbf{j} = -200\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(-200)^2 + \left(100\sqrt{3}\right)^2} = \sqrt{70{,}000} = 100\sqrt{7} \approx 264.6$ N. Let θ be the angle $\mathbf{F}$ makes with the positive x-axis. Then $\tan\theta = \dfrac{100\sqrt{3}}{-200} = -\dfrac{\sqrt{3}}{2}$ and the terminal point of $\mathbf{F}$ lies in the second quadrant, so $\theta = \tan^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) + 180^\circ \approx -40.9^\circ + 180^\circ = 139.1^\circ$.

34. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the tension vectors corresponding to the support cables as shown in the figure. In terms of vertical and horizontal components,



$$\mathbf{T}_1 = |\mathbf{T}_1|\cos 60^\circ\mathbf{i} + |\mathbf{T}_1|\sin 60^\circ\mathbf{j} = \frac{1}{2}\,|\mathbf{T}_1|\;\mathbf{i} + \frac{\sqrt{3}}{2}\,|\mathbf{T}_1|\;\mathbf{j}$$

$$\mathbf{T}_2 = -\,|\mathbf{T}_2|\cos 60^\circ\mathbf{i} + |\mathbf{T}_2|\sin 60^\circ\mathbf{j} = -\frac{1}{2}\,|\mathbf{T}_2|\;\mathbf{i} + \frac{\sqrt{3}}{2}\,|\mathbf{T}_2|\;\mathbf{j}$$

The resultant of these tensions, $\mathbf{T}_1 + \mathbf{T}_2$, counterbalances the weight $\mathbf{w} = -500\,\mathbf{j}$. So $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 500\,\mathbf{j} \quad\Rightarrow$

$$\left(\frac{1}{2}\,|\mathbf{T}_1|\;\mathbf{i} + \frac{\sqrt{3}}{2}\,|\mathbf{T}_1|\;\mathbf{j}\right) + \left(-\frac{1}{2}\,|\mathbf{T}_2|\;\mathbf{i} + \frac{\sqrt{3}}{2}\,|\mathbf{T}_2|\;\mathbf{j}\right) = 500\,\mathbf{j}.$$

Equating x-components gives $\frac{1}{2}\,|\mathbf{T}_1|\;\mathbf{i} - \frac{1}{2}\,|\mathbf{T}_2|\;\mathbf{i} = 0$, so $|\mathbf{T}_1| = |\mathbf{T}_2|$ (as we would expect from the symmetry of the problem). Equating y-components, we have $\dfrac{\sqrt{3}}{2}\,|\mathbf{T}_1|\;\mathbf{j} + \dfrac{\sqrt{3}}{2}\,|\mathbf{T}_2|\;\mathbf{j} = \sqrt{3}\,|\mathbf{T}_1|\;\mathbf{j} = 500\,\mathbf{j} \quad\Rightarrow\quad |\mathbf{T}_1| = \dfrac{500}{\sqrt{3}}$. Thus the magnitude of each tension is $|\mathbf{T}_1| = |\mathbf{T}_2| = \dfrac{500}{\sqrt{3}} \approx 288.68$ lb. The tension vectors are

$\mathbf{T}_1 = \dfrac{1}{2}\,|\mathbf{T}_1|\;\mathbf{i} + \dfrac{\sqrt{3}}{2}\,|\mathbf{T}_1|\;\mathbf{j} = \dfrac{250}{\sqrt{3}}\,\mathbf{i} + 250\,\mathbf{j} \approx 144.34\,\mathbf{i} + 250\,\mathbf{j}$ and $\mathbf{T}_2 = -\dfrac{250}{\sqrt{3}}\,\mathbf{i} + 250\,\mathbf{j} \approx -144.34\,\mathbf{i} + 250\,\mathbf{j}$.

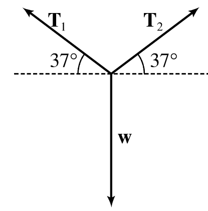
35. Call the two tension vectors $\mathbf{T}_2$ and $\mathbf{T}_3$, corresponding to the ropes of length 2 m and 3 m. In terms of vertical and horizontal components,

$$\mathbf{T}_2 = -\,|\mathbf{T}_2|\cos 50^\circ\mathbf{i} + |\mathbf{T}_2|\sin 50^\circ\mathbf{j} \quad (\mathbf{1}) \qquad \text{and} \qquad \mathbf{T}_3 = |\mathbf{T}_3|\cos 38^\circ\mathbf{i} + |\mathbf{T}_3|\sin 38^\circ\mathbf{j} \quad (\mathbf{2})$$

The resultant of these forces, $\mathbf{T}_2 + \mathbf{T}_3$, counterbalances the weight of the hoist (which is $-350\,\mathbf{j}$), so $\mathbf{T}_2 + \mathbf{T}_3 = 350\,\mathbf{j} \quad\Rightarrow$ $\left(-\,|\mathbf{T}_2|\cos 50^\circ + |\mathbf{T}_3|\cos 38^\circ\right)\mathbf{i} + \left(|\mathbf{T}_2|\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ\right)\mathbf{j} = 350\,\mathbf{j}$. Equating components, we have $-\,|\mathbf{T}_2|\cos 50^\circ + |\mathbf{T}_3|\cos 38^\circ = 0 \quad\Rightarrow\quad |\mathbf{T}_2| = |\mathbf{T}_3|\,\dfrac{\cos 38^\circ}{\cos 50^\circ}$ and $|\mathbf{T}_2|\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ = 350$. Substituting the first equation into the second gives $|\mathbf{T}_3|\,\dfrac{\cos 38^\circ}{\cos 50^\circ}\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ = 350 \quad\Rightarrow\quad |\mathbf{T}_3|\left(\cos 38^\circ\tan 50^\circ + \sin 38^\circ\right) = 350$, so the magnitudes of the tensions are $|\mathbf{T}_3| = \dfrac{350}{\cos 38^\circ\tan 50^\circ + \sin 38^\circ} \approx 225.11$ N and $|\mathbf{T}_2| = |\mathbf{T}_3|\,\dfrac{\cos 38^\circ}{\cos 50^\circ} \approx 275.97$ N.

Finally, from (**1**) and (**2**), the tension vectors are $\mathbf{T}_2 \approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$ and $\mathbf{T}_3 \approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$.

36. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25$ N so, in terms of vertical and horizontal components, we have

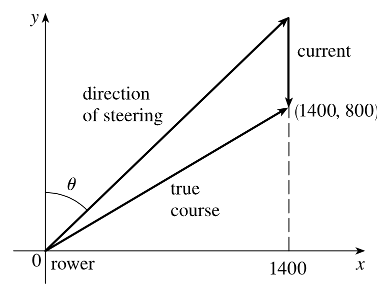


$$\mathbf{T}_1 = -25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j} \qquad \mathbf{T}_2 = 25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$

The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -|\mathbf{w}|\,\mathbf{j}$, we have $(-25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) + (25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) = |\mathbf{w}|\,\mathbf{j} \;\Rightarrow\; 50\sin 37^\circ\,\mathbf{j} = |\mathbf{w}|\,\mathbf{j} \;\Rightarrow$ $|\mathbf{w}| = 50\sin 37^\circ \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07$ kg.

37. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the force vectors where $|\mathbf{v}_1| = 25$, $|\mathbf{v}_2| = 12$, and $|\mathbf{v}_3| = 4$. Set up coordinate axes so that the object is at the origin and $\mathbf{v}_1$, $\mathbf{v}_2$ lie in the xy-plane. We can position the vectors so that $\mathbf{v}_1 = 25\,\mathbf{i}$, $\mathbf{v}_2 = 12\cos 100^\circ\,\mathbf{i} + 12\sin 100^\circ\,\mathbf{j}$, and $\mathbf{v}_3 = 4\,\mathbf{k}$. The magnitude of a force that counterbalances the three given forces must match the magnitude of the resultant force. We have $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (25 + 12\cos 100^\circ)\,\mathbf{i} + 12\sin 100^\circ\,\mathbf{j} + 4\,\mathbf{k}$, so the counterbalancing force must have magnitude $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| = \sqrt{(25 + 12\cos 100^\circ)^2 + (12\sin 100^\circ)^2 + 4^2} \approx 26.1$ N.

38. (a) Set up coordinate axes so that the rower is at the origin, the channel is bordered by the y-axis and the line $x = 1400$, and the current flows in the negative y direction. The rower wants to reach the point $(1400, 800)$. Let θ be the angle between the positive y-axis in the direction she should steer. (See the figure.)



In still water, the rower has velocity $\mathbf{v}_r = \langle 7\sin\theta, 7\cos\theta\rangle$ and the velocity of the current is $\mathbf{v}_c = \langle 0, -3\rangle$, so the true course of the rower is determined by the velocity vector $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_c = \langle 7\sin\theta, 7\cos\theta - 3\rangle$. Let t be the time in seconds after the rower departs. Then the position of the rower is given by $t\mathbf{v}$ and the rower crosses the channel when

$$t\mathbf{v} = t\,\langle 7\sin\theta, 7\cos\theta - 3\rangle = \langle 1400, 800\rangle \;\Rightarrow\; 7t\sin\theta = 1400 \quad \text{and} \quad (7\cos\theta - 3)t = 800$$

Then $t = \dfrac{1400}{7\sin\theta} = \dfrac{200}{\sin\theta}$ and substituting gives

$$(7\cos\theta - 3)\left(\frac{200}{\sin\theta}\right) = 800 \;\Rightarrow\; 7\cos\theta - 3 = 4\sin\theta \qquad \textbf{(1)}$$

Squaring both sides, we have

$$49\cos^2\theta - 42\cos\theta + 9 = 16\sin^2\theta = 16(1 - \cos^2\theta)$$

$$65\cos^2\theta - 42\cos\theta - 7 = 0$$

The quadratic formula gives

$$\cos\theta = \frac{42 \pm \sqrt{(-42)^2 - 4(65)(-7)}}{2(65)} = \frac{42 \pm \sqrt{3584}}{130} \approx 0.78359 \text{ or } -0.13743$$

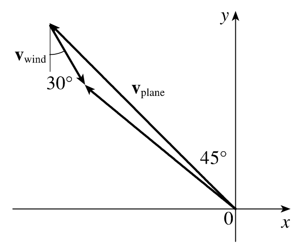
[continued]

The acute value for θ is approximately $\cos^{-1}(0.78359) \approx 38.4°$. Thus, the rower should steer in the direction that is $38.4°$ from the bank, toward upstream.

Alternate solution: We could solve **(1)** graphically by plotting $y_1 = 7\cos\theta - 3$ and $y_2 = 4\sin\theta$ on a graphing device and finding the approximate intersection point $(0.6704, 4.9702)$. Thus, $\theta \approx 0.6704$ radians, or equivalently, $38.4°$.

(b) From part (a) we know the trip is completed when $t = \dfrac{200}{\sin\theta}$. As $\theta \approx 38.4°$, the time required is approximately $\dfrac{200}{\sin(38.4°)} \approx 321.9$ seconds or 5.4 minutes.

39. Set up the coordinate axes so that north is the positive y direction and west is the negative x direction. With respect to the still air, the velocity of the plane can be written as $\mathbf{v}_{\text{plane}} = \langle -180\sin 45°, 180\cos 45° \rangle$ and the velocity of the wind is given by $\mathbf{v}_{\text{wind}} = \langle 35\sin 30°, -35\cos 30° \rangle$. (See the figure.)



Then the velocity vector of the plane relative to the ground is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_{\text{plane}} + \mathbf{v}_{\text{wind}} &= \langle -180\sin 45°, 180\cos 45° \rangle + \langle 35\sin 30°, -35\cos 30° \rangle \\ &= \left\langle -90\sqrt{2} + 35/2, 90\sqrt{2} - 35\sqrt{3}/2 \right\rangle \approx \langle -109.8, 97.0 \rangle\end{aligned}$$

The ground speed is $|\mathbf{v}| \approx \sqrt{(-109.8)^2 + (97.0)^2} \approx 146.5$ mi/h. The angle the velocity vector makes with the x-axis is about $\tan^{-1}\left(\dfrac{97.0}{-109.8}\right) \approx -41.5°$ and $-41.5° + 180° = 138.5°$. Therefore, the course of the plane is about N $(138.5 - 90)°$ W or N $48.5°$ W.

40. With respect to the water's surface, the dog's velocity is the sum of the velocity of the ship with respect to the water and the velocity of the dog with respect to the ship. If we let north be the positive y direction and west be the negative x direction, we have $\mathbf{v} = \langle -32, 0 \rangle + \langle 0, 4 \rangle = \langle -32, 4 \rangle$. Then, the speed of the dog is $|\mathbf{v}| = \sqrt{(-32)^2 + 4^2} \approx 32.2$ km/h. The vector $\mathbf{v}$ makes an angle of $\tan^{-1}\left(\dfrac{4}{-32}\right) \approx -7.1°$ and $-7.1° + 180° = 172.9°$. Therefore, the dog's direction is N $(172.9 - 90)°$ W or N $82.9°$ W.

41. The slope of the tangent line to the graph of $y = x^2$ at the point $(2, 4)$ is

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.2x\right|_{x=2} = 4$$

Thus, a parallel vector is $\mathbf{i} + 4\,\mathbf{j}$, which has length $|\mathbf{i} + 4\,\mathbf{j}| = \sqrt{1^2 + 4^2} = \sqrt{17}$, and so unit vectors parallel to the tangent line are $\pm\frac{1}{\sqrt{17}}\,(\mathbf{i} + 4\,\mathbf{j})$.

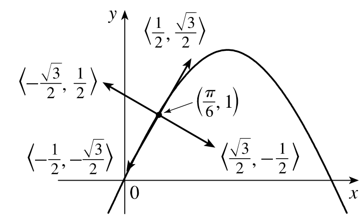
42. (a) The slope of the tangent line to the graph of $y = 2\sin x$ at the point $(\pi/6, 1)$ is

$$\left.\frac{dy}{dx}\right|_{x=\pi/6} = 2\cos x\Big|_{x=\pi/6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, a parallel vector is $\mathbf{i} + \sqrt{3}\,\mathbf{j}$, which has length $\left|\mathbf{i} + \sqrt{3}\,\mathbf{j}\right| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$, and so unit vectors parallel to the tangent line are $\pm\frac{1}{2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j}\right)$.

(b) The slope of the tangent line is $\sqrt{3}$, so the slope of a line perpendicular to the tangent line is $-\frac{1}{\sqrt{3}}$ and a vector in this direction is $\sqrt{3}\,\mathbf{i} - \mathbf{j}$. Since $\left|\sqrt{3}\,\mathbf{i} - \mathbf{j}\right| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, unit vectors perpendicular to the tangent line are $\pm\frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right)$.

(c)

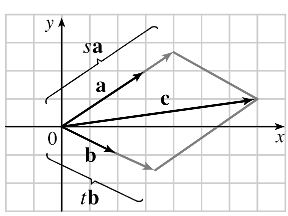


43. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$.

So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

44. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \;\Rightarrow\; \overrightarrow{AB} = 3\,\mathbf{c} - 3\,\mathbf{a}$. $\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BA} \;\Rightarrow\; \overrightarrow{BA} = \frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b}$. $\overrightarrow{BA} = -\overrightarrow{AB}$, so $\frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b} = 3\,\mathbf{a} - 3\,\mathbf{c} \;\Leftrightarrow\; \mathbf{c} + 2\,\mathbf{c} = 2\,\mathbf{a} + \mathbf{b} \;\Leftrightarrow\; \mathbf{c} = \frac{2}{3}\,\mathbf{a} + \frac{1}{3}\,\mathbf{b}$.

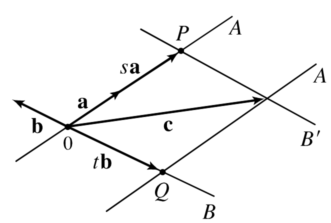
45. (a), (b)



(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \;\Leftrightarrow\; 7 = 3s + 2t$ and $1 = 2s - t$.
Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

46. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if



$s = \dfrac{\left|\overrightarrow{OP}\right|}{|\mathbf{a}|}$ (or its negative, if $\mathbf{a}$ points in the direction opposite $\overrightarrow{OP}$) and $t = \dfrac{\left|\overrightarrow{OQ}\right|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly $s = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$.

Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, the denominator is not zero.

47. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate method: $|\mathbf{r} - \mathbf{r}_0| = 1 \quad \Leftrightarrow \quad \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \quad \Leftrightarrow$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

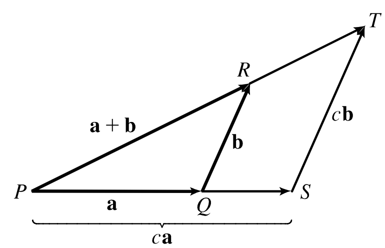
48. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2. Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2. The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

49.
$$\begin{aligned}\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle \\ &= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2 \rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle \\ &= \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle \\ &= (\mathbf{a} + \mathbf{b}) + \mathbf{c}\end{aligned}$$

50. *Algebraically:*
$$\begin{aligned}c(\mathbf{a} + \mathbf{b}) &= c\left(\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle\right) = c\,\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ &= \langle c\,(a_1 + b_1)\,, c\,(a_2 + b_2)\,, c\,(a_3 + b_3) \rangle = \langle ca_1 + cb_1, ca_2 + cb_2, ca_3 + cb_3 \rangle \\ &= \langle ca_1, ca_2, ca_3 \rangle + \langle cb_1, cb_2, cb_3 \rangle = c\,\mathbf{a} + c\,\mathbf{b}\end{aligned}$$

Geometrically:

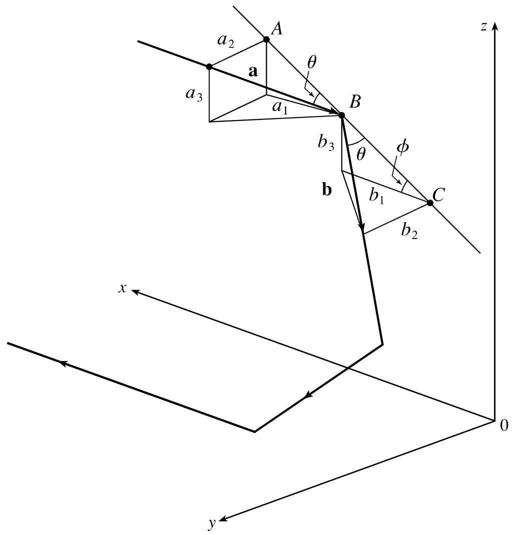
According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then $\mathbf{a} + \mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\,\mathbf{a}$ and $\overrightarrow{ST} = c\,\mathbf{b}$. (We have drawn the case where $c > 1$.) By the Triangle Law, $\overrightarrow{PT} = c\,\mathbf{a} + c\,\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\mathbf{b}$ is parallel to $\mathbf{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$. Thus, $c\,\mathbf{a} + c\,\mathbf{b} = c(\mathbf{a} + \mathbf{b})$.



51. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ **(1)** and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ **(2)**. However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into **(2)** gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with **(1)** gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

52. The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \quad \Rightarrow \quad |b_2| = |a_2|$.

[continued]



From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so

$|b_3| = \sin\phi\ |BC| = \sin\phi\ |AB| = |a_3|$, and

$|b_1| = \cos\phi\ |BC| = \cos\phi\ |AB| = |a_1|$.

$b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3\rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

DISCOVERY PROJECT The Shape of a Hanging Chain

1. As $s(x)$ is the length of the chain with uniform density ρ, the mass of the chain is given by $\rho s(x)$. Then the downward gravitational force is given by $\mathbf{w} = \langle 0, -g\rho s(x)\rangle$. Also, $\mathbf{T}_0 = \langle |\mathbf{T}_0|\cos 180°, |\mathbf{T}_0|\sin 180°\rangle = \langle -|\mathbf{T}_0|, 0\rangle$. As the system is in equilibrium, we have

$$\begin{aligned}\mathbf{T}_0 + \mathbf{T} + \mathbf{w} &= \mathbf{0}\\ \mathbf{T} &= -\mathbf{T}_0 - \mathbf{w}\\ &= -\langle -|\mathbf{T}_0|, 0\rangle - \langle 0, -g\rho s(x)\rangle\\ &= \langle |\mathbf{T}_0|, g\rho s(x)\rangle\end{aligned}$$

2. Note that the vector $\mathbf{T}$ is parallel to the tangent line to the curve at the point (x, y). Thus, the slope of the tangent line can be written as

$$\frac{dy}{dx} = \frac{g\rho s(x)}{|\mathbf{T}_0|} = \frac{s(x)}{|\mathbf{T}_0|/(g\rho)} = \frac{s(x)}{a} \qquad \text{where} \quad a = \frac{|\mathbf{T}_0|}{g\rho}$$

3. By Equation 8.1.6, $s'(x) = \int_0^x \sqrt{1 + \left(\dfrac{dy}{dt}\right)^2}$, so differentiating both sides of the equation from Problem 2 gives

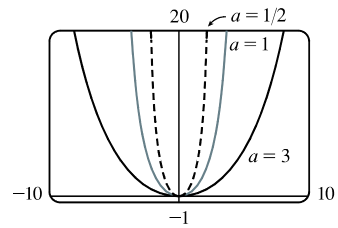
$\dfrac{d^2y}{dx^2} = \dfrac{1}{a}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Making the substitution $z = \dfrac{dy}{dx}$, we have $\dfrac{dz}{dx} = \dfrac{1}{a}\sqrt{1 + z^2} \quad\Rightarrow\quad \dfrac{dz}{\sqrt{1 + z^2}} = \dfrac{dx}{a}$.

From Table 3.11.6 we know that an antiderivative of $1/\sqrt{1 + x^2}$ is $\sinh^{-1} x$, so integrating both sides of the preceding equation gives $\sinh^{-1} z = \dfrac{x}{a} + C$. We are given that $y'(0) = 0 \quad\Rightarrow\quad z(0) = 0 \quad\Rightarrow\quad C = 0$, so $\sinh^{-1} z = \dfrac{x}{a} \quad\Rightarrow$ $z = \sinh\dfrac{x}{a}$.

[continued]

As $z = \dfrac{dy}{dx}, \dfrac{dy}{dx} = \sinh\dfrac{x}{a} \quad\Rightarrow\quad dy = \sinh\dfrac{x}{a}\,dx \quad\Rightarrow\quad \int dy = \int \sinh\dfrac{x}{a}\,dx \quad\Rightarrow\quad y = a\cosh\dfrac{x}{a} + C$. From the initial condition $y(0) = 0$, we have $0 = a\cosh 0 + C \quad\Rightarrow\quad 0 = a + C \quad\Rightarrow\quad -a = C$. Therefore, the equation of the curve is $y = a\cosh\dfrac{x}{a} - a$.

4. As the value of a increases, the graph of $y = a\cosh\dfrac{x}{a} - a$ is stretched horizontally.



12.3 The Dot Product

1. (a) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ has no meaning.

(b) $(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b}\cdot\mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b}\cdot\mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ are vectors, so the dot product $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ has meaning.

(e) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$ has no meaning.

2. $\mathbf{a}\cdot\mathbf{b} = \langle 5,-2\rangle\cdot\langle 3,4\rangle = (5)(3) + (-2)(4) = 15 - 8 = 7$

3. $\mathbf{a}\cdot\mathbf{b} = \langle 1.5, 0.4\rangle\cdot\langle -4, 6\rangle = (1.5)(-4) + (0.4)(6) = -6 + 2.4 = -3.6$

4. $\mathbf{a}\cdot\mathbf{b} = \langle 6,-2,3\rangle\cdot\langle 2,5,-1\rangle = (6)(2) + (-2)(5) + (3)(-1) = 12 - 10 - 3 = -1$

5. $\mathbf{a}\cdot\mathbf{b} = \left\langle 4, 1, \frac{1}{4}\right\rangle\cdot\langle 6,-3,-8\rangle = (4)(6) + (1)(-3) + \left(\frac{1}{4}\right)(-8) = 19$

6. $\mathbf{a}\cdot\mathbf{b} = \langle p, -p, 2p\rangle\cdot\langle 2q, q, -q\rangle = (p)(2q) + (-p)(q) + (2p)(-q) = 2pq - pq - 2pq = -pq$

7. $\mathbf{a}\cdot\mathbf{b} = (2\,\mathbf{i} + \mathbf{j})\cdot(\mathbf{i} - \mathbf{j} + \mathbf{k}) = (2)(1) + (1)(-1) + (0)(1) = 1$

8. $\mathbf{a}\cdot\mathbf{b} = (3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})\cdot(4\,\mathbf{i} + 5\,\mathbf{k}) = (3)(4) + (2)(0) + (-1)(5) = 7$

9. By Theorem 3, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = (7)(4)\cos 30^\circ = 28\left(\frac{\sqrt{3}}{2}\right) = 14\sqrt{3}$.

10. By Theorem 3, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = (80)(50)\cos\frac{3\pi}{4} = 4000\left(-\frac{\sqrt{2}}{2}\right) = -2000\sqrt{2}$.

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u}\cdot\mathbf{w} = |\mathbf{u}|\,|\mathbf{w}|\cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is $45°$, we have $|\mathbf{v}| = |\mathbf{u}|\cos 45° = \frac{\sqrt{2}}{2}$. Then $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 45° = (1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u}\cdot\mathbf{w} = 0$.

13. (a) $\mathbf{i}\cdot\mathbf{j} = \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle = (1)(0)+(0)(1)+(0)(0) = 0$. Similarly, $\mathbf{j}\cdot\mathbf{k} = (0)(0)+(1)(0)+(0)(1) = 0$ and $\mathbf{k}\cdot\mathbf{i} = (0)(1)+(0)(0)+(1)(0) = 0$.

Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2} = 0$.

(b) By Property 1 of the dot product, $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}\cdot\mathbf{k} = |\mathbf{k}|^2 = 1$.

14. The dot product $\mathbf{A}\cdot\mathbf{P}$ is

$$\begin{aligned}\langle a,b,c\rangle\cdot\langle 4,2.5,1\rangle &= a(4)+b(2.5)+c(1)\\ &= (\text{number of hamburgers sold})(\text{price per hamburger})\\ &\quad + (\text{number of hot dogs sold})(\text{price per hot dog})\\ &\quad + (\text{number of bottles sold})(\text{price per bottle})\end{aligned}$$

so it is equal to the vendor's total revenue for that day.

15. $\mathbf{u} = \langle 5,1\rangle$, $\mathbf{v} = \langle 3,2\rangle \quad\Rightarrow\quad |\mathbf{u}| = \sqrt{5^2+1^2} = \sqrt{26}$, $|\mathbf{v}| = \sqrt{3^2+2^2} = \sqrt{13}$, and $\mathbf{u}\cdot\mathbf{v} = 5(3)+1(2) = 17$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{17}{\sqrt{26}\sqrt{13}} = \dfrac{17}{13\sqrt{2}}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{17}{13\sqrt{2}}\right) \approx 22°$.

16. $\mathbf{a} = \mathbf{i} - 3\,\mathbf{j}$, $\mathbf{b} = -3\,\mathbf{i} + 4\,\mathbf{j} \quad\Rightarrow\quad |\mathbf{a}| = \sqrt{1^2+(-3)^2} = \sqrt{10}$, $|\mathbf{b}| = \sqrt{(-3)^2+4^2} = 5$, and $\mathbf{a}\cdot\mathbf{b} = 1(-3)+(-3)(4) = -15$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-15}{5\sqrt{10}} = \dfrac{-3}{\sqrt{10}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{10}}\right) \approx 162°$.

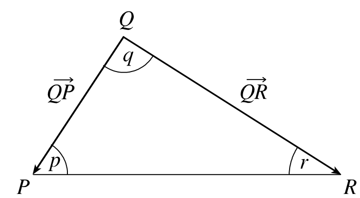
17. $\mathbf{a} = \langle 1,-4,1\rangle$, $\mathbf{b} = \langle 0,2,-2\rangle \quad\Rightarrow\quad |\mathbf{a}| = \sqrt{1^2+(-4)^2+1^2} = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{b}| = \sqrt{0^2+2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$, and $\mathbf{a}\cdot\mathbf{b} = (1)(0)+(-4)(2)+(1)(-2) = -10$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-10}{3\sqrt{2}\cdot 2\sqrt{2}} = -\dfrac{10}{12} = -\dfrac{5}{6}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(-\frac{5}{6}\right) \approx 146°$.

18. $\mathbf{a} = \langle -1,3,4\rangle$, $\mathbf{b} = \langle 5,2,1\rangle \quad\Rightarrow\quad |\mathbf{a}| = \sqrt{(-1)^2+3^2+4^2} = \sqrt{26}$, $|\mathbf{b}| = \sqrt{5^2+2^2+1^2} = \sqrt{30}$, and $\mathbf{a}\cdot\mathbf{b} = (-1)(5)+(3)(2)+(4)(1) = 5$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{\sqrt{26}\cdot\sqrt{30}} = \dfrac{5}{\sqrt{780}} = \dfrac{5}{2\sqrt{195}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{5}{2\sqrt{195}}\right) \approx 80°$.

19. $\mathbf{u} = \mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -3\,\mathbf{i} + \mathbf{j} + 5\,\mathbf{k} \Rightarrow |\mathbf{u}| = \sqrt{1^2 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{v}| = \sqrt{(-3)^2 + 1^2 + 5^2} = \sqrt{35}$, and $\mathbf{u} \cdot \mathbf{v} = 1(-3) + (-4)(1) + 1(5) = -2$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{-2}{3\sqrt{2}\sqrt{35}} = \dfrac{-2}{3\sqrt{70}}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{-2}{3\sqrt{70}}\right) \approx 95°$.

20. $\mathbf{a} = 8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 4\,\mathbf{j} + 2\,\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{81} = 9$, $|\mathbf{b}| = \sqrt{0^2 + 4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$, and $\mathbf{a} \cdot \mathbf{b} = (8)(0) + (-1)(4) + (4)(2) = 4$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{4}{9 \cdot 2\sqrt{5}} = \dfrac{2}{9\sqrt{5}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{2}{9\sqrt{5}}\right) \approx 84°$.

21. Let p, q, and r be the angles at vertices P, Q, and R respectively. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.



Thus $\cos p = \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{PR}\right|} = \dfrac{\langle -2, 3\rangle \cdot \langle 1, 4\rangle}{\sqrt{(-2)^2 + 3^2}\,\sqrt{1^2 + 4^2}} = \dfrac{-2 + 12}{\sqrt{13}\,\sqrt{17}} = \dfrac{10}{\sqrt{221}}$ and $p = \cos^{-1}\left(\dfrac{10}{\sqrt{221}}\right) \approx 48°$. Similarly,

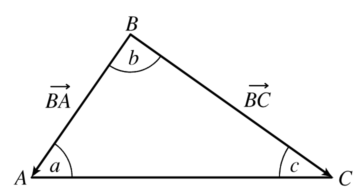
$\cos q = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\,\left|\overrightarrow{QR}\right|} = \dfrac{\langle 2, -3\rangle \cdot \langle 3, 1\rangle}{\sqrt{4+9}\,\sqrt{9+1}} = \dfrac{6-3}{\sqrt{13}\,\sqrt{10}} = \dfrac{3}{\sqrt{130}}$ so $q = \cos^{-1}\left(\dfrac{3}{\sqrt{130}}\right) \approx 75°$ and

$r \approx 180° - (48° + 75°) = 57°$.

Alternate solution: Apply the Law of Cosines three times as follows:

$$\cos p = \frac{\left|\overrightarrow{QR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2}{2\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{PR}\right|} \qquad \cos q = \frac{\left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{QR}\right|} \qquad \cos r = \frac{\left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PR}\right|\,\left|\overrightarrow{QR}\right|}$$

22. Let a, b, and c be the angles at vertices A, B, and C. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|} = \dfrac{\langle 2, -2, 1\rangle \cdot \langle 0, 3, 4\rangle}{\sqrt{2^2 + (-2)^2 + 1^2}\,\sqrt{0^2 + 3^2 + 4^2}} = \dfrac{0 - 6 + 4}{3 \cdot 5} = -\dfrac{2}{15}$ and $a = \cos^{-1}\left(-\frac{2}{15}\right) \approx 98°$.

Similarly, $\cos b = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\,\left|\overrightarrow{BC}\right|} = \dfrac{\langle -2, 2, -1\rangle \cdot \langle -2, 5, 3\rangle}{\sqrt{4+4+1}\,\sqrt{4+25+9}} = \dfrac{4 + 10 - 3}{3 \cdot \sqrt{38}} = \dfrac{11}{3\sqrt{38}}$ so $b = \cos^{-1}\left(\dfrac{11}{3\sqrt{38}}\right) \approx 54°$ and

$c \approx 180° - (98° + 54°) = 28°$.

[continued]

Alternate solution: Apply the Law of Cosines three times as follows:

$$\cos a = \frac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} \qquad \cos b = \frac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} \qquad \cos c = \frac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|}$$

23. (a) $\mathbf{a}\cdot\mathbf{b} = (9)(-2) + (3)(6) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(b) $\mathbf{a}\cdot\mathbf{b} = (4)(3) + (5)(-1) + (-2)(5) = -3 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(c) $\mathbf{a}\cdot\mathbf{b} = (-8)(6) + (12)(-9) + (4)(-3) = -168 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Because $\mathbf{a} = -\frac{4}{3}\,\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(d) $\mathbf{a}\cdot\mathbf{b} = (3)(5) + (-1)(9) + (3)(-2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

24. (a) $\mathbf{u}\cdot\mathbf{v} = (-5)(3) + (4)(4) + (-2)(-1) = 3 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Also, $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

(b) $\mathbf{u}\cdot\mathbf{v} = (9)(-6) + (-6)(4) + (3)(-2) = -84 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Because $\mathbf{u} = -\frac{3}{2}\,\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel.

(c) $\mathbf{u}\cdot\mathbf{v} = (c)(c) + (c)(0) + (c)(-c) = c^2 + 0 - c^2 = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (and not parallel). (Note that if $c = 0$ then $\mathbf{u} = \mathbf{v} = \mathbf{0}$, and the zero vector is considered orthogonal to all vectors. Although in this case $\mathbf{u}$ and $\mathbf{v}$ are identical, they are not considered parallel, as only nonzero vectors can be parallel.)

25. $\overrightarrow{QP} = \langle -1, -3, 2\rangle$, $\overrightarrow{QR} = \langle 4, -2, -1\rangle$, and $\overrightarrow{QP}\cdot\overrightarrow{QR} = -4 + 6 - 2 = 0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. By Theorem 3, vectors $\langle 2, 1, -1\rangle$ and $\langle 1, x, 0\rangle$ meet at an angle of 45° when

$\langle 2, 1, -1\rangle \cdot \langle 1, x, 0\rangle = \sqrt{4+1+1}\,\sqrt{1+x^2+0}\,\cos 45^\circ$ or $2 + x - 0 = \sqrt{6}\,\sqrt{1+x^2}\cdot\frac{\sqrt{2}}{2} \quad\Leftrightarrow\quad 2 + x = \sqrt{3}\sqrt{1+x^2}$.

Squaring both sides gives $4 + 4x + x^2 = 3 + 3x^2 \quad\Leftrightarrow\quad 2x^2 - 4x - 1 = 0$. By the quadratic formula,

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)} = \dfrac{4 \pm \sqrt{24}}{4} = \dfrac{4 \pm 2\sqrt{6}}{4} = 1 \pm \dfrac{\sqrt{6}}{2}$. (You can verify that both values are valid.)

27. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$. Then $\mathbf{a}\cdot(\mathbf{i}+\mathbf{j}) = 0 \quad\Leftrightarrow\quad a_1 + a_2 = 0$ and $\mathbf{a}\cdot(\mathbf{i}+\mathbf{k}) = 0 \quad\Leftrightarrow\quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

28. Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector. By Theorem 3 we need $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\;|\mathbf{v}|\cos 60^\circ \quad\Leftrightarrow\quad 3a + 4b = (1)(5)\frac{1}{2} \quad\Leftrightarrow$

$b = \frac{5}{8} - \frac{3}{4}a$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = \sqrt{a^2+b^2} = 1 \quad\Leftrightarrow\quad a^2 + b^2 = 1 \quad\Leftrightarrow\quad a^2 + \left(\frac{5}{8} - \frac{3}{4}a\right)^2 = 1 \quad\Leftrightarrow$

$\frac{25}{16}a^2 - \frac{15}{16}a + \frac{25}{64} = 1 \quad \Leftrightarrow \quad 100a^2 - 60a - 39 = 0$. By the quadratic formula,

$a = \dfrac{-(-60) \pm \sqrt{(-60)^2 - 4(100)(-39)}}{2(100)} = \dfrac{60 \pm \sqrt{19{,}200}}{200} = \dfrac{3 \pm 4\sqrt{3}}{10}$. If $a = \dfrac{3 + 4\sqrt{3}}{10}$ then

$b = \dfrac{5}{8} - \dfrac{3}{4}\left(\dfrac{3 + 4\sqrt{3}}{10}\right) = \dfrac{4 - 3\sqrt{3}}{10}$, and if $a = \dfrac{3 - 4\sqrt{3}}{10}$ then $b = \dfrac{5}{8} - \dfrac{3}{4}\left(\dfrac{3 - 4\sqrt{3}}{10}\right) = \dfrac{4 + 3\sqrt{3}}{10}$. Thus the two

unit vectors are $\left\langle \dfrac{3 + 4\sqrt{3}}{10}, \dfrac{4 - 3\sqrt{3}}{10} \right\rangle \approx \langle 0.9928, -0.1196 \rangle$ and $\left\langle \dfrac{3 - 4\sqrt{3}}{10}, \dfrac{4 + 3\sqrt{3}}{10} \right\rangle \approx \langle -0.3928, 0.9196 \rangle$.

29. The line $y = 4 - 3x \quad \Leftrightarrow \quad y = -3x + 4$ has slope -3, so a vector parallel to the line is $\mathbf{a} = \langle 1, -3 \rangle$. The line $y = 3x + 2$ has slope 3, so a vector parallel to the line is $\mathbf{b} = \langle 1, 3 \rangle$. The angle between the lines is the same as the angle θ between the vectors. Here we have $\mathbf{a} \cdot \mathbf{b} = 1(1) + 3(-3) = -8$, $|\mathbf{a}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$, and $|\mathbf{b}| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-8}{\sqrt{10}\sqrt{10}} = -\dfrac{4}{5}$ and $\theta = \cos^{-1}\left(-\dfrac{4}{5}\right) \approx 143.1^\circ$. Therefore, the acute angle between the two lines is approximately $180^\circ - 143.1^\circ = 36.9^\circ$.

30. The line $5x - y = 8 \quad \Leftrightarrow \quad y = 5x - 8$ has slope 5, so a vector parallel to the line is $\mathbf{a} = \langle 1, 5 \rangle$. The line $x + 3y = 15 \quad \Leftrightarrow \quad y = -\frac{1}{3}x + 5$ has slope $-\frac{1}{3}$, so a vector parallel to the line is $\mathbf{b} = \langle 3, -1 \rangle$. The angle between the lines is the same as the angle θ between the vectors. Here we have $\mathbf{a} \cdot \mathbf{b} = 1(3) + 5(-1) = -2$, $|\mathbf{a}| = \sqrt{1^2 + 5^2} = \sqrt{26}$, and $|\mathbf{b}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-2}{\sqrt{26}\sqrt{10}} = -\dfrac{1}{\sqrt{65}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{\sqrt{65}}\right) \approx 97.1^\circ$. Therefore, the acute angle between the two lines is approximately $180^\circ - 97.1^\circ = 82.9^\circ$.

31. The curves $y = x^2$ and $y = x^3$ meet when $x^2 = x^3 \quad \Leftrightarrow \quad x^3 - x^2 = 0 \quad \Leftrightarrow \quad x^2(x - 1) = 0 \quad \Leftrightarrow \quad x = 0, x = 1$. We have $\dfrac{d}{dx}x^2 = 2x$ and $\dfrac{d}{dx}x^3 = 3x^2$, so the tangent lines of both curves have slope 0 at $x = 0$. Thus the angle between the curves is 0° at the point $(0, 0)$. For $x = 1$, $\dfrac{d}{dx}\,x^2\Big|_{x=1} = 2$ and $\dfrac{d}{dx}\,x^3\Big|_{x=1} = 3$ so the tangent lines at the point $(1, 1)$ have slopes 2 and 3. Vectors parallel to the tangent lines are $\langle 1, 2 \rangle$ and $\langle 1, 3 \rangle$, and the angle θ between them is given by

$$\cos\theta = \frac{\langle 1, 2 \rangle \cdot \langle 1, 3 \rangle}{|\langle 1, 2 \rangle|\;|\langle 1, 3 \rangle|} = \frac{1 + 6}{\sqrt{5}\sqrt{10}} = \frac{7}{5\sqrt{2}}$$

Thus $\theta = \cos^{-1}\left(\dfrac{7}{5\sqrt{2}}\right) \approx 8.1^\circ$.

32. The curves $y = \sin x$ and $y = \cos x$ meet when $\sin x = \cos x \quad \Leftrightarrow \quad \tan x = 1 \quad \Leftrightarrow \quad x = \pi/4\;\; [0 \le x \le \pi/2]$. Thus the point of intersection is $\left(\pi/4, \sqrt{2}/2\right)$. We have $\dfrac{d}{dx}\,\sin x\Big|_{x=\pi/4} = \cos x\Big|_{x=\pi/4} = \dfrac{\sqrt{2}}{2}$ and $\dfrac{d}{dx}\,\cos x\Big|_{x=\pi/4} = -\sin x\Big|_{x=\pi/4} = -\dfrac{\sqrt{2}}{2}$, so the tangent lines at that point have slopes $\dfrac{\sqrt{2}}{2}$ and $-\dfrac{\sqrt{2}}{2}$. Vectors parallel to

the tangent lines are $\left\langle 1, \dfrac{\sqrt{2}}{2} \right\rangle$ and $\left\langle 1, -\dfrac{\sqrt{2}}{2} \right\rangle$, and the angle θ between them is given by

$$\cos\theta = \frac{\langle 1, \sqrt{2}/2\rangle \cdot \langle 1, -\sqrt{2}/2\rangle}{|\langle 1, \sqrt{2}/2\rangle|\,|\langle 1, -\sqrt{2}/2\rangle|} = \frac{1-\frac{1}{2}}{\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}} = \frac{1/2}{3/2} = \frac{1}{3}$$

Thus $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5°$.

33. $|\langle 4, 1, 8\rangle| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$. Using Equations 8 and 9, we have $\cos\alpha = \frac{4}{9}$, $\cos\beta = \frac{1}{9}$, and $\cos\gamma = \frac{8}{9}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{4}{9}\right) \approx 63.6°$, $\beta = \cos^{-1}\left(\frac{1}{9}\right) \approx 83.6°$, and $\gamma = \cos^{-1}\left(\frac{8}{9}\right) \approx 27.3°$.

34. $|\langle -6, 2, 9\rangle| = \sqrt{(-6)^2 + 2^2 + 9^2} = \sqrt{121} = 11$. Using Equations 8 and 9, we have $\cos\alpha = -\frac{6}{11}$, $\cos\beta = \frac{2}{11}$, and $\cos\gamma = \frac{9}{11}$. The direction angles are given by $\alpha = \cos^{-1}\left(-\frac{6}{11}\right) \approx 123.1°$, $\beta = \cos^{-1}\left(\frac{2}{11}\right) \approx 79.5°$, and $\gamma = \cos^{-1}\left(\frac{9}{11}\right) \approx 35.1°$.

35. $|3\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$. Using Equations 8 and 9, we have $\cos\alpha = \dfrac{3}{\sqrt{14}}$, $\cos\beta = -\dfrac{1}{\sqrt{14}}$, and $\cos\gamma = -\dfrac{2}{\sqrt{14}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{14}}\right) \approx 36.7°$, $\beta = \cos^{-1}\left(-\dfrac{1}{\sqrt{14}}\right) \approx 105.5°$, and $\gamma = \cos^{-1}\left(-\dfrac{2}{\sqrt{14}}\right) \approx 122.3°$.

36. $|-0.7\mathbf{i} + 1.2\mathbf{j} - 0.8\mathbf{k}| = \sqrt{(-0.7)^2 + (1.2)^2 + (-0.8)^2} = \sqrt{2.57}$. Using Equations 8 and 9, we have $\cos\alpha = -\dfrac{0.7}{\sqrt{2.57}}$, $\cos\beta = \dfrac{1.2}{\sqrt{2.57}}$, and $\cos\gamma = -\dfrac{0.8}{\sqrt{2.57}}$. The direction angles are given by $\alpha = \cos^{-1}\left(-\dfrac{0.7}{\sqrt{2.57}}\right) \approx 115.9°$, $\beta = \cos^{-1}\left(\dfrac{1.2}{\sqrt{2.57}}\right) \approx 41.5°$, and $\gamma = \cos^{-1}\left(-\dfrac{0.8}{\sqrt{2.57}}\right) \approx 119.9°$.

37. $|\langle c, c, c\rangle| = \sqrt{c^2 + c^2 + c^2} = \sqrt{3c^2} = c\sqrt{3}$. Using Equations 8 and 9, we have $\cos\alpha = \cos\beta = \cos\gamma = \dfrac{c}{c\sqrt{3}} = \dfrac{1}{\sqrt{3}}$. The direction angles are given by $\alpha = \beta = \gamma = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx 54.7°$.

38. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta = 1 - \cos^2\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{3}\right) = 1 - \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Thus $\cos\gamma = \pm\frac{1}{2}$ and $\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$.

39. $|\mathbf{a}| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{-5\cdot 4 + 12\cdot 6}{13} = 4$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = 4\cdot\frac{1}{13}\langle -5, 12\rangle = \left\langle -\frac{20}{13}, \frac{48}{13}\right\rangle$.

40. $|\mathbf{a}| = \sqrt{1^2 + 4^2} = \sqrt{17}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{1\cdot 2 + 4\cdot 3}{\sqrt{17}} = \dfrac{14}{\sqrt{17}}$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{14}{\sqrt{17}}\cdot\frac{1}{\sqrt{17}}\langle 1, 4\rangle = \left\langle \frac{14}{17}, \frac{56}{17}\right\rangle$.

41. $|\mathbf{a}| = \sqrt{4^2 + 7^2 + (-4)^2} = \sqrt{81} = 9$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{(4)(3) + (7)(-1) + (-4)(1)}{9} = \dfrac{1}{9}$. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{9} \cdot \frac{1}{9} \langle 4, 7, -4 \rangle = \frac{1}{81} \langle 4, 7, -4 \rangle = \left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle$.

42. $|\mathbf{a}| = \sqrt{1 + 16 + 64} = \sqrt{81} = 9$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1}{9}(-12 + 4 + 16) = \frac{8}{9}$, while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{8}{9} \cdot \frac{1}{9} \langle -1, 4, 8 \rangle = \frac{8}{81} \langle -1, 4, 8 \rangle = \left\langle -\frac{8}{81}, \frac{32}{81}, \frac{64}{81} \right\rangle$.

43. $|\mathbf{a}| = \sqrt{9 + 9 + 1} = \sqrt{19}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{6 - 12 - 1}{\sqrt{19}} = -\dfrac{7}{\sqrt{19}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = -\dfrac{7}{\sqrt{19}} \dfrac{\mathbf{a}}{|\mathbf{a}|} = -\frac{7}{\sqrt{19}} \cdot \frac{1}{\sqrt{19}} (3\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}) = -\frac{7}{19}(3\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}) = -\frac{21}{19}\,\mathbf{i} + \frac{21}{19}\,\mathbf{j} - \frac{7}{19}\,\mathbf{k}$.

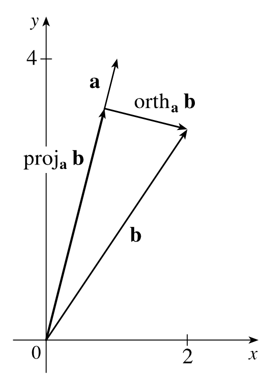
44. $|\mathbf{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{5 + 0 - 3}{\sqrt{14}} = \dfrac{2}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \dfrac{2}{\sqrt{14}} \dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{2}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}} (\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = \frac{1}{7}(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = \frac{1}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$.

45. $(\operatorname{orth}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = (\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - (\operatorname{proj}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\, \mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} |\mathbf{a}|^2 = \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} = 0$.

So they are orthogonal by (7).

46. Using the formula in Exercise 45 and the result of Exercise 40, we have

$\operatorname{orth}_{\mathbf{a}} \mathbf{b} = \mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \langle 2, 3 \rangle - \left\langle \frac{14}{17}, \frac{56}{17} \right\rangle = \left\langle \frac{20}{17}, -\frac{5}{17} \right\rangle$.



47. $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = 2 \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 2\,|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$. In general, $\mathbf{b} = \left\langle s, t, 3s - 2\sqrt{10} \right\rangle$, $s, t \in \mathbb{R}$.

48. (a) $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \operatorname{comp}_{\mathbf{b}} \mathbf{a} \quad \Leftrightarrow \quad \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|} \quad \Leftrightarrow \quad \dfrac{1}{|\mathbf{a}|} = \dfrac{1}{|\mathbf{b}|}$ or $\mathbf{a} \cdot \mathbf{b} = 0 \quad \Leftrightarrow \quad |\mathbf{b}| = |\mathbf{a}|$ or $\mathbf{a} \cdot \mathbf{b} = 0$.

That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

(b) $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{proj}_{\mathbf{b}} \mathbf{a} \quad \Leftrightarrow \quad \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\, \mathbf{a} = \dfrac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2}\, \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$ or $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2}$.

But $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2} \quad \Rightarrow \quad \dfrac{|\mathbf{a}|}{|\mathbf{a}|^2} = \dfrac{|\mathbf{b}|}{|\mathbf{b}|^2} \quad \Rightarrow \quad |\mathbf{a}| = |\mathbf{b}|$. Substituting this into the previous equation gives $\mathbf{a} = \mathbf{b}$.

So $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{proj}_{\mathbf{b}} \mathbf{a} \quad \Leftrightarrow \quad \mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

49. The displacement vector is $\mathbf{D} = (6-0)\,\mathbf{i} + (12-10)\,\mathbf{j} + (20-8)\,\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$ so, by Equation 12, the work done is $W = \mathbf{F} \cdot \mathbf{D} = (8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k}) \cdot (6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}) = 48 - 12 + 108 = 144$ joules.

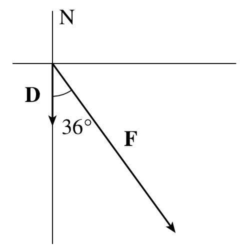
50. Here $|\mathbf{D}| = 1000$ m, $|\mathbf{F}| = 1500$ N, and $\theta = 30^\circ$. Thus

$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (1500)(1000)\left(\frac{\sqrt{3}}{2}\right) = 750{,}000\sqrt{3}$ joules.

51. Here $|\mathbf{D}| = 80$ ft, $|\mathbf{F}| = 30$ lb, and $\theta = 40^\circ$. Thus

$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (30)(80)\cos 40^\circ = 2400\cos 40^\circ \approx 1839$ ft-lb.

52. $W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (400)(120)\cos 36^\circ \approx 38{,}833$ ft-lb

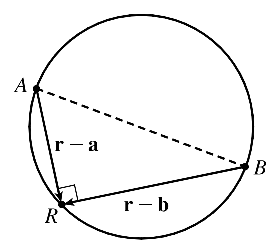


53. First note that $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then $\mathbf{n} \cdot \overrightarrow{Q_1Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line.

Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\operatorname{comp}_{\mathbf{n}}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n} \cdot \langle x_2 - x_1, y_2 - y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

since $ax_2 + by_2 = -c$. The required distance is $\dfrac{|(3)(-2) + (-4)(3) + 5|}{\sqrt{3^2 + (-4)^2}} = \dfrac{13}{5}$.

54. $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ implies that the vectors $\mathbf{r} - \mathbf{a}$ and $\mathbf{r} - \mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B. The center of this circle is the midpoint of AB, that is, $\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \left\langle \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2), \frac{1}{2}(a_3 + b_3)\right\rangle$, and its radius is $\frac{1}{2}|\mathbf{a} - \mathbf{b}| = \frac{1}{2}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.



Or: Expand the given equation, substitute $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

55. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1, 1, 1)$ has vector representation $\langle 1, 1, 1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is, $\langle 1, 0, 0\rangle$] is given by $\cos\theta = \dfrac{\langle 1, 1, 1\rangle \cdot \langle 1, 0, 0\rangle}{|\langle 1, 1, 1\rangle|\,|\langle 1, 0, 0\rangle|} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$.

56. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta = \dfrac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}+\mathbf{j}|} = \dfrac{1+1}{\sqrt{3}\sqrt{2}} = \sqrt{\dfrac{2}{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\sqrt{\frac{2}{3}} \approx 35.3^\circ$.

57. Consider the H—C—H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H—C—H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1-\frac{1}{2}, 0-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left\langle 0-\frac{1}{2}, 1-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The bond angle, θ, is therefore given by

$$\cos\,\theta = \frac{\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\cdot\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle\right|} = \frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(-\tfrac{1}{3}\right) \approx 109.5^\circ.$$

58. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha=\beta$. Now

$\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot|\mathbf{a}|\,\mathbf{b}+\mathbf{a}\cdot|\mathbf{b}|\,\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2\,|\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$. Similarly,

$\cos\beta = \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,|\mathbf{b}|+\mathbf{b}\cdot\mathbf{a}}{|\mathbf{c}|}$. Thus $\cos\alpha=\cos\beta$. However $0^\circ \le \alpha \le 180^\circ$ and $0^\circ \le \beta \le 180^\circ$, so $\alpha=\beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

59. Let $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$.

Property 2: $\quad \mathbf{a}\cdot\mathbf{b} = \langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle = a_1b_1+a_2b_2+a_3b_3$

$$= b_1a_1+b_2a_2+b_3a_3 = \langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle = \mathbf{b}\cdot\mathbf{a}$$

Property 4: $\quad (c\,\mathbf{a})\cdot\mathbf{b} = \langle ca_1,ca_2,ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle = (ca_1)b_1+(ca_2)b_2+(ca_3)b_3$

$$= c\,(a_1b_1+a_2b_2+a_3b_3) = c\,(\mathbf{a}\cdot\mathbf{b}) = a_1(cb_1)+a_2(cb_2)+a_3(cb_3)$$

$$= \langle a_1,a_2,a_3\rangle\cdot\langle cb_1,cb_2,cb_3\rangle = \mathbf{a}\cdot(c\,\mathbf{b})$$

Property 5: $\quad \mathbf{0}\cdot\mathbf{a} = \langle 0,0,0\rangle\cdot\langle a_1,a_2,a_3\rangle = (0)(a_1)+(0)(a_2)+(0)(a_3) = 0$

60. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$ and $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BC}-\overrightarrow{DC}=\overrightarrow{BC}-\overrightarrow{AB}$ (Since opposite sides of the object are of the same length and parallel, $\overrightarrow{AB}=\overrightarrow{DC}$.) Thus

$$\overrightarrow{AC}\cdot\overrightarrow{BD} = \left(\overrightarrow{AB}+\overrightarrow{BC}\right)\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right) = \overrightarrow{AB}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)+\overrightarrow{BC}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)$$

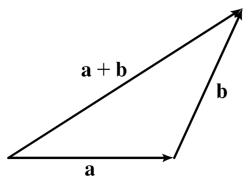
$$= \overrightarrow{AB}\cdot\overrightarrow{BC}-\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2-\overrightarrow{AB}\cdot\overrightarrow{BC} = \left|\overrightarrow{BC}\right|^2-\left|\overrightarrow{AB}\right|^2$$

But $\left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC}\cdot\overrightarrow{BD}=0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

61. $|\mathbf{a}\cdot\mathbf{b}| = \big|\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta\big| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta|$. Since $|\cos\theta| \le 1$, $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\,\theta| \le |\mathbf{a}|\,|\mathbf{b}|$.

Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

62. (a)

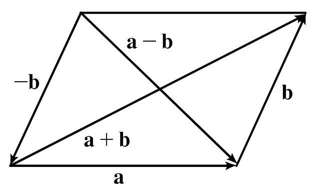


The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

(b) $$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a}) + 2(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2 \\ &\le |\mathbf{a}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| + |\mathbf{b}|^2 \qquad \text{[by the Cauchy-Schwartz Inequality]} \\ &= (|\mathbf{a}| + |\mathbf{b}|)^2 \end{aligned}$$

Thus, taking the square root of both sides, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.

63. (a)



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$ and $|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$.

Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2$.

64. If the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$. But

$$\begin{aligned} (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} && \text{by Property 3 of the dot product} \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} && \text{by Property 3} \\ &= |\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - |\mathbf{v}|^2 && \text{by Properties 1 and 2} \\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2 \end{aligned}$$

Thus $|\mathbf{u}|^2 - |\mathbf{v}|^2 = 0 \quad\Rightarrow\quad |\mathbf{u}|^2 = |\mathbf{v}|^2 \quad\Rightarrow\quad |\mathbf{u}| = |\mathbf{v}|$ [since $|\mathbf{u}|, |\mathbf{v}| \ge 0$].

65. $$\begin{aligned} \operatorname{proj}_{\mathbf{a}}\mathbf{b}\cdot\operatorname{proj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\cdot\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\cdot\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\,(\mathbf{a}\cdot\mathbf{b}) && \text{by Property 4 of the dot product} \\ &= \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{a}|^2\,|\mathbf{b}|^2}\,(\mathbf{a}\cdot\mathbf{b}) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right)^2(\mathbf{a}\cdot\mathbf{b}) && \text{by Property 2} \\ &= (\cos\theta)^2\,(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\cos^2\theta && \text{by Corollary 6} \end{aligned}$$

66. (a) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero orthogonal vectors. Then

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 + 0 + 0 + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \end{aligned}$$

(b) Suppose that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. From part (a), we know that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2$. Thus, $2(\mathbf{u}\cdot\mathbf{v}) = 0$, which implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

12.4 The Cross Product

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}\mathbf{k}$

$= (15-0)\,\mathbf{i} - (10-0)\,\mathbf{j} + (0-3)\,\mathbf{k} = 15\,\mathbf{i} - 10\,\mathbf{j} - 3\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 15,-10,-3\rangle\cdot\langle 2,3,0\rangle = 30-30+0 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 15,-10,-3\rangle\cdot\langle 1,0,5\rangle = 15+0-15 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{k}$

$= (3-2)\,\mathbf{i} - [4-(-4)]\,\mathbf{j} + (-4-6)\,\mathbf{k} = \mathbf{i} - 8\,\mathbf{j} - 10\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 1,-8,-10\rangle\cdot\langle 4,3,-2\rangle = 4-24+20 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 1,-8,-10\rangle\cdot\langle 2,-1,1\rangle = 2+8-10 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -4 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -4 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix}\mathbf{k}$

$= [2-(-12)]\,\mathbf{i} - (0-4)\,\mathbf{j} + [0-(-2)]\,\mathbf{k} = 14\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(2\,\mathbf{j}-4\,\mathbf{k}) = 0+8-8 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(-\mathbf{i}+3\,\mathbf{j}+\mathbf{k}) = -14+12+2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -3 \\ 3 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -3 \\ 3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 3 & -3 \end{vmatrix}\mathbf{k}$

$= (9-9)\,\mathbf{i} - [9-(-9)]\,\mathbf{j} + (-9-9)\,\mathbf{k} = -18\,\mathbf{j} - 18\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-18\,\mathbf{j}-18\,\mathbf{k})\cdot(3\,\mathbf{i}+3\,\mathbf{j}-3\,\mathbf{k}) = 0-54+54 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (-18\,\mathbf{j}-18\,\mathbf{k})\cdot(3\,\mathbf{i}-3\,\mathbf{j}+3\,\mathbf{k}) = 0+54-54 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{4} \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 2 \end{vmatrix}\mathbf{k}$

$= \left(-1-\frac{1}{2}\right)\mathbf{i} - \left(-\frac{3}{2}-\frac{1}{4}\right)\mathbf{j} + \left(1-\frac{1}{3}\right)\mathbf{k} = -\frac{3}{2}\,\mathbf{i} + \frac{7}{4}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \left(-\frac{3}{2}\,\mathbf{i}+\frac{7}{4}\,\mathbf{j}+\frac{2}{3}\,\mathbf{k}\right)\cdot\left(\frac{1}{2}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}+\frac{1}{4}\,\mathbf{k}\right) = -\frac{3}{4}+\frac{7}{12}+\frac{1}{6} = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \left(-\frac{3}{2}\,\mathbf{i}+\frac{7}{4}\,\mathbf{j}+\frac{2}{3}\,\mathbf{k}\right)\cdot(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k}) = -\frac{3}{2}+\frac{7}{2}-2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

6. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ t&\cos t&\sin t\\ 1&-\sin t&\cos t\end{vmatrix}=\begin{vmatrix}\cos t&\sin t\\ -\sin t&\cos t\end{vmatrix}\mathbf{i}-\begin{vmatrix}t&\sin t\\ 1&\cos t\end{vmatrix}\mathbf{j}+\begin{vmatrix}t&\cos t\\ 1&-\sin t\end{vmatrix}\mathbf{k}$

$=[\cos^2 t-(-\sin^2 t)]\,\mathbf{i}-(t\cos t-\sin t)\,\mathbf{j}+(-t\sin t-\cos t)\,\mathbf{k}=\mathbf{i}+(\sin t-t\cos t)\,\mathbf{j}+(-t\sin t-\cos t)\,\mathbf{k}$

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}&=[\,\mathbf{i}+(\sin t-t\cos t)\,\mathbf{j}+(-t\sin t-\cos t)\,\mathbf{k}\,]\cdot(t\,\mathbf{i}+\cos t\,\mathbf{j}+\sin t\,\mathbf{k})\\&=t+\sin t\cos t-t\cos^2 t-t\sin^2 t-\sin t\cos t\\&=t-t\left(\cos^2 t+\sin^2 t\right)=0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}&=[\,\mathbf{i}+(\sin t-t\cos t)\,\mathbf{j}+(-t\sin t-\cos t)\,\mathbf{k}\,]\cdot(\mathbf{i}-\sin t\,\mathbf{j}+\cos t\,\mathbf{k})\\&=1-\sin^2 t+t\sin t\cos t-t\sin t\cos t-\cos^2 t\\&=1-\left(\sin^2 t+\cos^2 t\right)=0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

7. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ t^3&t^2&t\\ t&2t&3t\end{vmatrix}=\begin{vmatrix}t^2&t\\ 2t&3t\end{vmatrix}\mathbf{i}-\begin{vmatrix}t^3&t\\ t&3t\end{vmatrix}\mathbf{j}+\begin{vmatrix}t^3&t^2\\ t&2t\end{vmatrix}\mathbf{k}$

$=(3t^3-2t^2)\mathbf{i}-(3t^4-t^2)\mathbf{j}+(2t^4-t^3)\mathbf{k}$

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}&=\left\langle 3t^3-2t^2,t^2-3t^4,2t^4-t^3\right\rangle\cdot\left\langle t^3,t^2,t\right\rangle\\&=3t^6-2t^5+t^4-3t^6+2t^5-t^4=0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

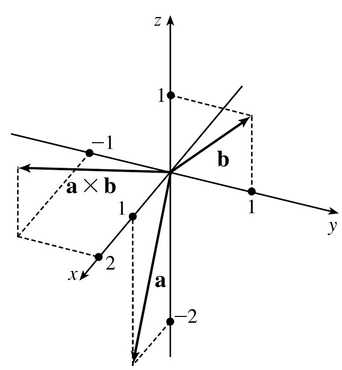
Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}&=\left\langle 3t^3-2t^2,t^2-3t^4,2t^4-t^3\right\rangle\cdot\langle t,2t,3t\rangle\\&=3t^4-2t^3+2t^3-6t^5+6t^5-3t^4=0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

8. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 1&0&-2\\ 0&1&1\end{vmatrix}$

$=\begin{vmatrix}0&-2\\ 1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&-2\\ 0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\ 0&1\end{vmatrix}\mathbf{k}$

$=2\,\mathbf{i}-\mathbf{j}+\mathbf{k}$



9. According to the discussion following Example 4, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, so $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$ [by Example 2].

10. $\mathbf{k}\times(\mathbf{i}-2\,\mathbf{j}) = \mathbf{k}\times\mathbf{i}+\mathbf{k}\times(-2\,\mathbf{j})$ by Property 3 of the cross product

$= \mathbf{k}\times\mathbf{i}+(-2)\,(\mathbf{k}\times\mathbf{j})$ by Property 2

$= \mathbf{j}+(-2)(-\mathbf{i}) = 2\,\mathbf{i}+\mathbf{j}$ by the discussion following Example 4

11. $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i}) = (\mathbf{j}-\mathbf{k})\times\mathbf{k}+(\mathbf{j}-\mathbf{k})\times(-\mathbf{i})$ by Property 3 of the cross product

$= \mathbf{j}\times\mathbf{k}+(-\mathbf{k})\times\mathbf{k}+\mathbf{j}\times(-\mathbf{i})+(-\mathbf{k})\times(-\mathbf{i})$ by Property 4

$= (\mathbf{j}\times\mathbf{k})+(-1)(\mathbf{k}\times\mathbf{k})+(-1)(\mathbf{j}\times\mathbf{i})+(-1)^2(\mathbf{k}\times\mathbf{i})$ by Property 2

$= \mathbf{i}+(-1)\,\mathbf{0}+(-1)(-\mathbf{k})+\mathbf{j} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ by Example 2 and the discussion following Example 4

12. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}) = (\mathbf{i}+\mathbf{j})\times\mathbf{i}+(\mathbf{i}+\mathbf{j})\times(-\mathbf{j})$ by Property 3 of the cross product

$= \mathbf{i}\times\mathbf{i}+\mathbf{j}\times\mathbf{i}+\mathbf{i}\times(-\mathbf{j})+\mathbf{j}\times(-\mathbf{j})$ by Property 4

$= (\mathbf{i}\times\mathbf{i})+(\mathbf{j}\times\mathbf{i})+(-1)(\mathbf{i}\times\mathbf{j})+(-1)(\mathbf{j}\times\mathbf{j})$ by Property 2

$= \mathbf{0}+(-\mathbf{k})+(-1)\,\mathbf{k}+(-1)\,\mathbf{0} = -2\,\mathbf{k}$ by Example 2 and the discussion following Example 4

13. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is meaningful and results in another vector.

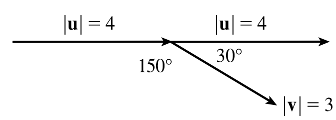
(d) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so the dot product $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the dot product is defined only for two vectors.

(e) Since $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{c}\cdot\mathbf{d})$ are both scalars, the cross product $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ is meaningless.

(f) $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both vectors, so the dot product $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is meaningful and is a scalar.

14. Using Theorem 9, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = (10)(8)\sin 60^\circ = 80\cdot\frac{\sqrt{3}}{2} = 40\sqrt{3}$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

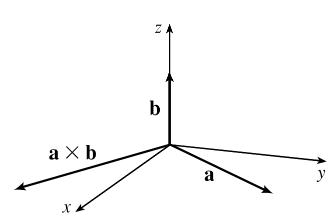
15. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30°.



Using Theorem 9, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = (4)(3)\sin 30^\circ = 12\cdot\frac{1}{2} = 6$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

16. (a) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta = 3\cdot 2\cdot\sin\frac{\pi}{2} = 6$

(b) $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy-plane, and its z-coordinate is 0. By the right-hand rule, its y-component is negative and its x-component is positive.



17. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&3\\4&2&1\end{vmatrix}=\begin{vmatrix}-1&3\\2&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&3\\4&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&-1\\4&2\end{vmatrix}\mathbf{k}=(-1-6)\,\mathbf{i}-(2-12)\,\mathbf{j}+[4-(-4)]\,\mathbf{k}=-7\,\mathbf{i}+10\,\mathbf{j}+8\,\mathbf{k}$

$\mathbf{b}\times\mathbf{a}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&2&1\\2&-1&3\end{vmatrix}=\begin{vmatrix}2&1\\-1&3\end{vmatrix}\mathbf{i}-\begin{vmatrix}4&1\\2&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}4&2\\2&-1\end{vmatrix}\mathbf{k}=[6-(-1)]\,\mathbf{i}-(12-2)\,\mathbf{j}+(-4-4)\,\mathbf{k}=7\,\mathbf{i}-10\,\mathbf{j}-8\,\mathbf{k}$

Notice $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Property 1 of the cross product.

18. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-1\\0&1&3\end{vmatrix}=\begin{vmatrix}1&-1\\1&3\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-1\\0&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\0&1\end{vmatrix}\mathbf{k}=4\,\mathbf{i}-6\,\mathbf{j}+2\,\mathbf{k}$ so

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\4&-6&2\end{vmatrix}=\begin{vmatrix}0&1\\-6&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\4&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\4&-6\end{vmatrix}\mathbf{k}=6\,\mathbf{i}+2\,\mathbf{j}-6\,\mathbf{k}.$

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\2&1&-1\end{vmatrix}=\begin{vmatrix}0&1\\1&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\2&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\2&1\end{vmatrix}\mathbf{k}=-\,\mathbf{i}+3\,\mathbf{j}+\mathbf{k}$ so

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&3&1\\0&1&3\end{vmatrix}=\begin{vmatrix}3&1\\1&3\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&1\\0&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&3\\0&1\end{vmatrix}\mathbf{k}=8\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}.$

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

19. By Theorem 8, the cross product of two vectors is orthogonal to both vectors. So we calculate

$$\langle 3,2,1\rangle\times\langle -1,1,0\rangle=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&1\\-1&1&0\end{vmatrix}=\begin{vmatrix}2&1\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&1\\-1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&2\\-1&1\end{vmatrix}\mathbf{k}=-\mathbf{i}-\mathbf{j}+5\,\mathbf{k}.$$

So two unit vectors orthogonal to both given vectors are $\pm\dfrac{\langle -1,-1,5\rangle}{\sqrt{1+1+25}}=\pm\dfrac{\langle -1,-1,5\rangle}{3\sqrt{3}}$, that is, $\left\langle -\frac{1}{3\sqrt{3}},-\frac{1}{3\sqrt{3}},\frac{5}{3\sqrt{3}}\right\rangle$ and $\left\langle \frac{1}{3\sqrt{3}},\frac{1}{3\sqrt{3}},-\frac{5}{3\sqrt{3}}\right\rangle$.

20. By Theorem 8, the cross product of two vectors is orthogonal to both vectors. So we calculate

$$(\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&-1\\1&1&0\end{vmatrix}=\begin{vmatrix}1&-1\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&-1\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&1\end{vmatrix}\mathbf{k}=\mathbf{i}-\mathbf{j}-\mathbf{k}$$

Thus two unit vectors orthogonal to both given vectors are $\pm\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$, that is, $\frac{1}{\sqrt{3}}\,\mathbf{i}-\frac{1}{\sqrt{3}}\,\mathbf{j}-\frac{1}{\sqrt{3}}\,\mathbf{k}$ and $-\frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k}$.

21. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then

$$\mathbf{0} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} = \mathbf{0},$$

$$\mathbf{a} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \mathbf{k} = \mathbf{0}.$$

22. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3$$

$$= (a_2b_3b_1 - a_3b_2b_1) - (a_1b_3b_2 - a_3b_1b_2) + (a_1b_2b_3 - a_2b_1b_3) = 0$$

23. $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

$$= \langle (-1)(b_2a_3 - b_3a_2), (-1)(b_3a_1 - b_1a_3), (-1)(b_1a_2 - b_2a_1) \rangle$$

$$= -\langle b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1 \rangle = -\mathbf{b} \times \mathbf{a}$$

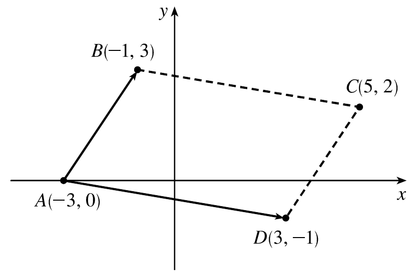
24. $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$, so

$$\begin{aligned}(c\,\mathbf{a}) \times \mathbf{b} &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1 \rangle \\ &= c\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = c(\mathbf{a} \times \mathbf{b}) \\ &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1 \rangle \\ &= \langle a_2(cb_3) - a_3(cb_2), a_3(cb_1) - a_1(cb_3), a_1(cb_2) - a_2(cb_1) \rangle \\ &= \mathbf{a} \times (c\,\mathbf{b})\end{aligned}$$

25.
$$\begin{aligned}\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle \\ &= \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1 \rangle \\ &= \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2), (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3), (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1) \rangle \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1 \rangle \\ &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})\end{aligned}$$

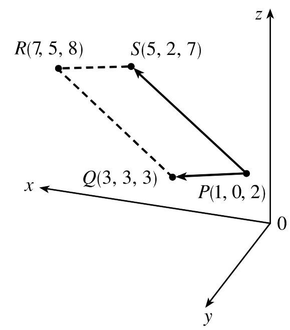
26.
$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= -\mathbf{c} \times (\mathbf{a} + \mathbf{b}) && \text{by Property 1 of the cross product} \\ &= -(\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}) && \text{by Property 3} \\ &= -(-\mathbf{a} \times \mathbf{c} + (-\mathbf{b} \times \mathbf{c})) && \text{by Property 1} \\ &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} && \text{by Property 2}\end{aligned}$$

27. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2, 3 \rangle$ and $\overrightarrow{AD} = \langle 6, -1 \rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2, 3, 0 \rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{matrix} \right\| = |(0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-2-18)\,\mathbf{k}| = |-20\,\mathbf{k}| = 20$$

28. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 2, 3, 1 \rangle$ and $\overrightarrow{PS} = \langle 4, 2, 5 \rangle$. Thus the area of parallelogram $PQRS$ is



$$\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{matrix} \right\| = |(15-2)\,\mathbf{i} - (10-4)\,\mathbf{j} + (4-12)\,\mathbf{k}|$$

$$= |13\,\mathbf{i} - 6\,\mathbf{j} - 8\,\mathbf{k}| = \sqrt{169 + 36 + 64} = \sqrt{269} \approx 16.40$$

29. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle 2, 1, 3 \rangle$ and $\overrightarrow{PR} = \langle 5, 4, 2 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(2) - (3)(4), (3)(5) - (2)(2), (2)(4) - (1)(5) \rangle = \langle -10, 11, 3 \rangle$$

Therefore, $\langle -10, 11, 3 \rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -10, 11, 3 \rangle| = \sqrt{(-10)^2 + 11^2 + 3^2} = \sqrt{230}$$

So the area of triangle PQR is $\frac{1}{2}\sqrt{230}$.

30. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle 3, 3, -6 \rangle$ and $\overrightarrow{PR} = \langle 2, 3, 1 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(1) - (-6)(3), (-6)(2) - (3)(1), (3)(3) - (3)(2) \rangle = \langle 21, -15, 3 \rangle$$

Therefore, $\langle 21, -15, 3 \rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle 21, -15, 3\rangle| = \sqrt{21^2 + (-15)^2 + 3^2} = \sqrt{675} = 15\sqrt{3}$$

So the area of triangle PQR is $\frac{15}{2}\sqrt{3}$.

31. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle -4, 3, 3\rangle$ and $\overrightarrow{PR} = \langle -3, -2, 2\rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(2) - (3)(-2), (3)(-3) - (-4)(2), (-4)(-2) - (3)(-3)\rangle = \langle 12, -1, 17\rangle$$

Therefore, $\langle 12, -1, 17\rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle 12, -1, 17\rangle| = \sqrt{12^2 + (-1)^2 + 17^2} = \sqrt{434}$$

So the area of triangle PQR is $\frac{1}{2}\sqrt{434}$.

32. (a) $\overrightarrow{PQ} = \langle -3, 1, -2\rangle$ and $\overrightarrow{PR} = \langle 1, 4, -7\rangle$, so a vector orthogonal to the plane through P, Q, and R is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(-7) - (-2)(4), (-2)(1) - (-3)(-7), (-3)(4) - (1)(1)\rangle = \langle 1, -23, -13\rangle$ (or any nonzero scalar multiple).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle 1, -23, -13\rangle| = \sqrt{1 + 529 + 169} = \sqrt{699}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{699}$.

33. By Equation 14, the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product,

which is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = 1(4-2) - 2(-4-4) + 3(-1-2) = 9.$

Thus the volume of the parallelepiped is 9 cubic units.

34. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 + 1 + 0 = 1.$

So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is 1 cubic unit.

35. $\mathbf{a} = \overrightarrow{PQ} = \langle 4, 2, 2\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 3, 3, -1\rangle$, and $\mathbf{c} = \overrightarrow{PS} = \langle 5, 5, 1\rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix} = 32 - 16 + 0 = 16,$$

so the volume of the parallelepiped is 16 cubic units.

36. $\mathbf{a} = \overrightarrow{PQ} = \langle -4, 2, 4 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2, 1, -2 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle -3, 4, 1 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16,$$

so the volume of the parallelepiped is 16 cubic units.

37. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4 + 60 - 64 = 0$, which says that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, and thus these three vectors are coplanar.

38. $\mathbf{u} = \overrightarrow{AB} = \langle 2, -4, 4 \rangle$, $\mathbf{v} = \overrightarrow{AC} = \langle 4, -1, -2 \rangle$ and $\mathbf{w} = \overrightarrow{AD} = \langle 2, 3, -6 \rangle$.

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2 \begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix} - (-4) \begin{vmatrix} 4 & -2 \\ 2 & -6 \end{vmatrix} + 4 \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 24 - 80 + 56 = 0$, so the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points A, B, C and D also lie in the same plane.

39. Using the notation of the text, $|\mathbf{r}| = 0.18$ m, $|\mathbf{F}| = 60$ N, and the angle between $\mathbf{r}$ and $\mathbf{F}$ is $\theta = 70^\circ + 10^\circ = 80^\circ$. (Move $\mathbf{F}$ so that both vectors start from the same point.) Then the magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.18)(60) \sin 80^\circ = 10.8 \sin 80^\circ \approx 10.6$ N·m.

40. (a) The position vector from the point P to the handle is $\mathbf{r} = \langle 1, 2 \rangle$ and has magnitude $|\mathbf{r}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ ft. Since the force vector $\mathbf{F}$ is parallel to the x-axis, the angle between $\mathbf{r}$ and $\mathbf{F}$ is $\theta = \tan^{-1}\left(\frac{2}{1}\right) \approx 63.43^\circ$ and the magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta \approx \left(\sqrt{5}\right)(20) \sin 63.43^\circ \approx 40.0$ ft-lb. (Alternatively, we can observe that $\sin\theta = \frac{2}{\sqrt{5}}$, so $|\mathbf{r}|\,|\mathbf{F}| \sin\theta = \sqrt{5} \cdot 20 \cdot \frac{2}{\sqrt{5}} = 40$.)

(b) In this case $\mathbf{r} = \overrightarrow{PQ} = \langle 0.6, 0.6 \rangle$, so $|\mathbf{r}| = \sqrt{(0.6)^2 + (0.6)^2} = \sqrt{0.72}$ and $\theta = 45^\circ$. The magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = \left(\sqrt{0.72}\right)(20) \sin 45^\circ = \left(\sqrt{0.72}\right)(20) \cdot \frac{\sqrt{2}}{2} = 10\sqrt{1.44} = 12$ ft-lb.

41. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0 \rangle$ (measuring in meters) and $\mathbf{F}$ has direction $\langle 0, 3, -4 \rangle$. The angle θ between them can be determined by $\cos\theta = \dfrac{\langle 0, 0.3, 0 \rangle \cdot \langle 0, 3, -4 \rangle}{|\langle 0, 0.3, 0 \rangle|\,|\langle 0, 3, -4 \rangle|} \Rightarrow \cos\theta = \dfrac{0.9}{(0.3)(5)} \Rightarrow \cos\theta = 0.6 \Rightarrow$ $\theta = \cos^{-1}(0.6) \approx 53.1^\circ$. Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta \Rightarrow 100 \approx 0.3\,|\mathbf{F}| \sin 53.1^\circ \Rightarrow |\mathbf{F}| \approx \dfrac{100}{0.3 \sin 53.1^\circ} \approx 417$ N.

42. Since $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, $0 \le \theta \le \pi$, $|\mathbf{u} \times \mathbf{v}|$ achieves its maximum value for $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$, in which case $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = 3\,|5\,\mathbf{j}| = 15$. The minimum value is zero, which occurs when $\sin\theta = 0 \Rightarrow \theta = 0$ or π, so when $\mathbf{u}$, $\mathbf{v}$ are parallel. Thus, when $\mathbf{u}$ points in the same direction as $\mathbf{v}$, so $\mathbf{u} = 3\,\mathbf{j}$, $|\mathbf{u} \times \mathbf{v}| = 0$. As $\mathbf{u}$ rotates counterclockwise,

$\mathbf{u}\times\mathbf{v}$ is directed in the negative z-direction (by the right-hand rule) and the length increases until $\theta = \frac{\pi}{2}$, in which case $\mathbf{u} = -3\,\mathbf{i}$ and $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = |-3\,\mathbf{i}|\,|5\,\mathbf{j}| = 15$. As $\mathbf{u}$ rotates to the negative y-axis, $\mathbf{u}\times\mathbf{v}$ remains pointed in the negative z-direction and the length of $\mathbf{u}\times\mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u}\times\mathbf{v}$ reverses to point in the positive z-direction and $|\mathbf{u}\times\mathbf{v}|$ increases. When $\mathbf{u} = 3\,\mathbf{i}$ (so $\theta = \frac{\pi}{2}$), $|\mathbf{u}\times\mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u}\times\mathbf{v}|$ decreases to 0 as $\mathbf{u}$ rotates to the positive y-axis.

43. From Theorem 9 we have $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and from Theorem 12.3.3 we have $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \quad\Rightarrow\quad |\mathbf{a}|\,|\mathbf{b}| = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\cos\theta}$. Substituting the second equation into the first gives $|\mathbf{a}\times\mathbf{b}| = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\cos\theta}\sin\theta$, so $\dfrac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}} = \tan\theta$. Here $|\mathbf{a}\times\mathbf{b}| = |\langle 1,2,2\rangle| = \sqrt{1+4+4} = 3$, so $\tan\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \quad\Rightarrow\quad \theta = 60^\circ$.

44. (a) Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

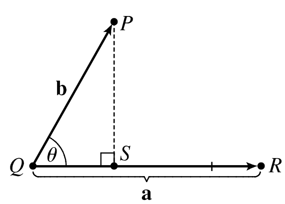
$$\langle 1,2,1\rangle\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} = (2v_3 - v_2)\,\mathbf{i} - (v_3 - v_1)\,\mathbf{j} + (v_2 - 2v_1)\,\mathbf{k}.$$

If $\langle 1,2,1\rangle\times\mathbf{v} = \langle 3,1,-5\rangle$ then $\langle 2v_3 - v_2, v_1 - v_3, v_2 - 2v_1\rangle = \langle 3,1,-5\rangle \quad\Leftrightarrow\quad 2v_3 - v_2 = 3$ **(1)**, $v_1 - v_3 = 1$ **(2)**, and $v_2 - 2v_1 = -5$ **(3)**. From **(3)** we have $v_2 = 2v_1 - 5$ and from **(2)** we have $v_3 = v_1 - 1$; substitution into **(1)** gives $2\,(v_1 - 1) - (2v_1 - 5) = 3 \quad\Rightarrow\quad 3 = 3$, so this is a dependent system. If we let $v_1 = a$ then $v_2 = 2a - 5$ and $v_3 = a - 1$, so $\mathbf{v}$ is any vector of the form $\langle a, 2a-5, a-1\rangle$.

(b) If $\langle 1,2,1\rangle\times\mathbf{v} = \langle 3,1,5\rangle$ then $2v_3 - v_2 = 3$ **(1)**, $v_1 - v_3 = 1$ **(2)**, and $v_2 - 2v_1 = 5$ **(3)**. From **(3)** we have $v_2 = 2v_1 + 5$ and from **(2)** we have $v_3 = v_1 - 1$; substitution into **(1)** gives $2\,(v_1 - 1) - (2v_1 + 5) = 3 \quad\Rightarrow\quad -7 = 3$, so this is an inconsistent system and has no solution.

Alternatively, if we use matrices to solve the system we could show that the determinant is 0 (and hence the system has no solution).

45. (a)

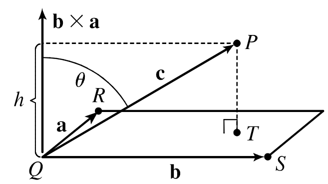


The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right| = d$. But referring to triangle PQS, $d = \left|\overrightarrow{PS}\right| = \left|\overrightarrow{QP}\right|\sin\theta = |\mathbf{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by Theorem 9, $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$ and so $d = |\mathbf{b}|\sin\theta = \dfrac{|\mathbf{b}|\,|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}$.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1,-2,-1\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1,-5,-7\rangle$. Then

$\mathbf{a}\times\mathbf{b} = \langle (-2)(-7) - (-1)(-5), (-1)(1) - (-1)(-7), (-1)(-5) - (-2)(1)\rangle = \langle 9,-8,7\rangle$.

Thus the distance is $d = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}}\sqrt{81+64+49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$.



46. (a) The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right| = d$. But $\overrightarrow{TP}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (because $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$) and $d = \left|\overrightarrow{TP}\right| =$ the absolute value of the scalar projection of $\mathbf{c}$ along $\mathbf{b} \times \mathbf{a}$, which is $|\mathbf{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h = |\mathbf{c}|\,|\cos\theta|$). Thus $d = |\mathbf{c}|\,|\cos\theta| = h = V/A$ where $A = |\mathbf{a} \times \mathbf{b}|$, the area of the base. So finally $d = \dfrac{V}{A} = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, 2, 0\rangle$, $\mathbf{b} = \overrightarrow{QS} = \langle -1, 0, 3\rangle$ and $\mathbf{c} = \overrightarrow{QP} = \langle 1, 1, 4\rangle$. Then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{k} = 6\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$$

Thus $d = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{17}{\sqrt{36 + 9 + 4}} = \dfrac{17}{7}$.

47. From Theorem 9 we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ so

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2\,|\mathbf{b}|^2\left(1 - \cos^2\theta\right) \\ &= |\mathbf{a}|^2\,|\mathbf{b}|^2 - (|\mathbf{a}|\,|\mathbf{b}|\cos\theta)^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \end{aligned}$$

by Theorem 12.3.3.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ then $\mathbf{b} = -(\mathbf{a} + \mathbf{c})$, so

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times [-(\mathbf{a} + \mathbf{c})] = -[\mathbf{a} \times (\mathbf{a} + \mathbf{c})]$	by Property 2 of the cross product (with $c = -1$)
$= -[(\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{c})]$	by Property 3
$= -[\mathbf{0} + (\mathbf{a} \times \mathbf{c})] = -\mathbf{a} \times \mathbf{c}$	by Example 2
$= \mathbf{c} \times \mathbf{a}$	by Property 1

Similarly, $\mathbf{a} = -(\mathbf{b} + \mathbf{c})$ so

$$\begin{aligned} \mathbf{c} \times \mathbf{a} &= \mathbf{c} \times [-(\mathbf{b} + \mathbf{c})] = -[\mathbf{c} \times (\mathbf{b} + \mathbf{c})] \\ &= -[(\mathbf{c} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{c})] = -[(\mathbf{c} \times \mathbf{b}) + \mathbf{0}] \\ &= -\mathbf{c} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} \end{aligned}$$

Thus $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

49. $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\times\mathbf{a} + (\mathbf{a}-\mathbf{b})\times\mathbf{b}$ by Property 3 of the cross product

$= \mathbf{a}\times\mathbf{a} + (-\mathbf{b})\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + (-\mathbf{b})\times\mathbf{b}$ by Property 4

$= (\mathbf{a}\times\mathbf{a}) - (\mathbf{b}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{b})$ by Property 2 (with $c=-1$)

$= \mathbf{0} - (\mathbf{b}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) - \mathbf{0}$ by Example 2

$= (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{b})$ by Property 1

$= 2(\mathbf{a}\times\mathbf{b})$

50. Let $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ and $\mathbf{c}=\langle c_1,c_2,c_3\rangle$, so $\mathbf{b}\times\mathbf{c}=\langle b_2c_3-b_3c_2, b_3c_1-b_1c_3, b_1c_2-b_2c_1\rangle$ and

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \langle a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3),\, a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1),\\
&\qquad a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\rangle\\
&= \langle a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,\, a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\\
&\qquad a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2\rangle\\
&= \langle (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1,\, (a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2,\\
&\qquad (a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3\rangle\\
(\star)\quad &= \langle (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1+a_1b_1c_1-a_1b_1c_1,\\
&\qquad (a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2+a_2b_2c_2-a_2b_2c_2,\\
&\qquad (a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3+a_3b_3c_3-a_3b_3c_3\rangle\\
&= \langle (a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1,\\
&\qquad (a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2,\\
&\qquad (a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\rangle\\
&= (a_1c_1+a_2c_2+a_3c_3)\langle b_1,b_2,b_3\rangle-(a_1b_1+a_2b_2+a_3b_3)\langle c_1,c_2,c_3\rangle\\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}
\end{aligned}$$

$(\star)$ Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

51. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}]+[(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}]+[(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}]$ by Exercise 50

$= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}=\mathbf{0}$

52. Let $\mathbf{c}\times\mathbf{d}=\mathbf{v}$. Then

$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{v} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{v})$ by Property 5 of the cross product

$= \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}]$ by Exercise 50

$= (\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$ by Properties 3 and 4 of the dot product

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

53. (a) No. If $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$, then $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, which can happen if $\mathbf{b}\neq\mathbf{c}$. For example, let $\mathbf{a}=\langle 1,1,1\rangle$, $\mathbf{b}=\langle 1,0,0\rangle$ and $\mathbf{c}=\langle 0,1,0\rangle$.

(b) No. If $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then $\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$, which of course can happen if $\mathbf{b}\neq\mathbf{c}$.

(c) Yes. Since $\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b}-\mathbf{c}$. Thus since $\mathbf{a}\neq\mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b}-\mathbf{c}$, we have $\mathbf{b}-\mathbf{c}=\mathbf{0}$, so $\mathbf{b}=\mathbf{c}$.

54. (a) $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i\neq j$ by the definition of $\mathbf{k}_i$ and Theorem 8.

(b) $\mathbf{k}_1\cdot\mathbf{v}_1=\dfrac{\mathbf{v}_2\times\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_1=\dfrac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1$

$$\mathbf{k}_2\cdot\mathbf{v}_2=\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_2=\frac{\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1 \qquad \text{[by Property 5 of the cross product]}$$

$$\mathbf{k}_3\cdot\mathbf{v}_3=\frac{(\mathbf{v}_1\times\mathbf{v}_2)\cdot\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1 \qquad \text{[by Property 5]}$$

(c) $$\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)=\mathbf{k}_1\cdot\left(\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\times\frac{\mathbf{v}_1\times\mathbf{v}_2}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\right)=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[(\mathbf{v}_3\times\mathbf{v}_1)\times(\mathbf{v}_1\times\mathbf{v}_2)]$$

$$=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot([(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2]\,\mathbf{v}_1-[(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1]\,\mathbf{v}_2) \qquad \text{[by Exercise 50]}$$

But $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1=0$ since $\mathbf{v}_3\times\mathbf{v}_1$ is orthogonal to $\mathbf{v}_1$, and $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2=\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)=(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1=\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)$. Thus

$$\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]\,\mathbf{v}_1=\frac{\mathbf{k}_1\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} \qquad \text{[by part (b)]}$$

DISCOVERY PROJECT The Geometry of a Tetrahedron

1. Set up a coordinate system so that vertex S is at the origin, $R=(0,y_1,0)$, $Q=(x_2,y_2,0)$, $P=(x_3,y_3,z_3)$. Then $\overrightarrow{SR}=\langle 0,y_1,0\rangle$, $\overrightarrow{SQ}=\langle x_2,y_2,0\rangle$, $\overrightarrow{SP}=\langle x_3,y_3,z_3\rangle$, $\overrightarrow{QR}=\langle -x_2,y_1-y_2,0\rangle$, and $\overrightarrow{QP}=\langle x_3-x_2,y_3-y_2,z_3\rangle$. Let

$$\mathbf{v}_S=\overrightarrow{QR}\times\overrightarrow{QP}=(y_1z_3-y_2z_3)\,\mathbf{i}+x_2z_3\,\mathbf{j}+(-x_2y_3-x_3y_1+x_3y_2+x_2y_1)\,\mathbf{k}$$

Then $\mathbf{v}_S$ is an outward normal to the face opposite vertex S. Similarly,

$\mathbf{v}_R=\overrightarrow{SQ}\times\overrightarrow{SP}=y_2z_3\,\mathbf{i}-x_2z_3\,\mathbf{j}+(x_2y_3-x_3y_2)\,\mathbf{k}$, $\mathbf{v}_Q=\overrightarrow{SP}\times\overrightarrow{SR}=-y_1z_3\,\mathbf{i}+x_3y_1\,\mathbf{k}$, and

$\mathbf{v}_P=\overrightarrow{SR}\times\overrightarrow{SQ}=-x_2y_1\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}_S+\mathbf{v}_R+\mathbf{v}_Q+\mathbf{v}_P=\mathbf{0}$. Now

$$\begin{aligned}|\mathbf{v}_S|&=\text{ area of the parallelogram determined by }\overrightarrow{QR}\text{ and }\overrightarrow{QP}\\&=2\,(\text{area of triangle }RQP)=2|\mathbf{v}_1|\end{aligned}$$

So $\mathbf{v}_S=2\mathbf{v}_1$, and similarly $\mathbf{v}_R=2\mathbf{v}_2$, $\mathbf{v}_Q=2\mathbf{v}_3$, $\mathbf{v}_P=2\mathbf{v}_4$. Thus $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4=\mathbf{0}$.

2. (a) Let $S = (x_0, y_0, z_0)$, $R = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $P = (x_3, y_3, z_3)$ be the four vertices. Then

$$\begin{aligned}\text{Volume} &= \tfrac{1}{3}(\text{distance from } S \text{ to plane } RQP) \times (\text{area of triangle } RQP)\\ &= \tfrac{1}{3}\frac{\left|\mathbf{N}\cdot\overrightarrow{SR}\right|}{|\mathbf{N}|}\cdot\tfrac{1}{2}\left|\overrightarrow{RQ}\times\overrightarrow{RP}\right|\end{aligned}$$

where $\mathbf{N}$ is a vector which is normal to the face RQP. Thus $\mathbf{N} = \overrightarrow{RQ}\times\overrightarrow{RP}$. Therefore

$$V = \left|\tfrac{1}{6}\left(\overrightarrow{RQ}\times\overrightarrow{RP}\right)\cdot\overrightarrow{SR}\right| = \frac{1}{6}\left\|\begin{matrix} x_0 - x & y_0 - y_1 & z_0 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix}\right\|$$

(b) Using the formula from part (a), $V = \dfrac{1}{6}\left\|\begin{matrix} 1-1 & 1-2 & 1-3 \\ 1-1 & 1-2 & 2-3 \\ 3-1 & -1-2 & 2-3 \end{matrix}\right\| = \dfrac{1}{6}\left|2(1-2)\right| = \dfrac{1}{3}.$

3. We define a vector $\mathbf{v}_1$ to have length equal to the area of the face opposite vertex P, so we can say $|\mathbf{v}_1| = A$, and direction perpendicular to the face and pointing outward, as in Problem 1. Similarly, we define $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ so that $|\mathbf{v}_2| = B$, $|\mathbf{v}_3| = C$, and $|\mathbf{v}_4| = D$ and with the analogous directions. From Problem 1, we know $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \Rightarrow \mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \Rightarrow |\mathbf{v}_4| = |-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| \Rightarrow |\mathbf{v}_4|^2 = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|^2 \Rightarrow$

$$\begin{aligned}\mathbf{v}_4\cdot\mathbf{v}_4 &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\cdot(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\\ &= \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_1\cdot\mathbf{v}_3 + \mathbf{v}_2\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_3 + \mathbf{v}_3\cdot\mathbf{v}_1 + \mathbf{v}_3\cdot\mathbf{v}_2 + \mathbf{v}_3\cdot\mathbf{v}_3\end{aligned}$$

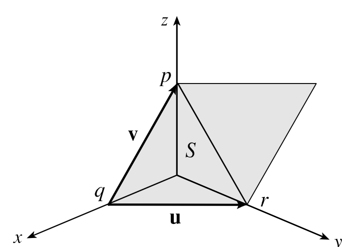
Since the vertex S is trirectangular, we know the three faces meeting at S are mutually perpendicular, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are also mutually perpendicular. Therefore, $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Thus we have $\mathbf{v}_4\cdot\mathbf{v}_4 = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 + \mathbf{v}_3\cdot\mathbf{v}_3 \Rightarrow |\mathbf{v}_4|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2 \Rightarrow D^2 = A^2 + B^2 + C^2.$

Another method: We introduce a coordinate system, as shown. Recall that the area of the parallelogram spanned by two vectors is equal to the length of their cross product, so since

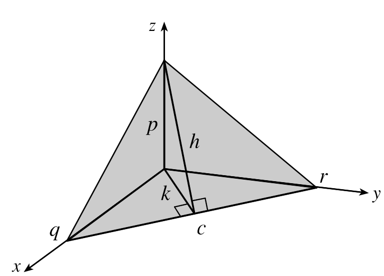
$\mathbf{u}\times\mathbf{v} = \langle -q, r, 0\rangle \times \langle -q, 0, p\rangle = \langle pr, pq, qr\rangle$, we have

$|\mathbf{u}\times\mathbf{v}| = \sqrt{(pr)^2 + (pq)^2 + (qr)^2}$, and therefore

$$\begin{aligned}D^2 &= \left(\tfrac{1}{2}|\mathbf{u}\times\mathbf{v}|\right)^2 = \tfrac{1}{4}[(pr)^2 + (pq)^2 + (qr)^2]\\ &= \left(\tfrac{1}{2}pr\right)^2 + \left(\tfrac{1}{2}pq\right)^2 + \left(\tfrac{1}{2}qr\right)^2 = A^2 + B^2 + C^2.\end{aligned}$$



A third method: We draw a line from S perpendicular to QR, as shown. Now $D = \frac{1}{2}ch$, so $D^2 = \frac{1}{4}c^2h^2$. Substituting $h^2 = p^2 + k^2$, we get $D^2 = \frac{1}{4}c^2\left(p^2 + k^2\right) = \frac{1}{4}c^2p^2 + \frac{1}{4}c^2k^2$. But $C = \frac{1}{2}ck$, so $D^2 = \frac{1}{4}c^2p^2 + C^2$. Now substituting $c^2 = q^2 + r^2$ gives $D^2 = \frac{1}{4}p^2q^2 + \frac{1}{4}q^2r^2 + C^2 = A^2 + B^2 + C^2$.



12.5 Equations of Lines and Planes

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \leq \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) = (4 + 2t)\,\mathbf{i} + (2 - t)\,\mathbf{j} + (-3 + 6t)\,\mathbf{k}$, and parametric equations are $x = 4 + 2t$, $y = 2 - t$, $z = -3 + 6t$.

3. For this line, we have $\mathbf{r}_0 = -\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$ and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (-\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}) + t\left(\frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}\right) = \left(-1 + \frac{1}{2}t\right)\mathbf{i} + \left(8 + \frac{1}{3}t\right)\mathbf{j} + \left(7 + \frac{1}{4}t\right)\mathbf{k}$, and parametric equations are $x = -1 + \frac{1}{2}t$, $y = 8 + \frac{1}{3}t$, $z = 7 + \frac{1}{4}t$.

4. The direction vector for this line is the same as the given line, $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$. Here $\mathbf{r}_0 = 6\mathbf{i} - 2\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (6\,\mathbf{i} - 2\,\mathbf{k}) + t(-3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}) = (6 - 3t)\,\mathbf{i} + 4t\,\mathbf{j} + (-2 + 5t)\,\mathbf{k}$, and parametric equations are $x = 6 - 3t$, $y = 4t$, $z = -2 + 5t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 3, -2, 2 \rangle$. So $\mathbf{r}_0 = 5\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ and we can take $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Then a vector equation is

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (5\,\mathbf{i} + 7\,\mathbf{j} + \mathbf{k}) + t(3\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}) = (5 + 3t)\,\mathbf{i} + (7 - 2t)\,\mathbf{j} + (1 + 2t)\,\mathbf{k}$, and parametric equations are $x = 5 + 3t$, $y = 7 - 2t$, $z = 1 + 2t$.

6. The vector $\mathbf{v} = \langle 1 - (-5), 6 - 2, -2 - 5\rangle = \langle 6, 4, -7\rangle$ is parallel to the line. Letting $P_0 = (-5, 2, 5)$, parametric equations are $x = -5 + 6t$, $y = 2 + 4t$, $z = 5 - 7t$ and symmetric equations are $\dfrac{x+5}{6} = \dfrac{y-2}{4} = \dfrac{z-5}{-7}$.

7. The vector $\mathbf{v} = \langle 8 - 0, -1 - 0, 3 - 0\rangle = \langle 8, -1, 3\rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 8t$, $y = -t$, $z = 3t$ and symmetric equations are $\dfrac{x}{8} = \dfrac{y}{-1} = \dfrac{z}{3}$ or $\dfrac{x}{8} = -y = \dfrac{z}{3}$.

8. The vector $\mathbf{v} = \langle 1.3 - 0.4, 0.8 - (-0.2), -2.3 - 1.1\rangle = \langle 0.9, 1, -3.4\rangle$ is parallel to the line. Letting $P_0 = (0.4, -0.2, 1.1)$, parametric equations are $x = 0.4 + 0.9t$, $y = -0.2 + t$, $z = 1.1 - 3.4t$ and symmetric equations are $\dfrac{x-0.4}{0.9} = \dfrac{y+0.2}{1} = \dfrac{z-1.1}{-3.4}$ or $\dfrac{x-0.4}{0.9} = y + 0.2 = \dfrac{z-1.1}{-3.4}$.

9. The vector $\mathbf{v} = \langle -7 - 12, 9 - 9, 11 - (-13)\rangle = \langle -19, 0, 24\rangle$ is parallel to the line. Letting $P_0 = (12, 9, -13)$, parametric equations are $x = 12 - 19t$, $y = 9$, $z = -13 + 24t$ and symmetric equations are $\dfrac{x-12}{-19} = \dfrac{z+13}{24}$, $y = 9$. Notice here that the direction number $b = 0$, so rather than writing $\dfrac{y-9}{0}$ in the symmetric equation, we must write $y = 9$ separately.

10. $\mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$ and symmetric equations are $x - 2 = \dfrac{y-1}{-1} = z$ or $x - 2 = 1 - y = z$.

11. The given line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$ has direction $\mathbf{v} = \langle 2, 3, 1\rangle$. Taking $(-6, 2, 3)$ as P_0, parametric equations are $x = -6 + 2t$, $y = 2 + 3t$, $z = 3 + t$ and symmetric equations are $\dfrac{x+6}{2} = \dfrac{y-2}{3} = z - 3$.

12. Setting $z = 0$ we see that $(1, 0, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. The line is perpendicular to the normal vectors of both planes, so a direction vector for the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, 3\rangle \times \langle 1, -1, 1\rangle = \langle 5, 2, -3\rangle$. Taking the point $(1, 0, 0)$ as P_0, parametric equations are $x = 1 + 5t$, $y = 2t$, $z = -3t$, and symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{2} = \dfrac{z}{-3}$.

13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2 - (-4), 0 - (-6), -3 - 1\rangle = \langle 2, 6, -4\rangle$ and $\mathbf{v}_2 = \langle 5 - 10, 3 - 18, 14 - 4\rangle = \langle -5, -15, 10\rangle$. Since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors, and thus the lines, are parallel.

14. Direction vectors of the lines are $\mathbf{v}_1 = \langle 1 - (-2), 1 - 4, 1 - 0\rangle = \langle 3, -3, 1\rangle$ and $\mathbf{v}_2 = \langle 3 - 2, -1 - 3, -8 - 4\rangle = \langle 1, -4, -12\rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 + 12 - 12 \neq 0$, the direction vectors, and thus the lines, are not perpendicular.

15. (a) The line passes through the point $(1,-5,6)$ and a direction vector for the line is $\langle -1,2,-3\rangle$, so symmetric equations for the line are $\dfrac{x-1}{-1}=\dfrac{y+5}{2}=\dfrac{z-6}{-3}$.

(b) The line intersects the xy-plane when $z=0$, so we need $\dfrac{x-1}{-1}=\dfrac{y+5}{2}=\dfrac{0-6}{-3}$ or $\dfrac{x-1}{-1}=2 \;\Rightarrow\; x=-1$, $\dfrac{y+5}{2}=2 \;\Rightarrow\; y=-1$. Thus the point of intersection with the xy-plane is $(-1,-1,0)$. Similarly for the yz-plane, we need $x=0 \;\Rightarrow\; 1=\dfrac{y+5}{2}=\dfrac{z-6}{-3} \;\Rightarrow\; y=-3$, $z=3$. Thus the line intersects the yz-plane at $(0,-3,3)$. For the xz-plane, we need $y=0 \;\Rightarrow\; \dfrac{x-1}{-1}=\dfrac{5}{2}=\dfrac{z-6}{-3} \;\Rightarrow\; x=-\frac{3}{2}$, $z=-\frac{3}{2}$. So the line intersects the xz-plane at $\left(-\frac{3}{2},0,-\frac{3}{2}\right)$.

16. (a) A vector normal to the plane $x-y+3z=7$ is $\mathbf{n}=\langle 1,-1,3\rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x=2+t$, $y=4-t$, $z=6+3t$.

(b) On the xy-plane, $z=0$. So $z=6+3t=0 \;\Rightarrow\; t=-2$ in the parametric equations of the line, and therefore $x=0$ and $y=6$, giving the point of intersection $(0,6,0)$. For the yz-plane, $x=0$ so we get the same point of interesection: $(0,6,0)$. For the xz-plane, $y=0$ which implies $t=4$, so $x=6$ and $z=18$ and the point of intersection is $(6,0,18)$.

17. From Equation 4, the line segment from $\mathbf{r}_0=6\,\mathbf{i}-\mathbf{j}+9\,\mathbf{k}$ to $\mathbf{r}_1=7\,\mathbf{i}+6\,\mathbf{j}$ has vector equation

$$\begin{aligned}\mathbf{r}(t)&=(1-t)\,\mathbf{r}_0+t\,\mathbf{r}_1=(1-t)(6\,\mathbf{i}-\mathbf{j}+9\,\mathbf{k})+t(7\,\mathbf{i}+6\,\mathbf{j})\\&=(6\,\mathbf{i}-\mathbf{j}+9\,\mathbf{k})-t(6\,\mathbf{i}-\mathbf{j}+9\,\mathbf{k})+t(7\,\mathbf{i}+6\,\mathbf{j})\\&=(6\,\mathbf{i}-\mathbf{j}+9\,\mathbf{k})+t(\,\mathbf{i}+7\,\mathbf{j}-9\,\mathbf{k}),\quad 0\le t\le 1.\end{aligned}$$

18. From Equation 4, the line segment from $\mathbf{r}_0=-2\,\mathbf{i}+18\,\mathbf{j}+31\,\mathbf{k}$ to $\mathbf{r}_1=11\,\mathbf{i}-4\,\mathbf{j}+48\,\mathbf{k}$ has vector equation

$$\begin{aligned}\mathbf{r}(t)&=(1-t)\,\mathbf{r}_0+t\,\mathbf{r}_1=(1-t)(-2\,\mathbf{i}+18\,\mathbf{j}+31\,\mathbf{k})+t(11\,\mathbf{i}-4\,\mathbf{j}+48\,\mathbf{k})\\&=(-2\,\mathbf{i}+18\,\mathbf{j}+31\,\mathbf{k})+t(13\,\mathbf{i}-22\,\mathbf{j}+17\,\mathbf{k}),\quad 0\le t\le 1.\end{aligned}$$

The corresponding parametric equations are $x=-2+13t$, $y=18-22t$, $z=31+17t$, $0\le t\le 1$.

19. Since the direction vectors $\langle 2,-1,3\rangle$ and $\langle 4,-2,5\rangle$ are not scalar multiples of each other, the lines aren't parallel. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $3+2t=1+4s$, $4-t=3-2s$, $1+3t=4+5s$. Solving the last two equations we get $t=1$, $s=0$ and checking, we see that these values don't satisfy the first equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

20. Since the direction vectors are $\mathbf{v}_1=\langle -12,9,-3\rangle$ and $\mathbf{v}_2=\langle 8,-6,2\rangle$, we have $\mathbf{v}_1=-\frac{3}{2}\mathbf{v}_2$ so the lines are parallel.

21. Since the direction vectors $\langle 1,-2,-3\rangle$ and $\langle 1,3,-7\rangle$ aren't scalar multiples of each other, the lines aren't parallel. Parametric equations of the lines are L_1: $x=2+t$, $y=3-2t$, $z=1-3t$ and L_2: $x=3+s$, $y=-4+3s$, $z=2-7s$. Thus, for the

lines to intersect, the three equations $2+t=3+s$, $3-2t=-4+3s$, and $1-3t=2-7s$ must be satisfied simultaneously. Solving the first two equations gives $t=2$, $s=1$ and checking, we see that these values do satisfy the third equation, so the lines intersect when $t=2$ and $s=1$, that is, at the point $(4,-1,-5)$.

22. The direction vectors $\langle 1,-1,3\rangle$ and $\langle 2,-2,7\rangle$ are not parallel, so neither are the lines. Parametric equations for the lines are L_1: $x=t$, $y=1-t$, $z=2+3t$ and L_2: $x=2+2s$, $y=3-2s$, $z=7s$. Thus, for the lines to interesect, the three equations $t=2+2s$, $1-t=3-2s$, and $2+3t=7s$ must be satisfied simultaneously. Solving the last two equations gives $t=-10$, $s=-4$ and checking, we see that these values don't satisfy the first equation. Thus the lines aren't parallel and don't intersect, so they must be skew.

23. $5\mathbf{i}+4\mathbf{j}+6\mathbf{k}$ is a normal vector to the plane. $(3,2,1)$ is a point on the plane. Setting $a=5$, $b=4$, $c=6$ and $x_0=3$, $y_0=2$, $z_0=1$ in Equation 7 gives $5(x-3)+4(y-2)+6(z-1)=0$, or $5x+4y+6z=29$, as an equation of the plane.

24. $\langle 6,1,-1\rangle$ is a normal vector to the plane. $(-3,4,2)$ is a point on the plane. Setting $a=6$, $b=1$, $c=-1$ and $x_0=-3$, $y_0=4$, $z_0=2$ in Equation 7 gives $6(x+3)+1(y-4)-(z-2)=0$, or $6x+y-z=-16$, as an equation of the plane.

25. Since the plane is perpendicular to the vector $-\mathbf{i}+2\,\mathbf{j}+3\,\mathbf{k}$, we can take $\langle -1,2,3\rangle$ as a normal vector to the plane. $(5,-2,4)$ is a point on the plane. Setting $a=-1$, $b=2$, $c=3$ and $x_0=5$, $y_0=-2$, $z_0=4$ in Equation 7 gives $-(x-5)+2(y+2)+3(z-4)=0$, or $-x+2y+3z=3$, as an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector, $\langle -8,-7,2\rangle$, is a normal vector to the plane. $(0,0,0)$ is a point on the plane. Setting $a=-8$, $b=-7$, $c=2$ and $x_0=0$, $y_0=0$, $z_0=0$ in Equation 7 gives $-8(x-0)-7(y-0)+2(z-0)=0$, or $-8x-7y+2z=0$, as an equation of the plane.

27. Since the line is perpendicular to the plane, its direction vector, $\langle 4,-1,5\rangle$, is a normal vector to the plane. $(1,3,-1)$ is a point on the plane. Setting $a=4$, $b=-1$, $c=5$ and $x_0=1$, $y_0=3$, $z_0=-1$ in Equation 7 gives $4(x-1)-1(y-3)+5(z+1)=0$, or $4x-y+5z=-4$, as an equation of the plane.

28. Since the two planes are parallel, they will have the same normal vectors. The plane is $z=2x-3y \quad\Leftrightarrow\quad 2x-3y-z=0$, so we can take $\mathbf{n}=\langle 2,-3,-1\rangle$, and an equation of the plane is $2(x-9)-3(y+4)-1(z+5)=0$, or $2x-3y-z=35$.

29. Since the two planes are parallel, they will have the same normal vectors. The plane is $2x-y+3z=1$, so we can take $\mathbf{n}=\langle 2,-1,3\rangle$, and an equation of the plane is $2(x-2.1)-1(y-1.7)+3(z+0.9)=0$, or $2x-y+3z=-0.2$, or $10x-5y+15z=-1$.

30. First, a normal vector for the plane $5x+2y+z=1$ is $\mathbf{n}=\langle 5,2,1\rangle$. A direction vector for the line is $\mathbf{v}=\langle 1,-1,-3\rangle$, and since $\mathbf{n}\cdot\mathbf{v}=0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t=0$, we know that the point $(1,2,4)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n}=\langle 5,2,1\rangle$, so an equation of the plane is $5(x-1)+2(y-2)+1(z-4)=0$ or $5x+2y+z=13$.

31. The vector from $(0,1,1)$ to $(1,0,1)$, namely $\mathbf{a}=\langle 1-0,0-1,1-1\rangle=\langle 1,-1,0\rangle$, and the vector from $(0,1,1)$ to $(1,1,0)$, $\mathbf{b}=\langle 1-0,1-1,0-1\rangle=\langle 1,0,-1\rangle$, both lie in the plane, so $\mathbf{a}\times\mathbf{b}$ is a normal vector to the plane. Thus, we can take

$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle (-1)((-1) - (0)(0), (0)(1) - (1)(-1), (1)(0) - (-1)(1) \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

32. Here the vectors $\mathbf{a} = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle (-2)(1) - (1)(1), (1)(1) - (3)(1), (3)(1) - (-2)(1) \rangle = \langle -3, -2, 5 \rangle$ is a normal vector to the plane. We can take the origin as P_0, so an equation of the plane is $-3(x - 0) - 2(y - 0) + 5(z - 0) = 0$ or $-3x - 2y + 5z = 0$ or $3x + 2y - 5z = 0$.

33. Here the vectors $\mathbf{a} = \langle 3 - 2, -8 - 1, 6 - 2 \rangle = \langle 1, -9, 4 \rangle$ and $\mathbf{b} = \langle -2 - 2, -3 - 1, 1 - 2 \rangle = \langle -4, -4, -1 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 9 + 16, -16 + 1, -4 - 36 \rangle = \langle 25, -15, -40 \rangle$ and an equation of the plane is $25(x - 2) - 15(y - 1) - 40(z - 2) = 0$ or $25x - 15y - 40z = -45$ or $5x - 3y - 8z = -9$.

34. The vectors $\mathbf{a} = \langle -2 - 3, -2 - 0, 3 - (-1) \rangle = \langle -5, -2, 4 \rangle$ and $\mathbf{b} = \langle 7 - 3, 1 - 0, -4 - (-1) \rangle = \langle 4, 1, -3 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 6 - 4, 16 - 15, -5 + 8 \rangle = \langle 2, 1, 3 \rangle$ and an equation of the plane is $2(x - 3) + 1(y - 0) + 3[z - (-1)] = 0$ or $2x + y + 3z = 3$.

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -1, 2, -3 \rangle$ is one vector in the plane. We can verify that the given point $(3, 5, -1)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, -1, 0)$ is on the line, so $\mathbf{b} = \langle 4 - 3, -1 - 5, 0 - (-1) \rangle = \langle 1, -6, 1 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 2 - 18, -3 + 1, 6 - 2 \rangle = \langle -16, -2, 4 \rangle$. Thus, an equation of the plane is $-16(x - 3) - 2(y - 5) + 4[z - (-1)] = 0$ or $-16x - 2y + 4z = -62$ or $8x + y - 2z = 31$.

36. Since the line $\dfrac{x}{3} = \dfrac{y + 4}{1} = \dfrac{z}{2}$ lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, 2 \rangle$ is parallel to the plane. The point $(0, -4, 0)$ is on the line (put $t = 0$ in the corresponding parametric equations), and we can verify that the given point $(6, -1, 3)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 6, 3, 3 \rangle$, is therefore parallel to the plane, but not parallel to $\mathbf{a}$. Then $\mathbf{a} \times \mathbf{b} = \langle 3 - 6, 12 - 9, 9 - 6 \rangle = \langle -3, 3, 3 \rangle$ is a normal vector to the plane, and an equation of the plane is $-3(x - 0) + 3[y - (-4)] + 3(z - 0) = 0$ or $-3x + 3y + 3z = -12$ or $x - y - z = 4$.

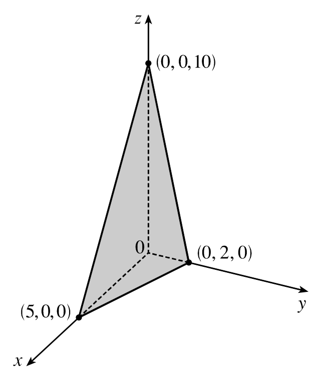
37. Normal vectors for the given planes are $\mathbf{n}_1 = \langle 1, 2, 3 \rangle$ and $\mathbf{n}_2 = \langle 2, -1, 1 \rangle$. A direction vector, then, for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2 + 3, 6 - 1, -1 - 4 \rangle = \langle 5, 5, -5 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(3, 1, 4)$ in the plane. Setting $z = 0$, the equations of the planes reduce to $x + 2y = 1$ and $2x - y = -3$ with simultaneous solution $x = -1$ and $y = 1$. So a point on the line is $(-1, 1, 0)$ and another vector parallel to the plane is $\mathbf{b} = \langle 3 - (-1), 1 - 1, 4 - 0 \rangle = \langle 4, 0, 4 \rangle$. Then a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 20 - 0, -20 - 20, 0 - 20 \rangle = \langle 20, -40, -20 \rangle$. Equivalently, we can take $\langle 1, -2, -1 \rangle$ as a normal vector, and an equation of the plane is $1(x - 3) - 2(y - 1) - 1(z - 4) = 0$ or $x - 2y - z = -3$.

38. The points $(0, -2, 5)$ and $(-1, 3, 1)$ lie in the desired plane, so the vector $\mathbf{v}_1 = \langle -1, 5, -4 \rangle$ connecting them is parallel to the plane. The desired plane is perpendicular to the plane $2z = 5x + 4y$ or $5x + 4y - 2z = 0$ and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so $\mathbf{v}_2 = \langle 5, 4, -2 \rangle$ is also parallel to the desired plane.
A normal vector to the desired plane is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -10 + 16, -20 - 2, -4 - 25 \rangle = \langle 6, -22, -29 \rangle$.
Taking $(x_0, y_0, z_0) = (0, -2, 5)$, the equation we are looking for is $6(x - 0) - 22(y + 2) - 29(z - 5) = 0$ or $6x - 22y - 29z = -101$.

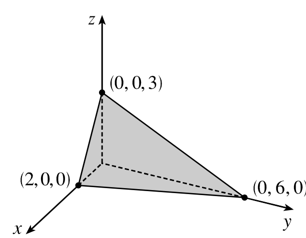
39. If a plane is perpendicular to two other planes, its normal vector is perpendicular to the normal vectors of the other two planes. Thus $\langle 2, 1, -2 \rangle \times \langle 1, 0, 3 \rangle = \langle 3 - 0, -2 - 6, 0 - 1 \rangle = \langle 3, -8, -1 \rangle$ is a normal vector to the desired plane. The point $(1, 5, 1)$ lies on the plane, so an equation is $3(x - 1) - 8(y - 5) - (z - 1) = 0$ or $3x - 8y - z = -38$.

40. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = \langle 3, 3, 3 \rangle$, or we can use $\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

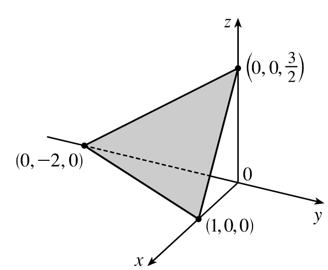
41. To find the x-intercept we set $y = z = 0$ in the equation $2x + 5y + z = 10$ and obtain $2x = 10 \quad \Rightarrow \quad x = 5$ so the x-intercept is $(5, 0, 0)$. When $x = z = 0$ we get $5y = 10 \quad \Rightarrow \quad y = 2$, so the y-intercept is $(0, 2, 0)$. Setting $x = y = 0$ gives $z = 10$, so the z-intercept is $(0, 0, 10)$ and we graph the portion of the plane that lies in the first octant.



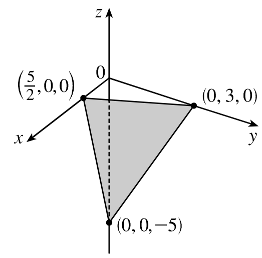
42. To find the x-intercept we set $y = z = 0$ in the equation $3x + y + 2z = 6$ and obtain $3x = 6 \quad \Rightarrow \quad x = 2$ so the x-intercept is $(2, 0, 0)$. When $x = z = 0$ we get $y = 6$ so the y-intercept is $(0, 6, 0)$. Setting $x = y = 0$ gives $2z = 6 \quad \Rightarrow \quad z = 3$, so the z-intercept is $(0, 0, 3)$. The figure shows the portion of the plane that lies in the first octant.



43. Setting $y = z = 0$ in the equation $6x - 3y + 4z = 6$ gives $6x = 6 \quad \Rightarrow$ $x = 1$, when $x = z = 0$ we have $-3y = 6 \quad \Rightarrow \quad y = -2$, and $x = y = 0$ implies $4z = 6 \quad \Rightarrow \quad z = \frac{3}{2}$, so the intercepts are $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, \frac{3}{2})$. The figure shows the portion of the plane cut off by the coordinate planes.



44. Setting $y = z = 0$ in the equation $6x + 5y - 3z = 15$ gives $6x = 15 \Rightarrow x = \frac{5}{2}$, when $x = z = 0$ we have $5y = 15 \Rightarrow y = 3$, and $x = y = 0$ implies $-3z = 15 \Rightarrow z = -5$, so the intercepts are $(\frac{5}{2}, 0, 0)$, $(0, 3, 0)$, and $(0, 0, -5)$. The figure shows the portion of the plane cut off by the coordinate planes.



45. Substitute the parametric equations of the line into the equation of the plane: $x + 2y - z = 7 \Rightarrow$ $(2 - 2t) + 2(3t) - (1 + t) = 7 \Rightarrow 3t + 1 = 7 \Rightarrow t = 2$. Therefore, the point of intersection of the line and the plane is given by $x = 2 - 2(2) = -2$, $y = 3(2) = 6$, and $z = 1 + 2 = 3$, that is, the point $(-2, 6, 3)$.

46. Substitute the parametric equations of the line into the equation of the plane: $3(t - 1) - (1 + 2t) + 2(3 - t) = 5 \Rightarrow$ $-t + 2 = 5 \Rightarrow t = -3$. Therefore, the point of intersection of the line and the plane is given by $x = -3 - 1 = -4$, $y = 1 + 2(-3) = -5$, and $z = 3 - (-3) = 6$, that is, the point $(-4, -5, 6)$.

47. Parametric equations for the line are $x = \frac{1}{5}t$, $y = 2t$, $z = t - 2$ and substitution into the equation of the plane gives $10\left(\frac{1}{5}t\right) - 7(2t) + 3(t - 2) + 24 = 0 \Rightarrow -9t + 18 = 0 \Rightarrow t = 2$. Thus $x = \frac{1}{5}(2) = \frac{2}{5}$, $y = 2(2) = 4$, $z = 2 - 2 = 0$ and the point of intersection is $\left(\frac{2}{5}, 4, 0\right)$.

48. A direction vector for the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ is $\mathbf{v} = \langle 2, 4, 6 \rangle$ and, taking $P_0 = (-3, 1, 0)$, parametric equations for the line are $x = -3 + 2t$, $y = 1 + 4t$, $z = 6t$. Substitution of the parametric equations into the equation of the plane gives $2(-3 + 2t) + (1 + 4t) - (6t) = -2 \Rightarrow 2t - 5 = -2 \Rightarrow t = \frac{3}{2}$. Then $x = -3 + 2\left(\frac{3}{2}\right) = 0$, $y = 1 + 4\left(\frac{3}{2}\right) = 7$, and $z = 6\left(\frac{3}{2}\right) = 9$, and the point of intersection is $(0, 7, 9)$.

49. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are 1, 0, -1.

50. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$. The cosine of the angle θ between these two planes is

$$\cos\theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 1, 1, 1 \rangle|\,|\langle 1, 2, 3 \rangle|} = \frac{1 + 2 + 3}{\sqrt{1 + 1 + 1}\,\sqrt{1 + 4 + 9}} = \frac{6}{\sqrt{42}} = \sqrt{\frac{6}{7}}.$$

51. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 4, -3 \rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7 \rangle$. The normals aren't parallel (they are not scalar multiples of each other), so neither are the planes. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals, and thus the planes, are perpendicular.

52. Normal vectors for the planes are $\mathbf{n}_1 = \langle 9, -3, 6 \rangle$ and $\mathbf{n}_2 = \langle 6, -2, 4 \rangle$ (the plane's equation is $6x - 2y + 4z = 0$). Since $\mathbf{n}_1 = \frac{3}{2}\mathbf{n}_2$, the normals, and thus the planes, are parallel.

53. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 2, -2, 1 \rangle$. The normals are not parallel (they are not scalar multiples of each other), so neither are the planes. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 - 1 = -3 \neq 0$, so the planes aren't perpendicular. The angle between the planes is the same as the angle between the normals, given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{-3}{\sqrt{6}\,\sqrt{9}} = -\frac{1}{\sqrt{6}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) \approx 114.1^\circ.$$

54. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -1, 3 \rangle$ and $\mathbf{n}_2 = \langle 3, 1, -1 \rangle$. The normals are not parallel, so neither are the planes. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 - 1 - 3 = -1 \neq 0$, the planes aren't perpendicular. The angle between the planes is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{-1}{\sqrt{11}\,\sqrt{11}} = -\frac{1}{11} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\tfrac{1}{11}\right) \approx 95.2^\circ.$$

55. The planes are $2x - 3y - z = 0$ and $4x - 6y - 2z = 3$ with normal vectors $\mathbf{n}_1 = \langle 2, -3, -1 \rangle$ and $\mathbf{n}_2 = \langle 4, -6, -2 \rangle$. Since $\mathbf{n}_2 = 2\mathbf{n}_1$, the normals, and thus the planes, are parallel.

56. The normals are $\mathbf{n}_1 = \langle 5, 2, 3 \rangle$ and $\mathbf{n}_2 = \langle 4, -1, -6 \rangle$ which are not scalar multiples of each other, so the planes aren't parallel. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 20 - 2 - 18 = 0$, the normals, and thus the planes, are perpendicular.

57. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will fail if the line of intersection does not cross the xy-plane; in that case, try setting x or y equal to 0.) The equations of the two planes reduce to $x + y = 1$ and $x + 2y = 1$. Solving these two equations gives $x = 1$, $y = 0$. Thus a point on the line is $(1, 0, 0)$.
A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so we can take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 2, 2 \rangle = \langle 2 - 2, 1 - 2, 2 - 1 \rangle = \langle 0, -1, 1 \rangle$. By Equations 2, parametric equations for the line are $x = 1$, $y = -t$, $z = t$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{1 + 2 + 2}{\sqrt{3}\,\sqrt{9}} = \dfrac{5}{3\sqrt{3}}$. Therefore $\theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8^\circ$.

58. (a) If we set $z = 0$ then the equations of the planes reduce to $3x - 2y = 1$ and $2x + y = 3$ and solving these two equations gives $x = 1$, $y = 1$. Thus a point on the line of intersection is $(1, 1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so let $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 3, -2, 1 \rangle \times \langle 2, 1, -3 \rangle = \langle 5, 11, 7 \rangle$. By Equations 2, parametric equations for the line are $x = 1 + 5t$, $y = 1 + 11t$, $z = 7t$.

(b) $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{6 - 2 - 3}{\sqrt{14}\,\sqrt{14}} = \dfrac{1}{14} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\tfrac{1}{14}\right) \approx 85.9^\circ$.

59. Setting $z = 0$, the equations of the two planes become $5x - 2y = 1$ and $4x + y = 6$. Solving these two equations gives $x = 1$, $y = 2$ so a point on the line of intersection is $(1, 2, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes. So we can use $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5, -2, -2 \rangle \times \langle 4, 1, 1 \rangle = \langle 0, -13, 13 \rangle$ or equivalently we can take $\mathbf{v} = \langle 0, -1, 1 \rangle$, and symmetric equations for the line are $x = 1$, $\dfrac{y - 2}{-1} = \dfrac{z}{1}$ or $x = 1$, $y - 2 = -z$.

60. If we set $z = 0$ then the equations of the planes reduce to $2x - y - 5 = 0$ and $4x + 3y - 5 = 0$ and solving these two equations gives $x = 2$, $y = -1$. Thus a point on the line of intersection is $(2, -1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -1, -1 \rangle \times \langle 4, 3, -1 \rangle = \langle 4, -2, 10 \rangle$ or equivalently we can take $\mathbf{v} = \langle 2, -1, 5 \rangle$. Symmetric equations for the line are $\dfrac{x-2}{2} = \dfrac{y+1}{-1} = \dfrac{z}{5}$.

61. The distance from a point (x, y, z) to $(1, 0, -2)$ is $d_1 = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ and the distance from (x, y, z) to $(3, 4, 0)$ is $d_2 = \sqrt{(x-3)^2 + (y-4)^2 + z^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \quad \Rightarrow \quad d_1^2 = d_2^2 \quad \Leftrightarrow$

$(x-1)^2 + y^2 + (z+2)^2 = (x-3)^2 + (y-4)^2 + z^2 \quad \Leftrightarrow$

$x^2 - 2x + y^2 + z^2 + 4z + 5 = x^2 - 6x + y^2 - 8y + z^2 + 25 \quad \Leftrightarrow \quad 4x + 8y + 4z = 20$ so an equation for the plane is $4x + 8y + 4z = 20$ or equivalently $x + 2y + z = 5$.

Alternatively, you can argue that the segment joining points $(1, 0, -2)$ and $(3, 4, 0)$ is perpendicular to the plane and the plane includes the midpoint of the segment.

62. The distance from a point (x, y, z) to $(2, 5, 5)$ is $d_1 = \sqrt{(x-2)^2 + (y-5)^2 + (z-5)^2}$ and the distance from (x, y, z) to $(-6, 3, 1)$ is $d_2 = \sqrt{(x+6)^2 + (y-3)^2 + (z-1)^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \quad \Rightarrow$

$d_1^2 = d_2^2 \quad \Leftrightarrow \quad (x-2)^2 + (y-5)^2 + (z-5)^2 = (x+6)^2 + (y-3)^2 + (z-1)^2 \quad \Leftrightarrow$

$x^2 - 4x + y^2 - 10y + z^2 - 10z + 54 = x^2 + 12x + y^2 - 6y + z^2 - 2z + 46 \quad \Leftrightarrow \quad 16x + 4y + 8z = 8$ so an equation for the plane is $16x + 4y + 8z = 8$ or equivalently $4x + y + 2z = 2$.

63. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

64. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2 \rangle$ and $\langle -1, 1, 0 \rangle$, so a normal vector for the plane is $\langle -1, 1, 0 \rangle \times \langle 1, -1, 2 \rangle = \langle 2, 2, 0 \rangle$ and it contains the point $(2, 0, 2)$. Then an equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \quad \Leftrightarrow \quad x + y = 2$.

65. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

66. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $\mathbf{a} = \langle 1, -1, 2\rangle$ and $\mathbf{b} = \langle -1, 0, 2\rangle$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then

$\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \langle -1, 0, 2\rangle - \dfrac{\langle 1, -1, 2\rangle \cdot \langle -1, 0, 2\rangle}{1^2 + (-1)^2 + 2^2} \langle 1, -1, 2\rangle = \langle -1, 0, 2\rangle - \frac{1}{2}\langle 1, -1, 2\rangle = \left\langle -\frac{3}{2}, \frac{1}{2}, 1\right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\frac{3}{2}, \frac{1}{2}, 1\right\rangle = \langle -3, 1, 2\rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1 + t$, $z = 2 + 2t$. (Notice that this is the same line as in Exercise 65.)

67. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 3, 6, -3\rangle$, $\mathbf{n}_2 = \langle 4, -12, 8\rangle$, $\mathbf{n}_3 = \langle 3, -9, 6\rangle$, $\mathbf{n}_4 = \langle 1, 2, -1\rangle$. Now $\mathbf{n}_1 = 3\mathbf{n}_4$, so $\mathbf{n}_1$ and $\mathbf{n}_4$ are parallel, and hence P_1 and P_4 are parallel; similarly P_2 and P_3 are parallel because $\mathbf{n}_2 = \frac{4}{3}\mathbf{n}_3$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel (so not all four planes are parallel). Notice that the point $(2, 0, 0)$ lies on both P_1 and P_4, so these two planes are identical. The point $\left(\frac{5}{4}, 0, 0\right)$ lies on P_2 but not on P_3, so these are different planes.

68. Let L_i have direction vector $\mathbf{v}_i$. Rewrite the symmetric equations for L_3 as $\dfrac{x-1}{1/2} = \dfrac{y-1}{-1/4} = \dfrac{z+1}{1}$; then $\mathbf{v}_1 = \langle 6, -3, 12\rangle$, $\mathbf{v}_2 = \langle 2, 1, 4\rangle$, $\mathbf{v}_3 = \left\langle \frac{1}{2}, -\frac{1}{4}, 1\right\rangle$, and $\mathbf{v}_4 = \langle 4, 2, 8\rangle$. $\mathbf{v}_1 = 12\mathbf{v}_3$, so L_1 and L_3 are parallel. $\mathbf{v}_4 = 2\mathbf{v}_2$, so L_2 and L_4 are parallel. (Note that L_1 and L_2 are not parallel.) L_1 contains the point $(1, 1, 5)$, but this point does not lie on L_3, so they're not identical. $(3, 1, 5)$ lies on L_4 and also on L_2 (for $t = 1$), so L_2 and L_4 are the same line.

69. Let $Q = (1, 3, 4)$ and $R = (2, 1, 1)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (4, 1, -2)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -2, -3\rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle 3, -2, -6\rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -2, -3\rangle \times \langle 3, -2, -6\rangle|}{|\langle 1, -2, -3\rangle|} = \frac{|\langle 6, -3, 4\rangle|}{|\langle 1, -2, -3\rangle|} = \frac{\sqrt{6^2 + (-3)^2 + 4^2}}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\sqrt{61}}{\sqrt{14}} = \sqrt{\frac{61}{14}}.$$

70. Let $Q = (0, 6, 3)$ and $R = (2, 4, 4)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (0, 1, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 2, -2, 1\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 0, -5, 0\rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 2, -2, 1\rangle \times \langle 0, -5, 0\rangle|}{|\langle 2, -2, 1\rangle|} = \frac{|\langle 5, 0, -10\rangle|}{|\langle 2, -2, 1\rangle|} = \frac{\sqrt{5^2 + 0^2 + (-10)^2}}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{\sqrt{125}}{\sqrt{9}} = \frac{5\sqrt{5}}{3}.$$

71. By Equation 9, the distance is $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \dfrac{|18|}{\sqrt{49}} = \dfrac{18}{7}$.

72. By Equation 9, the distance is $D = \dfrac{|1(-6) - 2(3) - 4(5) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \dfrac{|-40|}{\sqrt{21}} = \dfrac{40}{\sqrt{21}}$.

73. Put $y = z = 0$ in the equation of the first plane to get the point $(2, 0, 0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(2, 0, 0)$ to the second plane. By Equation 9,

$$D = \frac{|4(2) - 6(0) + 2(0) - 3|}{\sqrt{4^2 + (-6)^2 + (2)^2}} = \frac{5}{\sqrt{56}} = \frac{5}{2\sqrt{14}} \text{ or } \frac{5\sqrt{14}}{28}.$$

74. Put $x = y = 0$ in the equation of the first plane to get the point $(0, 0, 0)$ on the plane. Because the planes are parallel the distance D between them is the distance from $(0, 0, 0)$ to the second plane $3x - 6y + 9z - 1 = 0$. By Equation 9,

$$D = \frac{|3(0) - 6(0) + 9(0) - 1|}{\sqrt{3^2 + (-6)^2 + 9^2}} = \frac{1}{\sqrt{126}} = \frac{1}{3\sqrt{14}}.$$

75. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9,

$$D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

76. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c\rangle = t\langle 1, 2, -2\rangle = \langle t, 2t, -2t\rangle$. So this plane is given by the equation $x + 2y - 2z + k = 0$, where $k = d/t$. By Exercise 75, the distance between the planes is $2 = \dfrac{|1 - k|}{\sqrt{1^2 + 2^2 + (-2)^2}} \quad\Leftrightarrow\quad 6 = |1 - k| \quad\Leftrightarrow\quad k = 7$ or -5. So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

77. L_1: $x = y = z \quad\Rightarrow\quad x = y$ **(1)**. L_2: $x + 1 = y/2 = z/3 \quad\Rightarrow\quad x + 1 = y/2$ **(2)**. The solution of **(1)** and **(2)** is $x = y = -2$. However, when $x = -2$, $x = z \quad\Rightarrow\quad z = -2$, but $x + 1 = z/3 \quad\Rightarrow\quad z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1\rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 3 - 2, -3 + 1, 2 - 1\rangle = \langle 1, -2, 1\rangle$. From above, we know that $(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2) - 2(-2) + 1(-2) + d_1 = 0 \quad\Rightarrow\quad d_1 = 0$ and $1(-2) - 2(-2) + 1(-3) + d_2 = 0 \quad\Rightarrow\quad d_2 = 1$.

By Exercise 75, the distance between these two skew lines is $D = \dfrac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 1, -2, 1\rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$, and form the vector $\mathbf{b} = \langle 0, 0, 1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

78. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$, the direction vectors of the two lines, respectively. Thus set

$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12 \rangle = \langle 6, -2, 3 \rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \quad \Rightarrow \quad d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \quad \Rightarrow \quad d_2 = 10$.

Then by Exercise 75, the distance between the two skew lines is given by $D = \dfrac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \dfrac{14}{7} = 2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2 \rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}} \, |0 - 8 - 6| = \frac{14}{7} = 2.$$

79. A direction vector for L_1 is $\mathbf{v}_1 = \langle 2, 0, -1 \rangle$ and a direction vector for L_2 is $\mathbf{v}_2 = \langle 3, 2, 2 \rangle$. These vectors are not parallel so neither are the lines. Parametric equations for the lines are L_1: $x = 2t$, $y = 0$, $z = -t$, and L_2: $x = 1 + 3s$, $y = -1 + 2s$, $z = 1 + 2s$. No values of t and s satisfy these equations simultaneously, so the lines don't intersect and hence are skew. We can view the lines as lying in two parallel planes; a common normal vector to the planes is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -7, 4 \rangle$. Line L_1 passes through the origin, so $(0, 0, 0)$ lies on one of the planes, and $(1, -1, 1)$ is a point on L_2 and therefore on the other plane. Equations of the planes then are $2x - 7y + 4z = 0$ and $2x - 7y + 4z - 13 = 0$, and by Exercise 75, the distance between the two skew lines is $D = \dfrac{|0 - (-13)|}{\sqrt{4 + 49 + 16}} = \dfrac{13}{\sqrt{69}}$.

Alternate solution (without reference to planes): Direction vectors of the two lines are $\mathbf{v}_1 = \langle 2, 0, -1 \rangle$ and $\mathbf{v}_2 = \langle 3, 2, 2 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -7, 4 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(0, 0, 0)$ and $(1, -1, 1)$, and form the vector $\mathbf{b} = \langle 1, -1, 1 \rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|2 + 7 + 4|}{\sqrt{4 + 49 + 16}} = \dfrac{13}{\sqrt{69}}$.

80. A direction vector for the line L_1 is $\mathbf{v}_1 = \langle 1, 2, 2 \rangle$. A normal vector for the plane P_1 is $\mathbf{n}_1 = \langle 1, -1, 2 \rangle$. The vector from the point $(0, 0, 1)$ to $(3, 2, -1)$, $\langle 3, 2, -2 \rangle$, is parallel to the plane P_2, as is the vector from $(0, 0, 1)$ to $(1, 2, 1)$, namely $\langle 1, 2, 0 \rangle$. Thus a normal vector for P_2 is $\langle 3, 2, -2 \rangle \times \langle 1, 2, 0 \rangle = \langle 4, -2, 4 \rangle$, or we can use $\mathbf{n}_2 = \langle 2, -1, 2 \rangle$, and a direction vector for the line L_2 of intersection of these planes is $\mathbf{v}_2 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -1, 2 \rangle \times \langle 2, -1, 2 \rangle = \langle 0, 2, 1 \rangle$. Notice that the point $(3, 2, -1)$ lies on both planes, so it also lies on L_2. The lines are skew, so we can view them as lying in two parallel planes; a common normal vector to the planes is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -2, -1, 2 \rangle$. Line L_1 passes through the point $(1, 2, 6)$, so $(1, 2, 6)$ lies on one of the planes, and $(3, 2, -1)$ is a point on L_2 and therefore on the other plane. Equations of the planes then are $-2x - y + 2z - 8 = 0$ and $-2x - y + 2z + 10 = 0$, and by Exercise 75, the distance between the lines is

$$D = \frac{|-8 - 10|}{\sqrt{4 + 1 + 4}} = \frac{18}{3} = 6.$$

[continued]

Alternatively, direction vectors for the lines are $\mathbf{v}_1 = \langle 1, 2, 2 \rangle$ and $\mathbf{v}_2 = \langle 0, 2, 1 \rangle$, so $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -2, -1, 2 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 2, 6)$ and $(3, 2, -1)$, and form the vector $\mathbf{b} = \langle 2, 0, -7 \rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|-4 + 0 - 14|}{\sqrt{4 + 1 + 4}} = \dfrac{18}{3} = 6$.

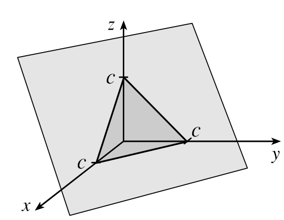
81. (a) A direction vector from tank A to tank B is $\langle 765 - 325, 675 - 810, 599 - 561 \rangle = \langle 440, -135, 38 \rangle$. Taking tank A's position $(325, 810, 561)$ as the initial point, parametric equations for the line of sight are $x = 325 + 440t$, $y = 810 - 135t$, $z = 561 + 38t$ for $0 \le t \le 1$.

(b) We divide the line of sight into 5 equal segments, corresponding to $\Delta t = 0.2$, and compute the elevation from the z-component of the parametric equations in part (a):

t	$z = 561 + 38t$	terrain elevation
0	561.0	
0.2	568.6	549
0.4	576.2	566
0.6	583.8	586
0.8	591.4	589
1.0	599.0	

Since the terrain is higher than the line of sight when $t = 0.6$, the tanks can't see each other.

82. (a) The planes $x + y + z = c$ have normal vector $\langle 1, 1, 1 \rangle$, so they are all parallel. Their x-, y-, and z-intercepts are all c. When $c > 0$ their intersection with the first octant is an equilateral triangle and when $c < 0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.



(b) The planes $x + y + cz = 1$ have x-intercept 1, y-intercept 1, and z-intercept $1/c$. The plane with $c = 0$ is parallel to the z-axis. As c gets larger, the planes get closer to the xy-plane.

(c) The planes $y \cos\theta + z \sin\theta = 1$ have normal vectors $\langle 0, \cos\theta, \sin\theta \rangle$, which are perpendicular to the x-axis, and so the planes are parallel to the x-axis. We look at their intersection with the yz-plane. These are lines that are perpendicular to $\langle \cos\theta, \sin\theta \rangle$ and pass through $(\cos\theta, \sin\theta)$, since $\cos^2\theta + \sin^2\theta = 1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x-axis.

83. If $a \neq 0$, then $ax + by + cz + d = 0 \quad \Rightarrow \quad a(x + d/a) + b(y - 0) + c(z - 0) = 0$ which by (7) is the scalar equation of the plane through the point $(-d/a, 0, 0)$ with normal vector $\langle a, b, c \rangle$. Similarly, if $b \neq 0$ (or if $c \neq 0$) the equation of the plane can be rewritten as $a(x - 0) + b(y + d/b) + c(z - 0) = 0$ [or as $a(x - 0) + b(y - 0) + c(z + d/c) = 0$] which by (7) is the scalar equation of a plane through the point $(0, -d/b, 0)$ [or the point $(0, 0, -d/c)$] with normal vector $\langle a, b, c \rangle$.

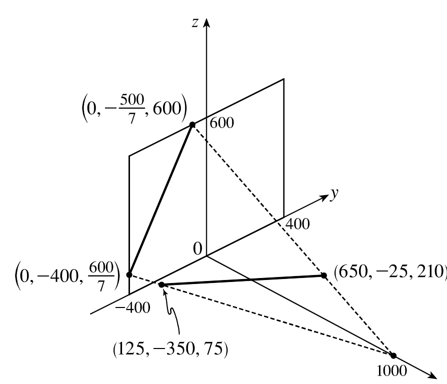
DISCOVERY PROJECT Putting 3D in Perspective

1. If we view the screen from the camera's location, the vertical clipping plane on the left passes through the points $(1000, 0, 0)$, $(0, -400, 0)$, and $(0, -400, 600)$. A vector from the first point to the second is $\mathbf{v}_1 = \langle -1000, -400, 0\rangle$ and a vector from the first point to the third is $\mathbf{v}_2 = \langle -1000, -400, 600\rangle$. A normal vector for the clipping plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -240{,}000\,\mathbf{i} + 600{,}000\,\mathbf{j}$ or $-2\,\mathbf{i} + 5\,\mathbf{j}$, and an equation for the plane is $-2(x - 1000) + 5(y - 0) + 0(z - 0) = 0 \quad\Rightarrow\quad 2x - 5y = 2000$. By symmetry, the vertical clipping plane on the right is given by $2x + 5y = 2000$. The lower clipping plane is $z = 0$. The upper clipping plane passes through the points $(1000, 0, 0)$, $(0, -400, 600)$, and $(0, 400, 600)$. Vectors from the first point to the second and third points are $\mathbf{v}_1 = \langle -1000, -400, 600\rangle$ and $\mathbf{v}_2 = \langle -1000, 400, 600\rangle$, and a normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -480{,}000\,\mathbf{i} - 800{,}000\,\mathbf{k}$ or $3\,\mathbf{i} + 5\,\mathbf{k}$. An equation for the plane is $3(x - 1000) + 0(y - 0) + 5(z - 0) = 0 \quad\Rightarrow\quad 3x + 5z = 3000$.

A direction vector for the line L is $\mathbf{v} = \langle 630, 390, 162\rangle$ and taking $P_0 = (230, -285, 102)$, parametric equations are $x = 230 + 630t$, $y = -285 + 390t$, $z = 102 + 162t$. L intersects the left clipping plane when $2(230 + 630t) - 5(-285 + 390t) = 2000 \quad\Rightarrow\quad t = -\frac{1}{6}$. The corresponding point is $(125, -350, 75)$. L intersects the right clipping plane when $2(230 + 630t) + 5(-285 + 390t) = 2000 \quad\Rightarrow\quad t = \frac{593}{642}$. The corresponding point is approximately $(811.9, 75.2, 251.6)$, but this point is not contained within the viewing volume. L intersects the upper clipping plane when $3(230 + 630t) + 5(102 + 162t) = 3000 \quad\Rightarrow\quad t = \frac{2}{3}$, corresponding to the point $(650, -25, 210)$, and L intersects the lower clipping plane when $z = 0 \quad\Rightarrow\quad 102 + 162t = 0 \quad\Rightarrow\quad t = -\frac{17}{27}$. The corresponding point is approximately $(-166.7, -530.6, 0)$, which is not contained within the viewing volume. Thus L should be clipped at the points $(125, -350, 75)$ and $(650, -25, 210)$.

2. A sight line from the camera at $(1000, 0, 0)$ to the left endpoint $(125, -350, 75)$ of the clipped line has direction $\mathbf{v} = \langle -875, -350, 75\rangle$. Parametric equations are $x = 1000 - 875t$, $y = -350t$, $z = 75t$. This line intersects the screen when $x = 0 \quad\Rightarrow\quad 1000 - 875t = 0 \quad\Rightarrow\quad t = \frac{8}{7}$, corresponding to the point $\left(0, -400, \frac{600}{7}\right)$. Similarly, a sight line from the camera to the right endpoint $(650, -25, 210)$ of the clipped line has direction $\langle -350, -25, 210\rangle$ and parametric equations are $x = 1000 - 350t$, $y = -25t$, $z = 210t$. $\quad x = 0 \quad\Rightarrow\quad 1000 - 350t = 0 \quad\Rightarrow\quad t = \frac{20}{7}$, corresponding to the point $\left(0, -\frac{500}{7}, 600\right)$. Thus the projection of the clipped line is the line segment between the points $\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$.

3. From Equation 12.5.4, equations for the four sides of the screen are $\mathbf{r}_1(t) = (1 - t)\langle 0, -400, 0\rangle + t\,\langle 0, -400, 600\rangle$, $\mathbf{r}_2(t) = (1 - t)\langle 0, -400, 600\rangle + t\,\langle 0, 400, 600\rangle$, $\mathbf{r}_3(t) = (1 - t)\langle 0, 400, 0\rangle + t\,\langle 0, 400, 600\rangle$, and $\mathbf{r}_4(t) = (1 - t)\langle 0, -400, 0\rangle + t\,\langle 0, 400, 0\rangle$. The clipped line segment connects the points $(125, -350, 75)$ and $(650, -25, 210)$, so an equation for the segment is $\mathbf{r}_5(t) = (1 - t)\langle 125, -350, 75\rangle + t\,\langle 650, -25, 210\rangle$. The projection of the clipped segment connects the points



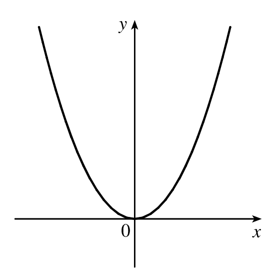
$\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_6(t) = (1-t)\left\langle 0, -400, \frac{600}{7}\right\rangle + t\left\langle 0, -\frac{500}{7}, 600\right\rangle$.

The sight line on the left connects the points $(1000, 0, 0)$ and $\left(0, -400, \frac{600}{7}\right)$, so an equation is $\mathbf{r}_7(t) = (1-t)\langle 1000, 0, 0\rangle + t\left\langle 0, -400, \frac{600}{7}\right\rangle$. The other sight line connects $(1000, 0, 0)$ to $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_8(t) = (1-t)\langle 1000, 0, 0\rangle + t\left\langle 0, -\frac{500}{7}, 600\right\rangle$.

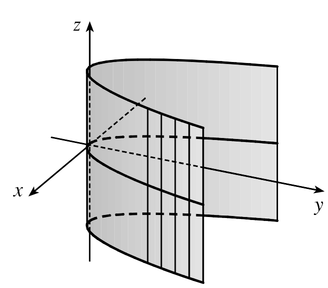
4. The vector from $(621, -147, 206)$ to $(563, 31, 242)$, $\mathbf{v}_1 = \langle -58, 178, 36\rangle$, lies in the plane of the rectangle, as does the vector from $(621, -147, 206)$ to $(657, -111, 86)$, $\mathbf{v}_2 = \langle 36, 36, -120\rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle -1888, -142, -708\rangle$ or $\langle 8, 2, 3\rangle$, and an equation of the plane is $8x + 2y + 3z = 5292$. The line L intersects this plane when $8(230 + 630t) + 2(-285 + 390t) + 3(102 + 162t) = 5292 \quad \Rightarrow \quad t = \frac{1858}{3153} \approx 0.589$. The corresponding point is approximately $(601.25, -55.18, 197.46)$. Starting at this point, a portion of the line is hidden behind the rectangle. The line becomes visible again at the left edge of the rectangle, specifically the edge between the points $(621, -147, 206)$ and $(657, -111, 86)$. (This is most easily determined by graphing the rectangle and the line.) A plane through these two points and the camera's location, $(1000, 0, 0)$, will clip the line at the point it becomes visible. Two vectors in this plane are $\mathbf{v}_1 = \langle -379, -147, 206\rangle$ and $\mathbf{v}_2 = \langle -343, -111, 86\rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 10224, -38064, -8352\rangle$ and an equation of the plane is $213x - 793y - 174z = 213{,}000$. L intersects this plane when $213(230 + 630t) - 793(-285 + 390t) - 174(102 + 162t) = 213{,}000 \quad \Rightarrow \quad t = \frac{44{,}247}{203{,}268} \approx 0.2177$. The corresponding point is approximately $(367.14, -200.11, 137.26)$. Thus the portion of L that should be removed is the segment between the points $(601.25, -55.18, 197.46)$ and $(367.14, -200.11, 137.26)$.

12.6 Cylinders and Quadric Surfaces

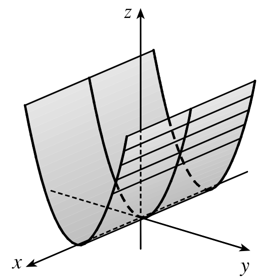
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



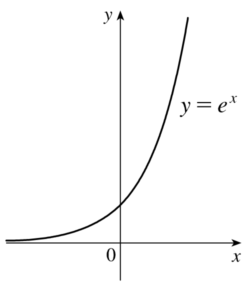
(b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z, so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



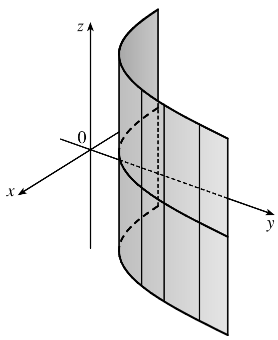
(c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



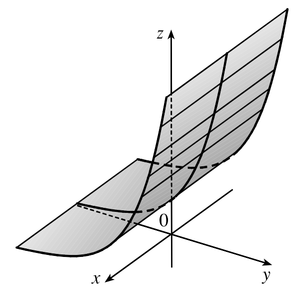
2. (a)



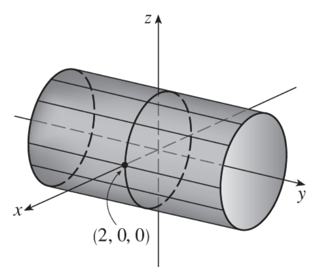
(b) Since the equation $y = e^x$ doesn't involve z, horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z-axis.



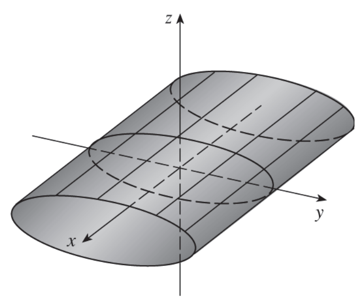
(c) The equation $z = e^y$ doesn't involve x, so vertical traces in $x = k$ (parallel to the yz-plane) are copies of the curve $z = e^y$. The rulings are parallel to the x-axis.



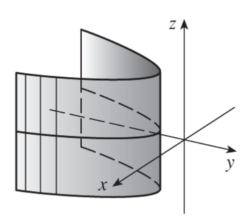
3. Since y is missing from the equation, the vertical traces $x^2 + z^2 = 4$, $y = k$ are copies of the same circle in the plane $y = k$. Thus, the surface $x^2 + z^2 = 4$ is a circular cylinder of radius 2 with rulings parallel to the y-axis.



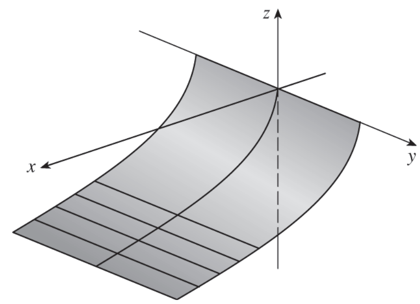
4. Since x is missing from the equation, the vertical traces $y^2 + 9z^2 = 9$, $x = k$ are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 9z^2 = 9$ is an elliptic cylinder with rulings parallel to the x-axis.



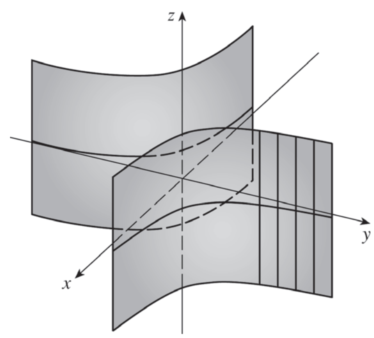
5. Since z is missing from the equation, the horizontal traces $x^2 + y + 1 = 0 \quad \Rightarrow \quad y = -x^2 - 1$, $z = k$ are copies of the same parabola in the plane $z = k$. Thus, the surface $x^2 + y + 1 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.



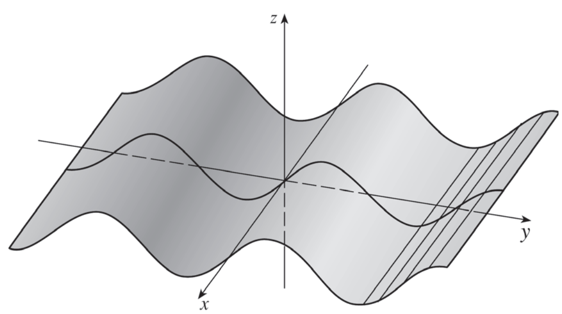
6. Since y is missing from the equation, the vertical traces $z = -\sqrt{x}$, $y = k$ are copies of the curve $z = -\sqrt{x}$ with rulings parallel to the y-axis.



7. Since z is missing, each horizontal trace $xy = 1$, $z = k$, is a copy of the same hyperbola in the plane $z = k$. Thus the surface $xy = 1$ is a hyperbolic cylinder with rulings parallel to the z-axis.



8. Since x is missing, each vertical trace $z = \sin y$, $x = k$, is a copy of a sine curve in the plane $x = k$. Thus the surface $z = \sin y$ is a cylindrical surface with rulings parallel to the x-axis.

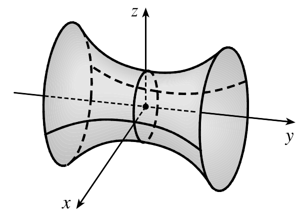


9. The trace in the xz-plane appears to be $z = \cos x$. The traces in the planes on the positive y-axis and negative y-axis are copies of the same graph. Therefore, an equation of the graph could be $z = \cos x$.

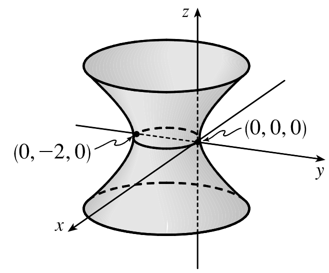
10. The trace in the yz-plane appears to be $z = y^3$. The traces in the planes on the positive x-axis and negative x-axis are copies of the same graph. Therefore, an equation of the graph could be $z = y^3$.

11. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

(b) If we change the equation $x^2 + y^2 - z^2 = 1$ to $x^2 - y^2 + z^2 = 1$, the shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

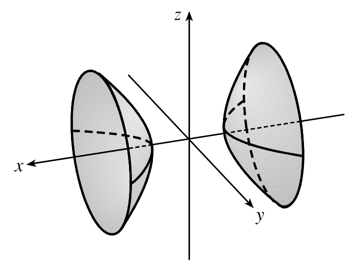


(c) Completing the square in y for $x^2 + y^2 + 2y - z^2 = 0$ gives $x^2 + (y+1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.

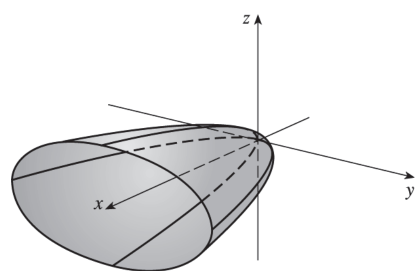


12. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

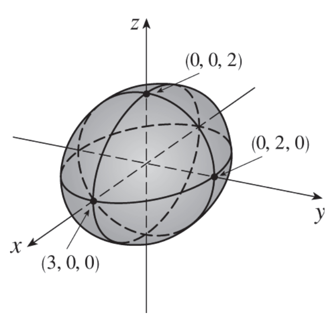
(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, the graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x-axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



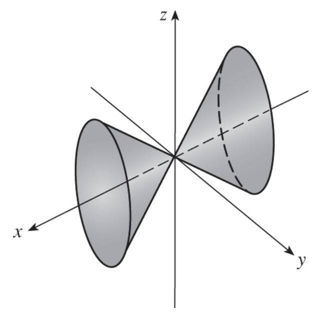
13. For $x = y^2 + 4z^2$, the traces in $x = k$ are $y^2 + 4z^2 = k$. When $k > 0$ we have a family of ellipses. When $k = 0$ we have just a point at the origin, and the trace is empty for $k < 0$. The traces in $y = k$ are $x = 4z^2 + k^2$, a family of parabolas opening in the positive x-direction. Similarly, the traces in $z = k$ are $x = y^2 + 4k^2$, a family of parabolas opening in the positive x-direction. We recognize the graph as an elliptic paraboloid with axis the x-axis and vertex the origin.



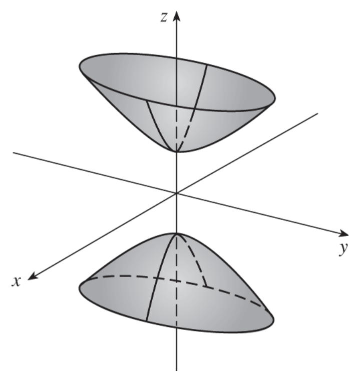
14. $4x^2 + 9y^2 + 9z^2 = 36$. The traces in $x = k$ are $9y^2 + 9z^2 = 36 - 4k^2 \Leftrightarrow y^2 + z^2 = 4 - \frac{4}{9}k^2$, a family of circles for $|k| < 3$. (The traces are a single point for $|k| = 3$ and are empty for $|k| > 3$.) The traces in $y = k$ are $4x^2 + 9z^2 = 36 - 9k^2$, a family of ellipses for $|k| < 2$. Similarly, the traces in $z = k$ are the ellipses $4x^2 + 9y^2 = 36 - 9k^2$, $|k| < 2$. The graph is an ellipsoid centered at the origin with intercepts $x = \pm 3$, $y = \pm 2$, $z = \pm 2$.



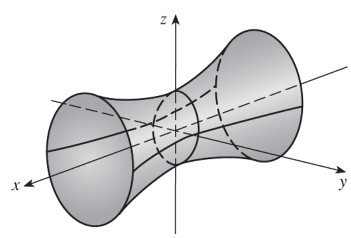
15. $x^2 = 4y^2 + z^2$. The traces in $x = k$ are the ellipses $4y^2 + z^2 = k^2$. The traces in $y = k$ are $x^2 - z^2 = 4k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. Similarly, the traces in $z = k$ are $x^2 - 4y^2 = k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. We recognize the graph as an elliptic cone with axis the x-axis and vertex the origin.



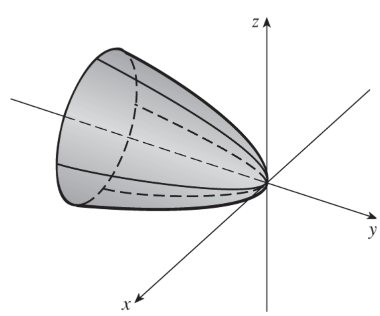
16. $z^2 - 4x^2 - y^2 = 4$. The traces in $x = k$ are the hyperbolas $z^2 - y^2 = 4 + 4k^2$, and the traces in $y = k$ are the hyperbolas $z^2 - 4x^2 = 4 + k^2$. The traces in $z = k$ are $4x^2 + y^2 = k^2 - 4$, a family of ellipses for $|k| > 2$. (The traces are a single point for $|k| = 2$ and are empty for $|k| < 2$.) The surface is a hyperboloid of two sheets with axis the z-axis.



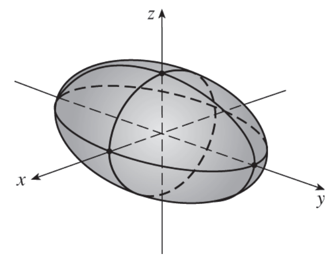
17. $9y^2 + 4z^2 = x^2 + 36$. The traces in $x = k$ are $9y^2 + 4z^2 = k^2 + 36$, a family of ellipses. The traces in $y = k$ are $4z^2 - x^2 = 9(4 - k^2)$, a family of hyperbolas for $|k| \neq 2$ and two intersecting lines when $|k| = 2$. (Note that the hyperbolas are oriented differently for $|k| < 2$ than for $|k| > 2$.) The traces in $z = k$ are $9y^2 - x^2 = 4(9 - k^2)$, a family of hyperbolas when $|k| \neq 3$ (oriented differently for $|k| < 3$ than for $|k| > 3$) and two intersecting lines when $|k| = 3$. We recognize the graph as a hyperboloid of one sheet with axis the x-axis.



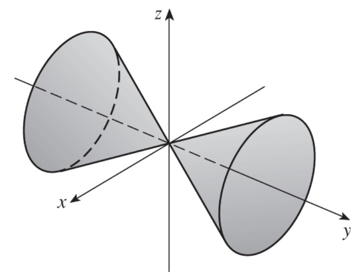
18. $3x^2 + y + 3z^2 = 0$. The traces in $x = k$ are the parabolas $y = -3z^2 - 3k^2$ which open to the left (in the negative y-direction). Traces in $y = k$ are $3x^2 + 3z^2 = -k \quad \Leftrightarrow \quad x^2 + z^2 = -\dfrac{k}{3}$, a family of circles for $k < 0$. (Traces are empty for $k > 0$ and a single point for $k = 0$.) Traces in $z = k$ are the parabolas $y = -3x^2 - 3k^2$ which open in the negative y-direction. The graph is a circular paraboloid with axis the y-axis, opening in the negative y-direction, and vertex the origin.



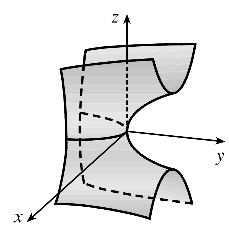
19. $\dfrac{x^2}{9}+\dfrac{y^2}{25}+\dfrac{z^2}{4}=1$. The traces in $x=k$ are $\dfrac{y^2}{25}+\dfrac{z^2}{4}=1-\dfrac{k^2}{9}$, a family of ellipses for $|k|<3$. (The traces are a single point for $|k|=3$ and are empty for $|k|>3$.) The traces in $y=k$ are the ellipses $\dfrac{x^2}{9}+\dfrac{z^2}{4}=1-\dfrac{k^2}{25}$, $|k|<5$, and the traces in $z=k$ are the ellipses $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1-\dfrac{k^2}{4}$, $|k|<2$. The surface is an ellipsoid centered at the origin with intercepts $x=\pm3$, $y=\pm5$, $z=\pm2$.



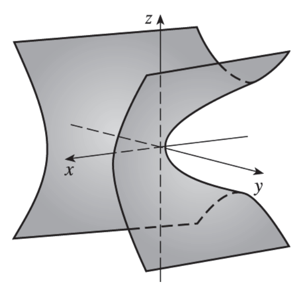
20. $3x^2-y^2+3z^2=0$. The traces in $x=k$ are $y^2-3z^2=3k^2$, a family of hyperbolas for $k\neq0$ and two intersecting lines if $k=0$. Traces in $y=k$ are the circles $3x^2+3z^2=k^2 \quad\Leftrightarrow\quad x^2+z^2=\frac{1}{3}k^2$. The traces in $z=k$ are $y^2-3x^2=3k^2$, hyperbolas for $k\neq0$ and two intersecting lines if $k=0$. We recognize the surface as a circular cone with axis the y-axis and vertex the origin.



21. $y=z^2-x^2$. The traces in $x=k$ are the parabolas $y=z^2-k^2$, opening in the positive y-direction. The traces in $y=k$ are $k=z^2-x^2$, two intersecting lines when $k=0$ and a family of hyperbolas for $k\neq0$ (note that the hyperbolas are oriented differently for $k>0$ than for $k<0$). The traces in $z=k$ are the parabolas $y=k^2-x^2$ which open in the negative y-direction. Thus the surface is a hyperbolic paraboloid centered at $(0,0,0)$.



22. $x=y^2-z^2$. The traces in $x=k$ are $y^2-z^2=k$, two intersecting lines when $k=0$ and a family of hyperbolas for $k\neq0$ (oriented differently for $k>0$ than for $k<0$). The traces in $y=k$ are the parabolas $x=-z^2+k^2$, opening in the negative x-direction, and the traces in $z=k$ are the parabolas $x=y^2-k^2$ which open in the positive x-direction. The graph is a hyperbolic paraboloid centered at $(0,0,0)$.



23. This is the equation of an ellipsoid: $x^2+4y^2+9z^2=x^2+\dfrac{y^2}{(1/2)^2}+\dfrac{z^2}{(1/3)^2}=1$, with x-intercepts ±1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

24. This is the equation of an ellipsoid: $9x^2+4y^2+z^2=\dfrac{x^2}{(1/3)^2}+\dfrac{y^2}{(1/2)^2}+z^2=1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ±1. So the major axis is the z-axis and the only possible graph is IV.

25. $x^2 - y^2 + z^2 = 1$ is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis. Hence, the correct graph is II.

26. $-x^2 + y^2 - z^2 = 1$ is the equation of a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis. Hence, the correct graph is III.

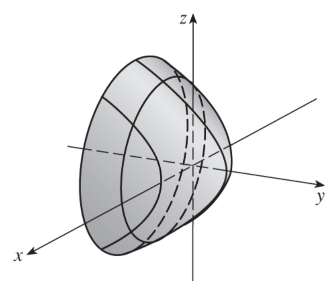
27. There are no real values of x and z that satisfy this equation, $y = 2x^2 + z^2$, for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

28. $y^2 = x^2 + 2z^2$ is the equation of a cone with axis the y-axis. Its graph is I.

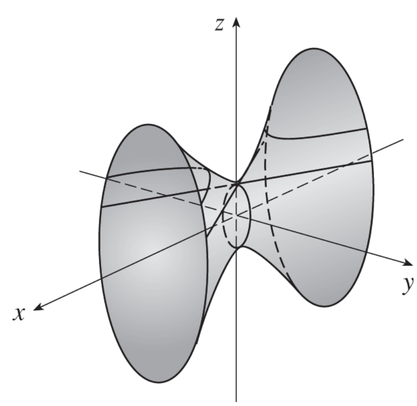
29. $x^2 + 2z^2 = 1$ is the equation of a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. Its graph is VIII.

30. $y = x^2 - z^2$ is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

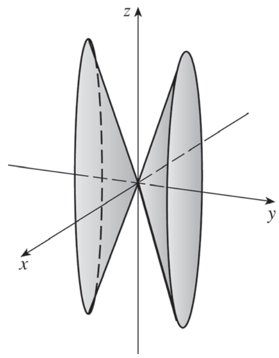
31. Vertical traces parallel to the xz-plane are circles centered at the origin whose radii increase as y decreases. (The trace in $y = 1$ is just a single point and the graph suggests that traces in $y = k$ are empty for $k > 1$.) The traces in vertical planes parallel to the yz-plane are parabolas opening to the left that shift to the left as $|x|$ increases. One surface that fits this description is a circular paraboloid, opening to the left, with vertex $(0, 1, 0)$.



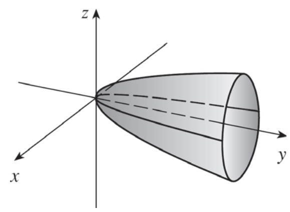
32. The vertical traces parallel to the yz-plane are ellipses that are smallest in the yz-plane and increase in size as $|x|$ increases. One surface that fits this description is a hyperboloid of one sheet with axis the x-axis. The horizontal traces in $z = k$ (hyperbolas and intersecting lines) also fit this surface, as shown in the figure.



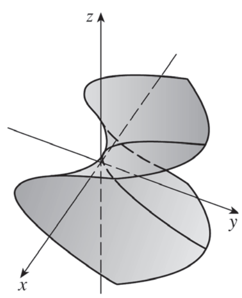
33. $y^2 = x^2 + \frac{1}{9}z^2$ or $y^2 = x^2 + \dfrac{z^2}{9}$ represents an elliptic cone with vertex $(0, 0, 0)$ and axis the y-axis.



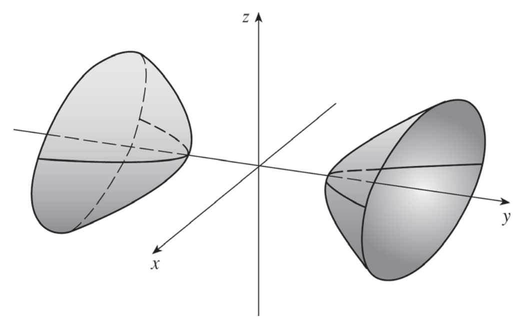
34. $4x^2 - y + 2z^2 = 0$ or $y = \dfrac{x^2}{1/4} + \dfrac{z^2}{1/2}$ or $\dfrac{y}{4} = x^2 + \dfrac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0, 0, 0)$ and axis the y-axis.



35. $x^2 + 2y - 2z^2 = 0$ or $2y = 2z^2 - x^2$ or $y = z^2 - \dfrac{x^2}{2}$ represents a hyperbolic paraboloid with center $(0, 0, 0)$.



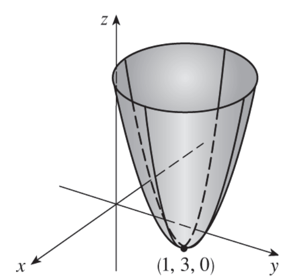
36. $y^2 = x^2 + 4z^2 + 4$ or $-x^2 + y^2 - 4z^2 = 4$ or $-\dfrac{x^2}{4} + \dfrac{y^2}{4} - z^2 = 1$ represents a hyperboloid of two sheets with axis the y-axis.



37. Completing squares in x and y gives

$\left(x^2 - 2x + 1\right) + \left(y^2 - 6y + 9\right) - z = 0 \quad \Leftrightarrow$

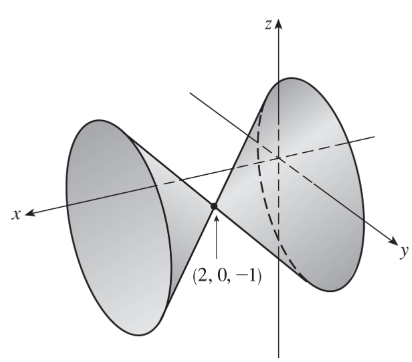
$(x-1)^2 + (y-3)^2 - z = 0$ or $z = (x-1)^2 + (y-3)^2$, a circular paraboloid opening upward with vertex $(1, 3, 0)$ and axis the vertical line $x = 1$, $y = 3$.



38. Completing squares in x and z gives

$\left(x^2 - 4x + 4\right) - y^2 - \left(z^2 + 2z + 1\right) + 3 = 0 + 4 - 1 \quad \Leftrightarrow$

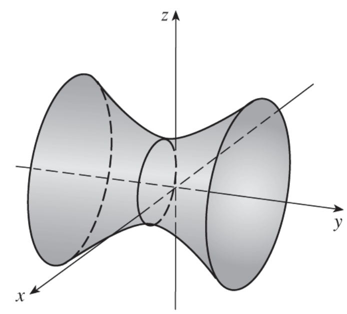
$(x-2)^2 - y^2 - (z+1)^2 = 0$ or $(x-2)^2 = y^2 + (z+1)^2$, a circular cone with vertex $(2, 0, -1)$ and axis the horizontal line $y = 0$, $z = -1$.



39. Completing squares in x and z gives

$$\left(x^2 - 4x + 4\right) - y^2 + \left(z^2 - 2z + 1\right) = 0 + 4 + 1 \quad \Leftrightarrow$$

$(x-2)^2 - y^2 + (z-1)^2 = 5$ or $\dfrac{(x-2)^2}{5} - \dfrac{y^2}{5} + \dfrac{(z-1)^2}{5} = 1$, a hyperboloid of one sheet with center $(2, 0, 1)$ and axis the horizontal line $x = 2$, $z = 1$.

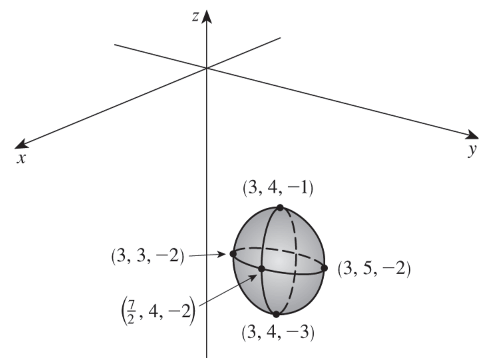


40. Completing squares in all three variables gives

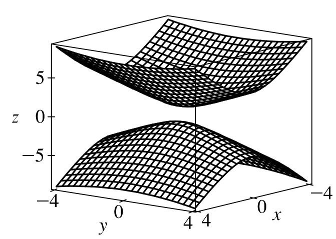
$$4(x^2 - 6x + 9) + (y^2 - 8y + 16) + (z^2 + 4z + 4) = -55 + 36 + 16 + 4 \quad \Leftrightarrow$$

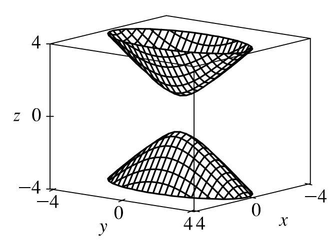
$4\,(x-3)^2 + (y-4)^2 + (z+2)^2 = 1$ or

$\dfrac{(x-3)^2}{1/4} + (y-4)^2 + (z+2)^2 = 1$, an ellipsoid with center $(3, 4, -2)$.



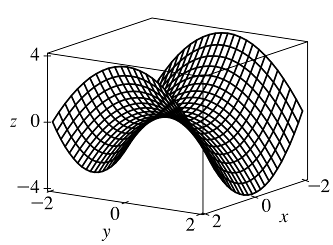
41. Solving the equation for z we get $z = \pm\sqrt{1 + 4x^2 + y^2}$, so we plot separately $z = \sqrt{1 + 4x^2 + y^2}$ and $z = -\sqrt{1 + 4x^2 + y^2}$.



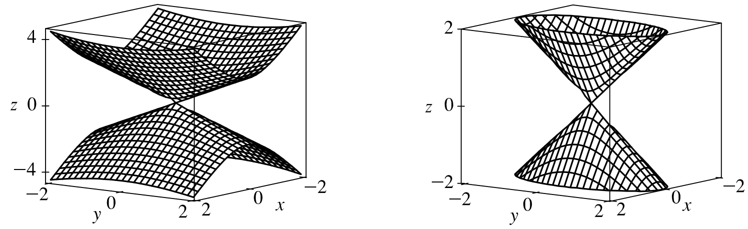


To restrict the z-range as in the second graph, we can use the option `view=-4..4` in Maple's `plot3d` command, or `PlotRange->{-4,4}` in Mathematica's `Plot3D` command.

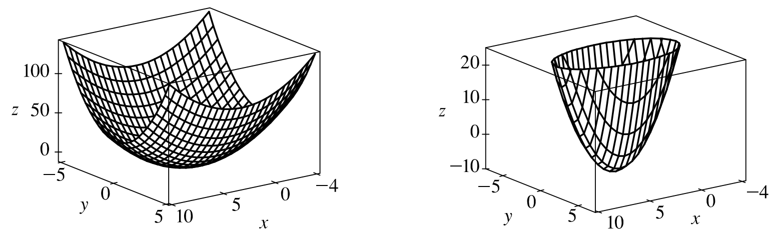
42. We plot the surface $z = x^2 - y^2$.



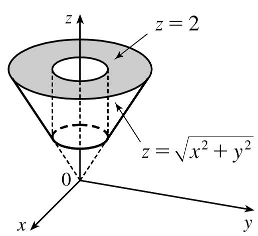
43. Solving the equation for z we get $z = \pm\sqrt{4x^2 + y^2}$, so we plot separately $z = \sqrt{4x^2 + y^2}$ and $z = -\sqrt{4x^2 + y^2}$.



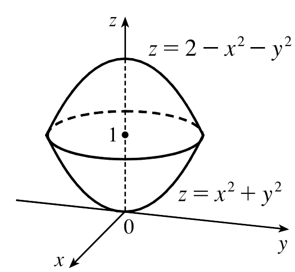
44. We plot the surface $z = x^2 - 6x + 4y^2$.



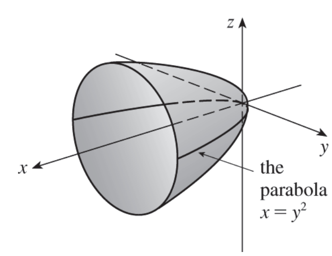
45.



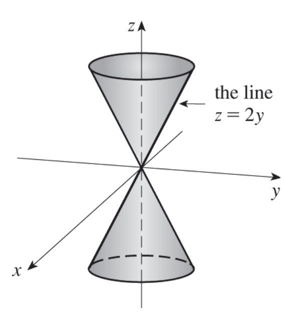
46.



47. The curve $y = \sqrt{x}$ is equivalent to $x = y^2$, $y \geq 0$. Rotating the curve about the x-axis creates a circular paraboloid with vertex at the origin, axis the x-axis, opening in the positive x-direction. The trace in the xy-plane is $x = y^2$, $z = 0$, and the trace in the xz-plane is a parabola of the same shape: $x = z^2$, $y = 0$. An equation for the surface is $x = y^2 + z^2$.



48. Rotating the line $z = 2y$ about the z-axis creates a (right) circular cone with vertex at the origin and axis the z-axis. Traces in $z = k$ ($k \neq 0$) are circles with center $(0, 0, k)$ and radius $y = z/2 = k/2$, so an equation for the trace is $x^2 + y^2 = (k/2)^2$, $z = k$. Thus an equation for the surface is $x^2 + y^2 = (z/2)^2$ or $4x^2 + 4y^2 = z^2$.



49. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x - 1| / \sqrt{1^2} = |x - 1|$ (by Equation 12.5.9). So $|x - 1| = \sqrt{(x+1)^2 + y^2 + z^2} \Leftrightarrow (x-1)^2 = (x+1)^2 + y^2 + z^2 \Leftrightarrow$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \Leftrightarrow -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative x-direction.

50. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plane ($x = 0$) is $|x|/1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \Leftrightarrow y^2 + z^2 = 4x^2 \Leftrightarrow x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

51. (a) An equation for an ellipsoid centered at the origin with intercepts $x = \pm a$, $y = \pm b$, and $z = \pm c$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Here the poles of the model intersect the z-axis at $z = \pm 6356.523$ and the equator intersects the x- and y-axes at $x = \pm 6378.137$, $y = \pm 6378.137$, so an equation is

$$\frac{x^2}{(6378.137)^2} + \frac{y^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} = 1$$

(b) Traces in $z = k$ are the circles $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} = 1 - \dfrac{k^2}{(6356.523)^2} \Leftrightarrow$

$x^2 + y^2 = (6378.137)^2 - \left(\dfrac{6378.137}{6356.523}\right)^2 k^2$.

(c) To identify the traces in $y = mx$ we substitute $y = mx$ into the equation of the ellipsoid:

$$\begin{aligned} \frac{x^2}{(6378.137)^2} + \frac{(mx)^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} &= 1 \\ \frac{(1+m^2)x^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} &= 1 \\ \frac{x^2}{(6378.137)^2/(1+m^2)} + \frac{z^2}{(6356.523)^2} &= 1 \end{aligned}$$

As expected, this is a family of ellipses.

52. If we position the hyperboloid on coordinate axes so that it is centered at the origin with axis the z-axis then its equation is given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. Horizontal traces in $z = k$ are $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{k^2}{c^2}$, a family of ellipses, but we know that the traces are circles so we must have $a = b$. The trace in $z = 0$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1 \Leftrightarrow x^2 + y^2 = a^2$ and since the minimum radius of 100 m occurs there, we must have $a = 100$. The base of the tower is the trace in $z = -500$ given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1 + \dfrac{(-500)^2}{c^2}$ but $a = 100$ so the trace is $x^2 + y^2 = 100^2 + 50{,}000^2 \dfrac{1}{c^2}$. We know the base is a circle of

radius 140, so we must have $100^2 + 50{,}000^2 \dfrac{1}{c^2} = 140^2 \quad \Rightarrow \quad c^2 = \dfrac{50{,}000^2}{140^2 - 100^2} = \dfrac{781{,}250}{3}$ and an equation for the tower is $\dfrac{x^2}{100^2} + \dfrac{y^2}{100^2} - \dfrac{z^2}{(781{,}250)/3} = 1$ or $\dfrac{x^2}{10{,}000} + \dfrac{y^2}{10{,}000} - \dfrac{3z^2}{781{,}250} = 1, \ -500 \le z \le 500.$

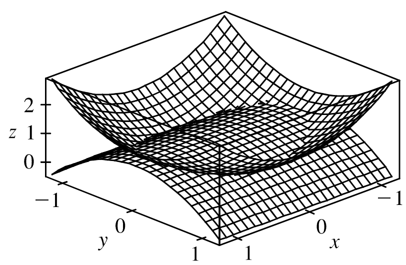
53. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \quad \Rightarrow$

$c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \quad \Rightarrow \quad c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \quad \Rightarrow$

$c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \quad \Rightarrow \quad c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

54. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

55.



The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 = 1 \quad \Leftrightarrow$

$x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$. This is an equation of an ellipse.

12 Review

TRUE-FALSE QUIZ

1. This is false, as the dot product of two vectors is a scalar, not a vector.

2. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = -\mathbf{i}$ then $|\mathbf{u} + \mathbf{v}| = |\mathbf{0}| = 0$ but $|\mathbf{u}| + |\mathbf{v}| = 1 + 1 = 2$.

3. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$ then $|\mathbf{u} \cdot \mathbf{v}| = |0| = 0$ but $|\mathbf{u}|\,|\mathbf{v}| = 1 \cdot 1 = 1$. In fact, by Theorem 12.3.3, $|\mathbf{u} \cdot \mathbf{v}| = \big||\mathbf{u}|\,|\mathbf{v}| \cos\theta\big|$.

4. False. For example, $|\mathbf{i} \times \mathbf{i}| = |\mathbf{0}| = 0$ (see Example 12.4.2) but $|\mathbf{i}|\,|\mathbf{i}| = 1 \cdot 1 = 1$. In fact, by Theorem 12.4.9, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$.

5. True, by Theorem 12.3.2, property 2.

6. False. Property 1 of Theorem 12.4.11 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

7. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 12.4.9, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{v}|\,|\mathbf{u}|\sin\theta = |\mathbf{v}\times\mathbf{u}|$.
(Or, by Theorem 12.4.11, $|\mathbf{u}\times\mathbf{v}| = |-\mathbf{v}\times\mathbf{u}| = |-1|\,|\mathbf{v}\times\mathbf{u}| = |\mathbf{v}\times\mathbf{u}|$.)

8. This is true by Theorem 12.3.2, property 4.

9. Theorem 12.4.11, property 2 tells us that this is true.

10. This is true by Theorem 12.4.11, property 4.

11. This is true by Theorem 12.4.11, property 5.

12. In general, this assertion is false; a counterexample is $\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) \neq (\mathbf{i}\times\mathbf{i})\times\mathbf{j}$. (See the paragraph preceding Theorem 12.4.11.)

13. This is true because $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 12.4.8), and the dot product of two orthogonal vectors is 0.

14.
$$\begin{aligned}(\mathbf{u}+\mathbf{v})\times\mathbf{v} &= \mathbf{u}\times\mathbf{v}+\mathbf{v}\times\mathbf{v} && \text{[by Theorem 12.4.11, property 4]}\\ &= \mathbf{u}\times\mathbf{v}+\mathbf{0} && \text{[by Example 12.4.2]}\\ &= \mathbf{u}\times\mathbf{v}, \text{ so this is true.}\end{aligned}$$

15. This is false. A normal vector to the plane is $\mathbf{n} = \langle 6,-2,4\rangle$. Because $\langle 3,-1,2\rangle = \frac{1}{2}\mathbf{n}$, the vector is parallel to $\mathbf{n}$ and hence perpendicular to the plane.

16. This is false, because according to Equation 12.5.8, $ax+by+cz+d=0$ is the general equation of a plane.

17. This is false. In $\mathbb{R}^2$, $x^2+y^2=1$ represents a circle, but $\{(x,y,z)\mid x^2+y^2=1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

18. This is false. In $\mathbb{R}^3$ the graph of $y=x^2$ is a parabolic cylinder (see Example 12.6.1). A paraboloid has an equation such as $z=x^2+y^2$.

19. False. For example, $\mathbf{i}\cdot\mathbf{j}=\mathbf{0}$ but $\mathbf{i}\neq\mathbf{0}$ and $\mathbf{j}\neq\mathbf{0}$.

20. This is false. By Corollary 12.4.10, $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ for any nonzero parallel vectors $\mathbf{u}$, $\mathbf{v}$. For instance, $\mathbf{i}\times\mathbf{i}=\mathbf{0}$.

21. This is true. If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then by (7) in Section 12.3, $\mathbf{u}\cdot\mathbf{v}=0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. But $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel (see Corollary 12.4.10). Two nonzero vectors can't be both parallel and orthogonal, so at least one of $\mathbf{u}$, $\mathbf{v}$ must be $\mathbf{0}$.

22. This is true. We know $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\;|\mathbf{v}|\;\cos\theta$ where $|\mathbf{u}|\geq 0$, $|\mathbf{v}|\geq 0$, and $|\cos\theta|\leq 1$, so $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\;|\mathbf{v}|\;|\cos\theta| \leq |\mathbf{u}|\;|\mathbf{v}|$.

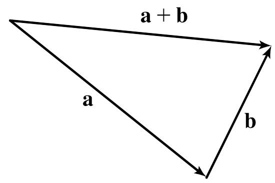
EXERCISES

1. (a) The radius of the sphere is the distance between the points $(-1, 2, 1)$ and $(6, -2, 3)$, namely, $\sqrt{[6-(-1)]^2 + (-2-2)^2 + (3-1)^2} = \sqrt{69}$. By the formula for an equation of a sphere (following Example 12.1.4), an equation of the sphere with center $(-1, 2, 1)$ and radius $\sqrt{69}$ is $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$.

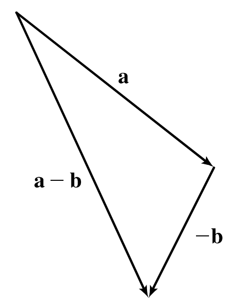
(b) The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $(y-2)^2 + (z-1)^2 = 68$, $x = 0$ which represents a circle in the yz-plane with center $(0, 2, 1)$ and radius $\sqrt{68}$.

(c) Completing squares gives $(x-4)^2 + (y+1)^2 + (z+3)^2 = -1 + 16 + 1 + 9 = 25$. Thus the sphere is centered at $(4, -1, -3)$ and has radius 5.

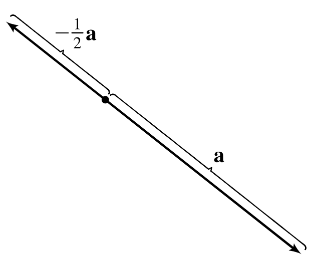
2. (a)



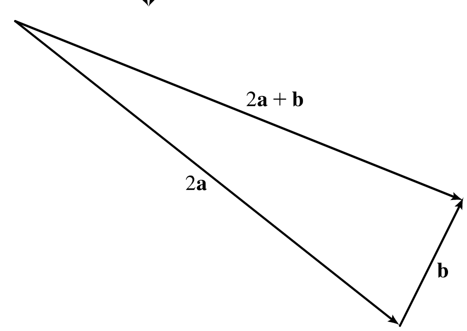
(b)



(c)



(d)



3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $\quad |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$.
By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

4. (a) $2\,\mathbf{a} + 3\,\mathbf{b} = 2\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} + 9\,\mathbf{i} - 6\,\mathbf{j} + 3\,\mathbf{k} = 11\,\mathbf{i} - 4\,\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9+4+1} = \sqrt{14}$

(c) $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\,\mathbf{i} - 7\,\mathbf{j} - 5\,\mathbf{k}$

(e) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}$, $\quad |\mathbf{b} \times \mathbf{c}| = 3\sqrt{9+25+1} = 3\sqrt{35}$

(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix}$$

$$= (3 + 30)\,\mathbf{i} - (3 + 18)\,\mathbf{j} + (15 - 9)\,\mathbf{k} = 33\,\mathbf{i} - 21\,\mathbf{j} + 6\,\mathbf{k}$$

(i) The scalar projection is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/\,|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\dfrac{1}{\sqrt{6}}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|}\right) = -\frac{1}{6}(\mathbf{i} + \mathbf{j} - 2\,\mathbf{k})$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\,\sqrt{14}} = \dfrac{-1}{2\,\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{-1}{2\,\sqrt{21}}\right) \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3, 2, x\rangle\cdot\langle 2x, 4, x\rangle = 0 \quad\Leftrightarrow\quad (3)(2x) + (2)(4) + (x)(x) = 0 \quad\Leftrightarrow$

$x^2 + 6x + 8 = 0 \quad\Leftrightarrow\quad (x + 2)(x + 4) = 0 \quad\Leftrightarrow\quad x = -2$ or $x = -4$.

6. We know that the cross product of two vectors is orthogonal to both given vectors. So we calculate

$(\mathbf{j} + 2\,\mathbf{k})\times(\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) = [3 - (-4)]\,\mathbf{i} - (0 - 2)\,\mathbf{j} + (0 - 1)\,\mathbf{k} = 7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{\sqrt{7^2 + 2^2 + (-1)^2}} = \pm\dfrac{1}{3\,\sqrt{6}}\,(7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$,

that is, $\frac{7}{3\sqrt{6}}\,\mathbf{i} + \frac{2}{3\sqrt{6}}\,\mathbf{j} - \frac{1}{3\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\,\mathbf{i} - \frac{2}{3\sqrt{6}}\,\mathbf{j} + \frac{1}{3\sqrt{6}}\,\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8. $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a})$

[by Property 6 of the cross product]

$$= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

$$= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0, 0, 0)$ to $(1, 1, 1)$ and $(1, 0, 0)$ to $(0, 1, 1)$ are $\langle 1, 1, 1\rangle$ and $\langle -1, 1, 1\rangle$. Let θ be the angle between these two vectors. $\langle 1, 1, 1\rangle\cdot\langle -1, 1, 1\rangle = -1 + 1 + 1 = 1 = |\langle 1, 1, 1\rangle|\,|\langle -1, 1, 1\rangle|\cos\theta = 3\cos\theta \quad\Rightarrow\quad \cos\theta = \frac{1}{3} \quad\Rightarrow$

$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1, 3, -1 \rangle$, $\overrightarrow{AC} = \langle -2, 1, 3 \rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1 \rangle$. By Equation 12.4.13,

$$\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD} \right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$

The volume is $\left| \overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD} \right) \right| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \frac{1}{2} \sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = (5 - 1)\,\mathbf{i} + (3 - 0)\,\mathbf{j} + (8 - 2)\,\mathbf{k} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$.

$W = \mathbf{F} \cdot \mathbf{D} = (3\,\mathbf{i} + 5\,\mathbf{j} + 10\,\mathbf{k}) \cdot (4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}) = (3)(4) + (5)(3) + (10)(6) = 12 + 15 + 60 = 87$ J

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives

$F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ **(1)**, and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \quad \Rightarrow \quad F_1 = F_2 \dfrac{\sin 30^\circ}{\sin 20^\circ}$ **(2)**. Substituting **(2)**

into **(1)** gives $F_2 (\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \quad \Rightarrow \quad F_2 \approx 114$ N. Substituting this into **(2)** gives $F_1 \approx 166$ N.

14. $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50) \sin(90^\circ - 30^\circ) \approx 17.3$ N·m.

15. The line has direction $\mathbf{v} = \langle -3, 2, 3 \rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are

$x = 4 - 3t,\ \ y = -1 + 2t,\ \ z = 2 + 3t.$

16. The line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$, or $\dfrac{x - 4}{3} = \dfrac{y}{2} = \dfrac{z + 2}{1}$, has direction vector $\mathbf{v} = \langle 3, 2, 1 \rangle$ (or a nonzero scalar multiple).

So parametric equations for the line through $(1, 0, -1)$ are $x = 1 + 3t,\ \ y = 2t,\ \ z = -1 + t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5 \rangle$, and parametric equations for the line are

$x = -2 + 2t,\ \ y = 2 - t,\ \ z = 4 + 5t.$

18. Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3 \rangle$ and an equation of the plane is $1(x - 2) + 4(y - 1) - 3(z - 0) = 0$ or $x + 4y - 3z = 6$.

19. Here the vectors $\mathbf{a} = \langle 4 - 3, 0 - (-1), 2 - 1 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle 6 - 3, 3 - (-1), 1 - 1 \rangle = \langle 3, 4, 0 \rangle$ lie in the plane,

so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -4, 3, 1 \rangle$ is a normal vector to the plane and an equation of the plane is

$-4(x - 3) + 3(y - (-1)) + 1(z - 1) = 0$ or $-4x + 3y + z = -14$.

20. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0 - 1, 3 - 2, 1 - (-2) \rangle = \langle -1, 1, 3 \rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1 \rangle$. Thus an equation of the plane is $-6(x - 1) - 9(y - 2) + (z + 2) = 0$ or $6x + 9y - z = 26$.

21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2 - t) - (1 + 3t) + 4t = 2 \quad \Rightarrow$ $-t + 3 = 2 \quad \Rightarrow \quad t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

22. Use the formula proven in Exercise 12.4.45(a). In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle$. Hence $d = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \dfrac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1 + 1 + 4}} = \dfrac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \sqrt{\dfrac{27}{6}} = \dfrac{3}{\sqrt{2}}$.

23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

(b) $\cos\theta = \dfrac{\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle}{\sqrt{3}\,\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ [or we can say $\approx 58^\circ$].

25. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, the normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = 3\,\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

26. (a) The vectors $\overrightarrow{AB} = \langle -1 - 2, -1 - 1, 10 - 1 \rangle = \langle -3, -2, 9 \rangle$ and $\overrightarrow{AC} = \langle 1 - 2, 3 - 1, -4 - 1 \rangle = \langle -1, 2, -5 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -3, -2, 9 \rangle \times \langle -1, 2, -5 \rangle = \langle -8, -24, -8 \rangle$ or equivalently $\langle 1, 3, 1 \rangle$ is a normal vector to the plane. The point $A(2, 1, 1)$ lies on the plane so an equation of the plane is $1(x - 2) + 3(y - 1) + 1(z - 1) = 0$ or $x + 3y + z = 6$.

(b) The line is perpendicular to the plane so it is parallel to a normal vector for the plane, namely $\langle 1, 3, 1 \rangle$. If the line passes through $B(-1, -1, 10)$ then symmetric equations are $\dfrac{x - (-1)}{1} = \dfrac{y - (-1)}{3} = \dfrac{z - 10}{1}$ or $x + 1 = \dfrac{y + 1}{3} = z - 10$.

(c) Normal vectors for the two planes are $\mathbf{n}_1 = \langle 1, 3, 1 \rangle$ and $\mathbf{n}_2 = \langle 2, -4, -3 \rangle$. The angle θ between the planes is given by

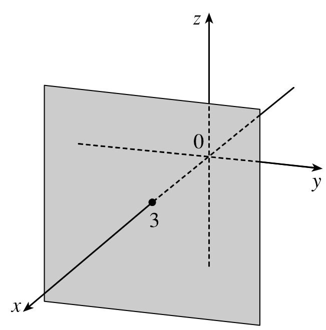
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\ |\mathbf{n}_2|} = \frac{\langle 1, 3, 1 \rangle \cdot \langle 2, -4, -3 \rangle}{\sqrt{1^2 + 3^2 + 1^2}\,\sqrt{2^2 + (-4)^2 + (-3)^2}} = \frac{2 - 12 - 3}{\sqrt{11}\,\sqrt{29}} = -\frac{13}{\sqrt{319}}$$

Thus $\theta = \cos^{-1}\left(-\dfrac{13}{\sqrt{319}}\right) \approx 137^\circ$ or $180^\circ - 137^\circ = 43^\circ$.

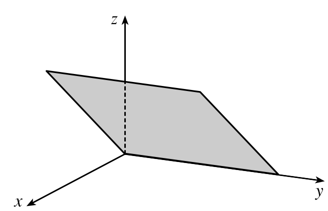
(d) From part (c), the point $(2, 0, 4)$ lies on the second plane, but notice that the point also satisfies the equation of the first plane, so the point lies on the line of intersection of the planes. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 3, 1\rangle \times \langle 2, -4, -3\rangle = \langle -5, 5, -10\rangle$ or equivalently we can take $\mathbf{v} = \langle 1, -1, 2\rangle$. Parametric equations for the line are $x = 2 + t$, $y = -t$, $z = 4 + 2t$.

27. By Exercise 12.5.75, $D = \dfrac{|-2 - (-24)|}{\sqrt{3^2 + 1^2 + (-4)^2}} = \dfrac{22}{\sqrt{26}}$.

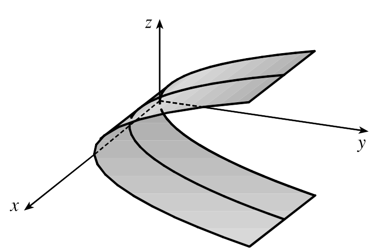
28. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



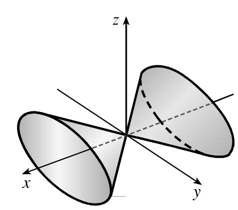
29. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



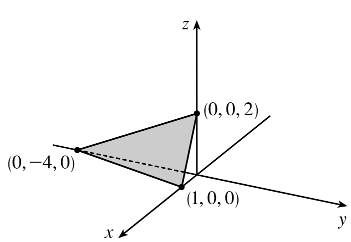
30. The equation $y = z^2$ represents a parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



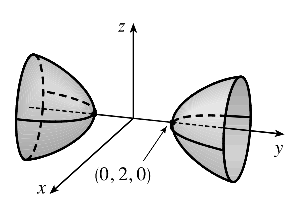
31. The equation $x^2 = y^2 + 4z^2$ represents a (right elliptical) cone with vertex at the origin and axis the x-axis.



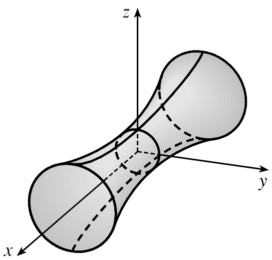
32. $4x - y + 2z = 4$ is a plane with intercepts $(1, 0, 0)$, $(0, -4, 0)$, and $(0, 0, 2)$.



33. An equivalent equation is $-x^2 + \dfrac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



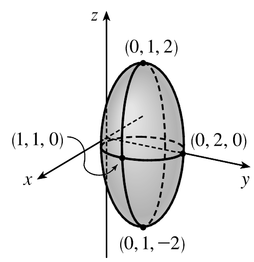
34. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



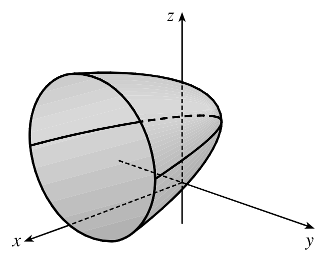
35. Completing the square in y gives

$4x^2 + 4(y-1)^2 + z^2 = 4$ or $x^2 + (y-1)^2 + \dfrac{z^2}{4} = 1$,

an ellipsoid centered at $(0, 1, 0)$.



36. Completing the square for $x = y^2 + z^2 - 2y - 4z + 5$ in y and z gives $x = (y-1)^2 + (z-2)^2$, which is a circular paraboloid with vertex $(0, 1, 2)$ and axis the horizontal line $y = 1$, $z = 2$.



37. $4x^2 + y^2 = 16 \quad\Leftrightarrow\quad \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. The equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1 \quad\Leftrightarrow$

$4x^2 + y^2 + z^2 = 16$.

38. The distance from a point $P(x, y, z)$ to the plane $y = 1$ is $|y - 1|$, so the given condition becomes

$|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \quad\Rightarrow\quad |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \quad\Rightarrow$

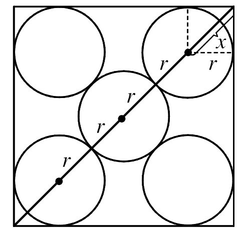
$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \quad\Leftrightarrow\quad -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \quad\Leftrightarrow$

$\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \quad\Rightarrow\quad \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1$.

This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.

PROBLEMS PLUS

1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.



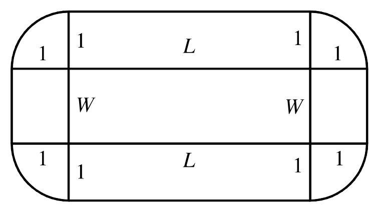
The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore $x = \sqrt{2}\,r \quad\Rightarrow\quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = \left(4 + 2\sqrt{2}\right)r \quad\Rightarrow\quad r = \frac{\sqrt{2}}{4+2\sqrt{2}}$.

Let's use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore

$x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r \quad\Rightarrow\quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = \left(4 + 2\sqrt{3}\right)r$. Thus $r = \dfrac{\sqrt{3}}{4 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{2}$.

The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W. Since S contains B, it has area



$$
\begin{aligned}
A(S) = LW + {} & \text{the area of two } L \times 1 \text{ rectangles} \\
& + \text{the area of two } 1 \times W \text{ rectangles} \\
& + \text{the area of four quarter-circles of radius } 1
\end{aligned}
$$

as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$
\begin{aligned}
& LWH + 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H) \\
& \quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } W \\
& \quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } L \\
& \quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } H \\
& \quad + \text{the volume of 8 eighths of a sphere of radius 1}
\end{aligned}
$$

So

$$
\begin{aligned}
V(S) &= LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \tfrac{4}{3}\pi \cdot 1^3 \\
&= LWH + 2(LW + WH + LH) + \pi(L + W + H) + \tfrac{4}{3}\pi.
\end{aligned}
$$

3. (a) We find the line of intersection L as in Example 12.5.7(b). Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2+1, -c^2-1 \rangle$$

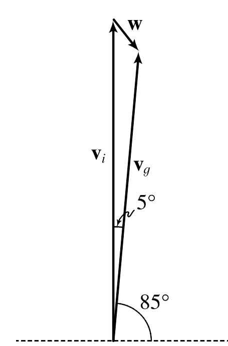
So symmetric equations of L can be written as $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t-c)(-2c)}{c^2+1}$, $y - c = \dfrac{(t-c)(c^2-1)}{c^2+1} \quad\Rightarrow\quad x = \dfrac{-2ct + (c^2-1)}{c^2+1}$, $y = \dfrac{(c^2-1)t + 2c}{c^2+1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2+1}$.

(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2+1)$, so $V = \int_0^1 A(z)\,dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\,\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus $\mathbf{v}_g = (160\cos 85^\circ)\,\mathbf{i} + (160 \sin 85^\circ)\,\mathbf{j} \approx 13.9\,\mathbf{i} + 159.4\,\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\,\mathbf{i} - 20.6\,\mathbf{j}$. Thus, the wind velocity is about $13.9\,\mathbf{i} - 20.6\,\mathbf{j}$, and the wind speed is $|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9$ km/h.



(b) Let $\mathbf{v}$ be the velocity the pilot should have taken. The pilot would need to head a little west of north to compensate for the wind, so let θ be the angle $\mathbf{v}$ makes with the north direction. Then we can write $\mathbf{v} = \langle 180\cos(\theta + 90^\circ), 180\sin(\theta + 90^\circ)\rangle$. With the effect of the wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be due north. Equating horizontal components of the vectors, we must have

$180\cos(\theta + 90^\circ) + 160\cos 85^\circ = 0 \quad\Rightarrow\quad \cos(\theta + 90^\circ) = -\frac{160}{180}\cos 85^\circ \approx -0.0775$, so

$\theta \approx \cos^{-1}(-0.0775) - 90^\circ \approx 4.4^\circ$. Thus the pilot should have headed about 4.4° west of north.

5. $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1}\mathbf{v}_2 = \dfrac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|^2}\,\mathbf{v}_1 = \dfrac{5}{2^2}\,\mathbf{v}_1$ so $|\mathbf{v}_3| = \dfrac{5}{2^2}\,|\mathbf{v}_1| = \dfrac{5}{2}$,

$$\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2}\mathbf{v}_3 = \frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{|\mathbf{v}_2|^2}\,\mathbf{v}_2 = \frac{\mathbf{v}_2 \cdot \frac{5}{2^2}\,\mathbf{v}_1}{|\mathbf{v}_2|^2}\,\mathbf{v}_2 = \frac{5}{2^2 \cdot 3^2}\,(\mathbf{v}_1 \cdot \mathbf{v}_2)\,\mathbf{v}_2 = \frac{5^2}{2^2 \cdot 3^2}\,\mathbf{v}_2 \quad \Rightarrow \quad |\mathbf{v}_4| = \frac{5^2}{2^2 \cdot 3^2}\,|\mathbf{v}_2| = \frac{5^2}{2^2 \cdot 3},$$

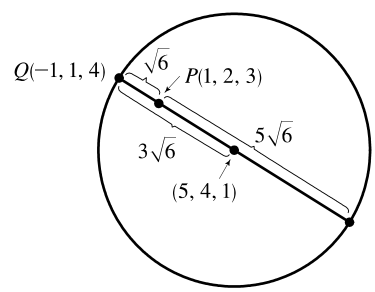
$$\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3}\mathbf{v}_4 = \frac{\mathbf{v}_3 \cdot \mathbf{v}_4}{|\mathbf{v}_3|^2}\,\mathbf{v}_3 = \frac{\frac{5}{2^2}\,\mathbf{v}_1 \cdot \frac{5^2}{2^2\,3^2}\,\mathbf{v}_2}{\left(\frac{5}{2}\right)^2}\left(\frac{5}{2^2}\,\mathbf{v}_1\right) = \frac{5^2}{2^4 \cdot 3^2}\,(\mathbf{v}_1 \cdot \mathbf{v}_2)\;\mathbf{v}_1 = \frac{5^3}{2^4 \cdot 3^2}\,\mathbf{v}_1 \quad \Rightarrow$$

$|\mathbf{v}_5| = \dfrac{5^3}{2^4 \cdot 3^2}\,|\mathbf{v}_1| = \dfrac{5^3}{2^3 \cdot 3^2}$. Similarly, $|\mathbf{v}_6| = \dfrac{5^4}{2^4 \cdot 3^3}$, $|\mathbf{v}_7| = \dfrac{5^5}{2^5 \cdot 3^4}$, and in general, $|\mathbf{v}_n| = \dfrac{5^{n-2}}{2^{n-2} \cdot 3^{n-3}} = 3\left(\frac{5}{6}\right)^{n-2}$.

Thus

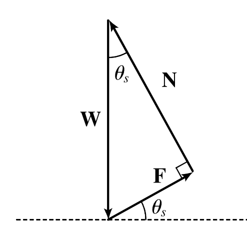
$$\begin{aligned}\sum_{n=1}^{\infty} |\mathbf{v}_n| &= |\mathbf{v}_1| + |\mathbf{v}_2| + \sum_{n=3}^{\infty} 3\left(\tfrac{5}{6}\right)^{n-2} = 2 + 3 + \sum_{n=1}^{\infty} 3\left(\tfrac{5}{6}\right)^n \\ &= 5 + \sum_{n=1}^{\infty} \tfrac{5}{2}\left(\tfrac{5}{6}\right)^{n-1} = 5 + \frac{\frac{5}{2}}{1 - \frac{5}{6}} \quad \text{[sum of a geometric series]} \quad = 5 + 15 = 20\end{aligned}$$

6. Completing squares in the inequality $x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$ gives $(x-1)^2 + (y-2)^2 + (z-3)^2 < 150$ which describes the set of all points (x, y, z) whose distance from the point $P\,(1, 2, 3)$ is less than $\sqrt{150} = 5\sqrt{6}$. The distance from P to $Q\,(-1, 1, 4)$ is $\sqrt{4 + 1 + 1} = \sqrt{6}$, so the largest possible sphere that passes through Q and satisfies the stated conditions extends $5\sqrt{6}$ units in the opposite direction, giving a diameter of $6\sqrt{6}$. (See the figure.)

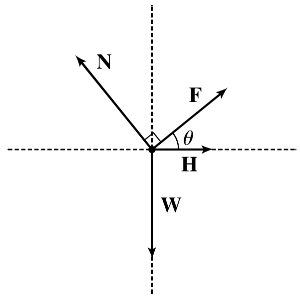


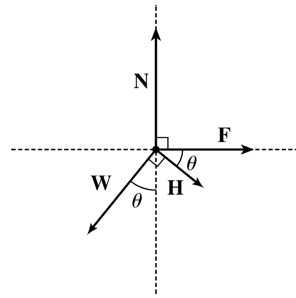
Thus the radius of the desired sphere is $3\sqrt{6}$, and its center is $3\sqrt{6}$ units from Q in the direction of P. A unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{6}}\,\langle 2, 1, -1\rangle$, so starting at $Q(-1, 1, 4)$ and following the vector $3\sqrt{6}\,\mathbf{u} = \langle 6, 3, -3\rangle$ we arrive at the center of the sphere, $(5, 4, 1)$. An equation of the sphere then is $(x-5)^2 + (y-4)^2 + (z-1)^2 = \left(3\sqrt{6}\right)^2$ or $(x-5)^2 + (y-4)^2 + (z-1)^2 = 54$.

7. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have $\tan(\theta_s) = \dfrac{|\mathbf{F}|}{|\mathbf{N}|} = \dfrac{\mu_s n}{n} = \mu_s$.



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis. (See the following figure.)





$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j} \qquad \mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad\Rightarrow\quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \tag{1}$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad\Rightarrow\quad h_{\min}\sin\theta + mg\cos\theta = n \tag{2}$$

(c) Since **(2)** is solved for n, we substitute into **(1)**:

$$h_{\min}\cos\theta + \mu_s(h_{\min}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg\tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90^\circ \quad\Rightarrow\quad h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we

equate components:

$$-\mu_s n - mg\sin\theta + h_{\max}\cos\theta = 0 \quad\Rightarrow\quad h_{\max}\cos\theta - \mu_s n = mg\sin\theta$$

$$n - mg\cos\theta - h_{\max}\sin\theta = 0 \quad\Rightarrow\quad h_{\max}\sin\theta + mg\cos\theta = n$$

Then substituting,

$$h_{\max}\cos\theta - \mu_s(h_{\max}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

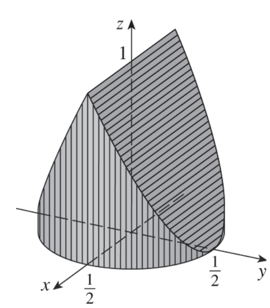
$$h_{\max}\cos\theta - h_{\max}\mu_s\sin\theta = mg\sin\theta + mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\max} = mg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)$$

$$= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s\tan\theta}\right) = mg\tan(\theta + \theta_s)$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \le \theta \le 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

8. (a) The largest possible solid is achieved by starting with a circular cylinder of diameter 1 with axis the z-axis and with a height of 1. This is the largest solid that creates a square shadow with side length 1 in the y-direction and a circular disk shadow in the z-direction. For convenience, we place the base of the cylinder on the xy-plane so that it intersects the x- and y-axes at $\pm\frac{1}{2}$. We then remove as little as possible from the solid that leaves an isosceles triangle shadow in the x-direction. This is achieved by cutting the solid with planes parallel to the x-axis that intersect the z-axis at 1 and the y-axis at $\pm\frac{1}{2}$ (see the figure).



To compute the volume of this solid, we take vertical slices parallel to the xz-plane. The equation of the base of the solid is $x^2 + y^2 = \frac{1}{4}$, so a cross-section y units to the right of the origin is a rectangle with base $2\sqrt{\frac{1}{4} - y^2}$. For $0 \le y \le \frac{1}{2}$, the solid is cut off on top by the plane $z = 1 - 2y$, so the height of the rectangular cross-section is $1 - 2y$ and the cross-sectional area is $A(y) = 2\sqrt{\frac{1}{4} - y^2}\,(1 - 2y)$. The volume of the right half of the solid is

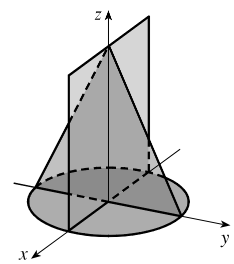
$$\int_0^{1/2} 2\sqrt{\tfrac{1}{4} - y^2}\,(1 - 2y)\,dy = 2\int_0^{1/2}\sqrt{\tfrac{1}{4} - y^2}\,dy - 4\int_0^{1/2} y\sqrt{\tfrac{1}{4} - y^2}\,dy$$

$$= 2\left[\tfrac{1}{4}\text{ area of a circle of radius }\tfrac{1}{2}\right] - 4\left[-\tfrac{1}{3}\left(\tfrac{1}{4} - y^2\right)^{3/2}\right]_0^{1/2}$$

$$= 2\left[\tfrac{1}{4}\cdot\pi\left(\tfrac{1}{2}\right)^2\right] + \tfrac{4}{3}\left[0 - \left(\tfrac{1}{4}\right)^{3/2}\right] = \tfrac{\pi}{8} - \tfrac{1}{6}$$

Thus the volume of the solid is $2\left(\frac{\pi}{8} - \frac{1}{6}\right) = \frac{\pi}{4} - \frac{1}{3} \approx 0.45$.

(b) There is not a smallest volume. We can remove portions of the solid from part (a) to create smaller and smaller volumes as long as we leave the "skeleton" shown in the figure intact (which has no volume at all and is not a solid). Thus we can create solids with arbitrarily small volume.



13 ☐ VECTOR FUNCTIONS

13.1 Vector Functions and Space Curves

1. The component functions $\ln(t+1)$, $\dfrac{t}{\sqrt{9-t^2}}$, and 2^t are all defined when $t+1>0 \quad\Rightarrow\quad t>-1$ and $9-t^2>0 \quad\Rightarrow$ $-3<t<3$, so the domain of $\mathbf{r}$ is $(-1,3)$.

2. The component functions $\cos t$, $\ln t$, and $\dfrac{1}{t-2}$ are all defined when $t>0$ and $t\neq 2$, so the domain of $\mathbf{r}$ is $(0,2)\cup(2,\infty)$.

3. $\displaystyle\lim_{t\to 0} e^{-3t}=e^0=1,\ \lim_{t\to 0}\frac{t^2}{\sin^2 t}=\lim_{t\to 0}\frac{1}{\dfrac{\sin^2 t}{t^2}}=\frac{1}{\displaystyle\lim_{t\to 0}\frac{\sin^2 t}{t^2}}=\frac{1}{\left(\displaystyle\lim_{t\to 0}\frac{\sin t}{t}\right)^2}=\frac{1}{1^2}=1,$

and $\displaystyle\lim_{t\to 0}\cos 2t=\cos 0=1$. Thus

$$\lim_{t\to 0}\left(e^{-3t}\,\mathbf{i}+\frac{t^2}{\sin^2 t}\,\mathbf{j}+\cos 2t\,\mathbf{k}\right)=\left[\lim_{t\to 0}e^{-3t}\right]\mathbf{i}+\left[\lim_{t\to 0}\frac{t^2}{\sin^2 t}\right]\mathbf{j}+\left[\lim_{t\to 0}\cos 2t\right]\mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

4. $\displaystyle\lim_{t\to 1}\frac{t^2-t}{t-1}=\lim_{t\to 1}\frac{t\,(t-1)}{t-1}=\lim_{t\to 1}t=1,\ \lim_{t\to 1}\sqrt{t+8}=3,\ \lim_{t\to 1}\frac{\sin \pi t}{\ln t}=\lim_{t\to 1}\frac{\pi\cos\pi t}{1/t}=-\pi$ [by l'Hospital's Rule].

Thus the given limit equals $\mathbf{i}+3\,\mathbf{j}-\pi\,\mathbf{k}$.

5. $\displaystyle\lim_{t\to\infty}\frac{1+t^2}{1-t^2}=\lim_{t\to\infty}\frac{(1/t^2)+1}{(1/t^2)-1}=\frac{0+1}{0-1}=-1,\ \lim_{t\to\infty}\tan^{-1}t=\tfrac{\pi}{2},\ \lim_{t\to\infty}\frac{1-e^{-2t}}{t}=\lim_{t\to\infty}\frac{1}{t}-\frac{1}{te^{2t}}=0-0=0.$ Thus

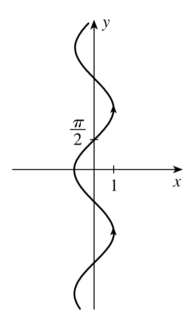
$\displaystyle\lim_{t\to\infty}\left\langle\frac{1+t^2}{1-t^2},\tan^{-1}t,\frac{1-e^{-2t}}{t}\right\rangle=\left\langle -1,\tfrac{\pi}{2},0\right\rangle.$

6. $\displaystyle\lim_{t\to\infty}te^{-t}=\lim_{t\to\infty}\frac{t}{e^t}=\lim_{t\to\infty}\frac{1}{e^t}=0$ [by l'Hospital's Rule], $\displaystyle\lim_{t\to\infty}\frac{t^3+t}{2t^3-1}=\lim_{t\to\infty}\frac{1+(1/t^2)}{2-(1/t^3)}=\frac{1+0}{2-0}=\frac{1}{2},$

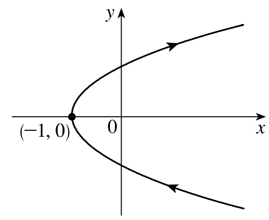
and $\displaystyle\lim_{t\to\infty}t\sin\frac{1}{t}=\lim_{t\to\infty}\frac{\sin(1/t)}{1/t}=\lim_{t\to\infty}\frac{\cos(1/t)(-1/t^2)}{-1/t^2}=\lim_{t\to\infty}\cos\frac{1}{t}=\cos 0=1$ [again by l'Hospital's Rule].

Thus $\displaystyle\lim_{t\to\infty}\left\langle te^{-t},\frac{t^3+t}{2t^3-1},t\sin\frac{1}{t}\right\rangle=\left\langle 0,\tfrac{1}{2},1\right\rangle.$

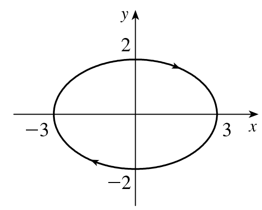
7. The corresponding parametric equations for this curve are $x=-\cos t$, $y=t$. We can make a table of values or we can eliminate the parameter: $t=y \quad\Rightarrow\quad x=-\cos y$, with $y\in\mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



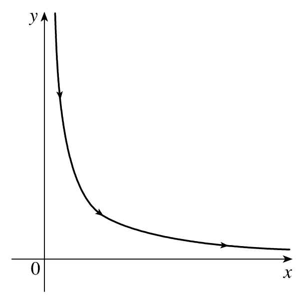
8. The corresponding parametric equations for this curve are $x = t^2 - 1,\ y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \quad \Rightarrow \quad x = y^2 - 1$, with $y \in \mathbb{R}$. Thus the curve is a parabola with vertex $(-1, 0)$ that opens to the right. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



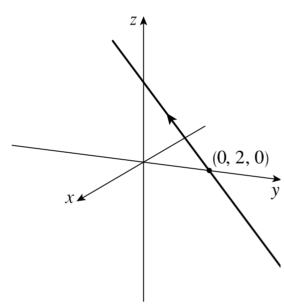
9. The corresponding parametric equations for this curve are $x = 3\sin t$, $y = 2\cos t$. We can make a table of values, or we can eliminate the parameter: $x = 3\sin t$, $y = 2\cos t \quad \Rightarrow \quad \dfrac{x}{3} = \sin t$, $\dfrac{y}{2} = \cos t \quad \Rightarrow$ $\dfrac{x^2}{9} + \dfrac{y^2}{4} = \sin^2 t + \cos^2 t = 1$, which we recognize as the equation of an ellipse with $x \in [-3, 3]$ and $y \in [-2, 2]$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



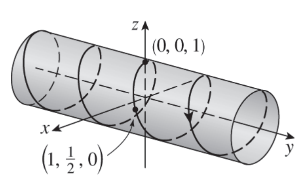
10. The corresponding parametric equations for this curve are $x = e^t$, $y = e^{-t}$. We can make a table of values, or we can eliminate the parameter: $y = e^{-t} = 1/e^t = 1/x$ with $x, y > 0$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



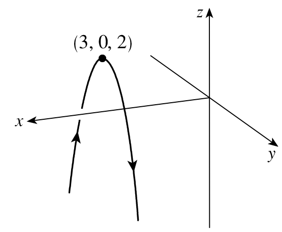
11. The corresponding parametric equations are $x = t,\ y = 2 - t,\ z = 2t$, which are parametric equations of a line through the point $(0, 2, 0)$ and with direction vector $\langle 1, -1, 2 \rangle$.



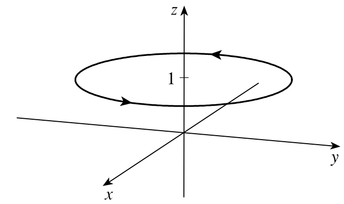
12. The corresponding parametric equations are $x = \sin \pi t,\ y = t,\ z = \cos \pi t$. Note that $x^2 + z^2 = \sin^2 \pi t + \cos^2 \pi t = 1$, so the curve lies on the circular cylinder $x^2 + z^2 = 1$. A point (x, y, z) on the curve lies directly to the left or right of the point $(x, 0, z)$ which moves clockwise (when viewed from the left) along the circle $x^2 + z^2 = 1$ in the xz-plane as t increases. Since $y = t$, the curve is a helix that spirals toward the right around the cylinder.



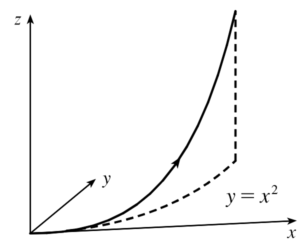
13. The corresponding parametric equations are $x = 3,\ y = t,\ z = 2 - t^2$. Eliminating the parameter in y and z gives $z = 2 - y^2$. Because $x = 3$, the curve is a parabola in the vertical plane $x = 3$ with vertex $(3, 0, 2)$.



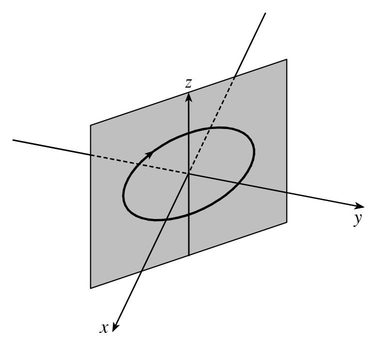
14. The corresponding parametric equations are $x = 2\cos t,\ y = 2\sin t$, $z = 1$. Eliminating the parameter in x and y gives $x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$. Since $z = 1$, the curve is a circle of radius 2 centered at $(0, 0, 1)$ in the horizontal plane $z = 1$.



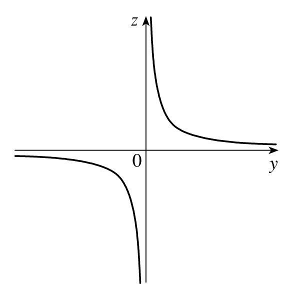
15. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first octant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. The projection onto the xz-plane is $z = x^3$, $z > 0$, a half cubic, and the projection onto the yz-plane is $y^3 = z^2$.



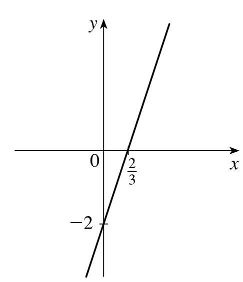
16. If $x = \cos t,\ y = -\cos t,\ z = \sin t$, then $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, so the curve is contained in the intersection of circular cylinders along the x- and y-axes. Furthermore, $y = -x$, so the curve is an ellipse in the plane $y = -x$, centered at the origin.



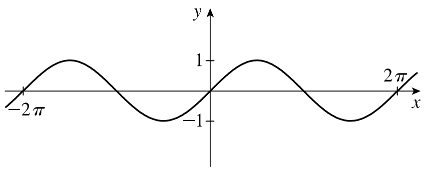
17. The projection of the curve defined by the vector function $\mathbf{r}(t) = \langle t^2, t^3, t^{-3} \rangle$ onto the yz-plane is given by $\mathbf{r}(t) = \langle 0, t^3, t^{-3} \rangle$ [we use 0 for the x-component], whose graph is the curve $z = 1/y$, $x = 0$, since $z = t^{-3} = 1/t^3$.



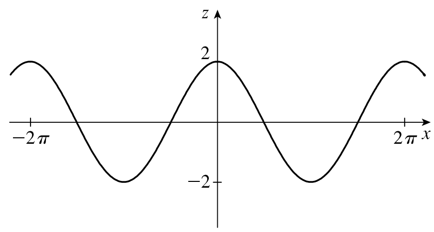
18. The projection of the curve defined by the vector function $\mathbf{r}(t) = \langle t + 1, 3t + 1, \cos(t/2) \rangle$ onto the xy-plane is given by $\mathbf{r}(t) = \langle t + 1, 3t + 1, 0 \rangle$ [we use 0 for the z-component], whose graph is the curve $y = 3x - 2$, $z = 0$, since $y = 3t + 1 = 3(x - 1) + 1 = 3x - 2$.



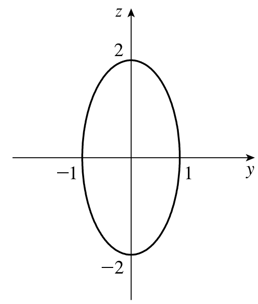
19. The projection of the curve onto the xy-plane is given by $\mathbf{r}(t) = \langle t, \sin t, 0 \rangle$ [we use 0 for the z-component] whose graph is the curve $y = \sin x$, $z = 0$. Similarly, the projection onto the xz-plane is $\mathbf{r}(t) = \langle t, 0, 2\cos t \rangle$, whose graph is the cosine wave $z = 2\cos x$, $y = 0$, and the projection onto the yz-plane is $\mathbf{r}(t) = \langle 0, \sin t, 2\cos t \rangle$ whose graph is the ellipse $y^2 + \frac{1}{4}z^2 = 1$, $x = 0$.



xy-plane

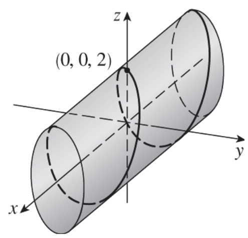


xz-plane

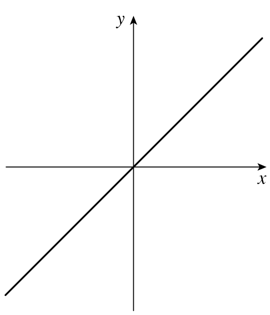


yz-plane

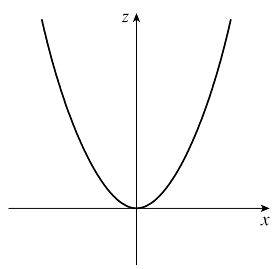
From the projection onto the yz-plane we see that the curve lies on an elliptical cylinder with axis the x-axis. The other two projections show that the curve oscillates both vertically and horizontally as we move in the x-direction, suggesting that the curve is an elliptical helix that spirals along the cylinder.



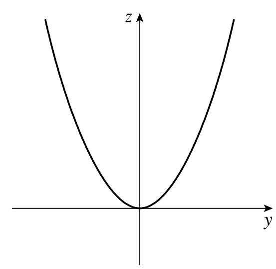
20. The projection of the curve onto the xy-plane is given by $\mathbf{r}(t) = \langle t, t, 0 \rangle$ whose graph is the line $y = x$, $z = 0$. The projection onto the xz-plane is $\mathbf{r}(t) = \langle t, 0, t^2 \rangle$ whose graph is the parabola $z = x^2$, $y = 0$. The projection onto the yz-plane is $\mathbf{r}(t) = \langle 0, t, t^2 \rangle$ whose graph is the parabola $z = y^2$, $x = 0$.



xy-plane

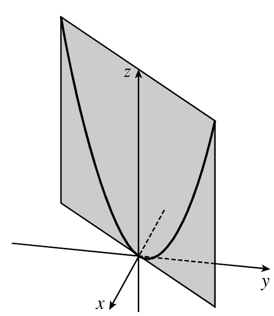


xz-plane



yz-plane

From the projection onto the xy-plane we see that the curve lies on the vertical plane $y = x$. The other two projections show that the curve is a parabola contained in this plane.



21. We take $\mathbf{r}_0 = \langle -2, 1, 0 \rangle$ and $\mathbf{r}_1 = \langle 5, 2, -3 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle -2,1,0\rangle + t\langle 5,2,-3\rangle \quad\Rightarrow\quad \mathbf{r}(t) = \langle -2+7t, 1+t, -3t\rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = -2 + 7t$, $y = 1 + t$, $z = -3t$, $0 \le t \le 1$.

22. We take $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$ and $\mathbf{r}_1 = \langle -7, 4, 6 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle 0,0,0\rangle + t\langle -7,4,6\rangle \quad\Rightarrow\quad \mathbf{r}(t) = \langle -7t, 4t, 6t\rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = -7t$, $y = 4t$, $z = 6t$, $0 \le t \le 1$.

23. We take $\mathbf{r}_0 = \langle 3.5, -1.4, 2.1 \rangle$ and $\mathbf{r}_1 = \langle 1.8, 0.3, 2.1 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle 3.5,-1.4,2.1\rangle + t\langle 1.8,0.3,2.1\rangle \quad\Rightarrow\quad \mathbf{r}(t) = \langle 3.5-1.7t, -1.4+1.7t, 2.1\rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = 3.5 - 1.7t$, $y = -1.4 + 1.7t$, $z = 2.1$, $0 \le t \le 1$.

24. We take $\mathbf{r}_0 = \langle a, b, c \rangle$ and $\mathbf{r}_1 = \langle u, v, w \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle a,b,c\rangle + t\langle u,v,w\rangle$$

$$\Rightarrow \qquad \mathbf{r}(t) = \langle a+(u-a)t, b+(v-b)t, c+(w-c)t\rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = a + (u-a)t,\ y = b + (v-b)t,\ z = c + (w-c)t,\ 0 \le t \le 1$.

25. $x = t\cos t,\ y = t,\ z = t\sin t,\ t \ge 0$. At any point (x, y, z) on the curve, $x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = y^2$ so the curve lies on the circular cone $x^2 + z^2 = y^2$ with axis the y-axis. Also notice that $y \ge 0$; the graph is II.

26. $x = \cos t,\ y = \sin t,\ z = 1/(1+t^2)$. At any point on the curve we have $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on the circular cylinder $x^2 + y^2 = 1$ with axis the z-axis. Notice that $0 < z \le 1$ and $z = 1$ only for $t = 0$. A point (x, y, z) on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the xy-plane as t increases, and $z \to 0$ as $t \to \pm\infty$. The graph must be VI.

27. $x = t,\ y = 1/(1+t^2),\ z = t^2$. At any point on the curve we have $z = x^2$, so the curve lies on a parabolic cylinder parallel to the y-axis. Notice that $0 < y \le 1$ and $z \ge 0$. Also the curve passes through $(0, 1, 0)$ when $t = 0$ and $y \to 0$, $z \to \infty$ as $t \to \pm\infty$, so the graph must be V.

28. $x = \cos t,\ y = \sin t,\ z = \cos 2t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. A point (x, y, z) on the curve lies directly above or below $(x, y, 0)$, which moves around the unit circle in the xy-plane with period 2π. At the same time, the z-value of the point (x, y, z) oscillates with a period of π. So the curve repeats itself and the graph is I.

29. $x = \cos 8t,\ y = \sin 8t,\ z = e^{0.8t},\ t \ge 0$. $x^2 + y^2 = \cos^2 8t + \sin^2 8t = 1$, so the curve lies on a circular cylinder with axis the z-axis. A point (x, y, z) on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the xy-plane as t increases. The curve starts at $(1, 0, 1)$, when $t = 0$, and $z \to \infty$ (at an increasing rate) as $t \to \infty$, so the graph is IV.

30. $x = \cos^2 t,\ y = \sin^2 t,\ z = t.\quad x + y = \cos^2 t + \sin^2 t = 1$, so the curve lies in the vertical plane $x + y = 1$. x and y are periodic, both with period π, and z increases as t increases, so the graph is III.

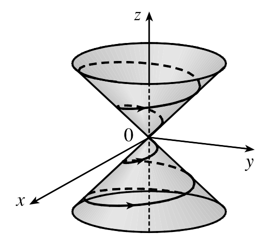
31. As $y = 4$ in the vector equation $\mathbf{r}(t) = \left\langle t, 4, t^2 \right\rangle$, the curve $z = x^2$ lies in the plane $y = 4$.

32. $\mathbf{r}(t) = \left\langle t, t^2, t \right\rangle$. Consider the projection of the curve in the xz-plane, $\mathbf{r}(t) = \langle t, 0, t \rangle$. This is the line $z = x$, $y = 0$. Thus, the curve is contained in the plane $z = x$.

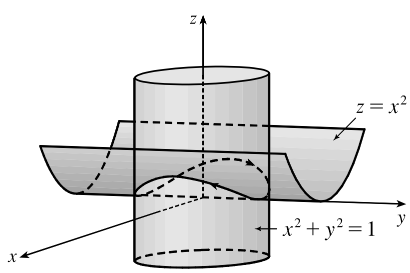
33. $\mathbf{r}(t) = \langle \sin t, \cos t, -\cos t \rangle$. Consider the projection of the curve in the yz-plane, $\mathbf{r}(t) = \langle 0, \cos t, -\cos t \rangle$. This is the line $z = -y$, $x = 0$. Thus, the curve is contained in the plane $z = -y$.

34. $\mathbf{r}(t) = \langle 2t, \sin t, t + 1 \rangle$. Consider the projection in the xz-plane, $\mathbf{r}(t) = \langle 2t, 0, t + 1 \rangle$. This is the line with parametric equations $x = 2t$, $z = t + 1$, $y = 0 \quad \Rightarrow \quad x = 2t = 2(z - 1) = 2z - 2, y = 0$. Thus, the curve is contained in the plane $x = 2z - 2$.

35. If $x = t\cos t,\ y = t\sin t,\ z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



36. If $x = \sin t$, $y = \cos t$, $z = \sin^2 t$, then $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is contained in the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$. We get the complete intersection for $0 \le t \le 2\pi$.



37. Here $x = 2t,\ y = e^t,\ z = e^{2t}$. Then $t = x/2 \quad \Rightarrow \quad y = e^t = e^{x/2}$, so the curve lies on the cylinder $y = e^{x/2}$. Also $z = e^{2t} = e^x$, so the curve lies on the cylinder $z = e^x$. Since $z = e^{2t} = \left(e^t\right)^2 = y^2$, the curve also lies on the parabolic cylinder $z = y^2$.

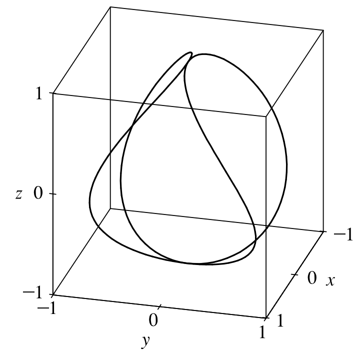
38. Here $x = t^2,\ y = \ln t,\ z = 1/t$. The domain of $\mathbf{r}$ is $(0, \infty)$, so $x = t^2 \quad \Rightarrow \quad t = \sqrt{x} \quad \Rightarrow \quad y = \ln\sqrt{x}$. Thus one surface containing the curve is the cylinder $y = \ln\sqrt{x}$ or $y = \ln x^{1/2} = \frac{1}{2}\ln x$. Also $z = 1/t = 1/\sqrt{x}$, so the curve also lies on the cylinder $z = 1/\sqrt{x}$ or $x = 1/z^2$, $z > 0$. Finally $z = 1/t \quad \Rightarrow \quad t = 1/z \quad \Rightarrow \quad y = \ln(1/z)$, so the curve also lies on the cylinder $y = \ln(1/z)$ or $y = \ln z^{-1} = -\ln z$. Note that the surface $y = \ln(xz)$ also contains the curve, since $\ln(xz) = \ln(t^2 \cdot 1/t) = \ln t = y$.

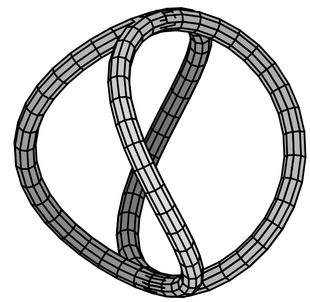
39. Parametric equations for the curve are $x = t,\ y = 0,\ z = 2t - t^2$. Substituting into the equation of the paraboloid gives $2t - t^2 = t^2 \quad \Rightarrow \quad 2t = 2t^2 \quad \Rightarrow \quad t = 0, 1$. Since $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$, the points of intersection are $(0, 0, 0)$ and $(1, 0, 1)$.

40. Parametric equations for the helix are $x = \sin t$, $y = \cos t$, $z = t$. Substituting into the equation of the sphere gives $\sin^2 t + \cos^2 t + t^2 = 5 \quad \Rightarrow \quad 1 + t^2 = 5 \quad \Rightarrow \quad t = \pm 2$. Since $\mathbf{r}(2) = \langle \sin 2, \cos 2, 2 \rangle$ and $\mathbf{r}(-2) = \langle \sin(-2), \cos(-2), -2 \rangle$, the points of intersection are $(\sin 2, \cos 2, 2) \approx (0.909, -0.416, 2)$ and $(\sin(-2), \cos(-2), -2) \approx (-0.909, -0.416, -2)$.

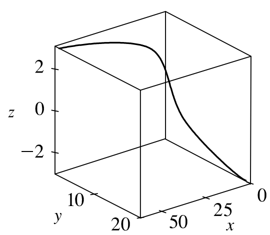
41. $\mathbf{r}(t) = \langle \cos t \; \sin 2t, \sin t \; \sin 2t, \cos 2t \rangle$.

We include both a regular plot and a plot showing a tube of radius 0.08 around the curve.

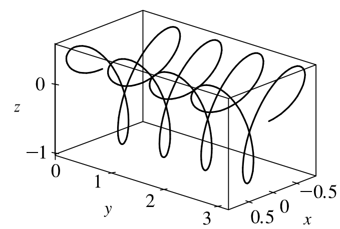




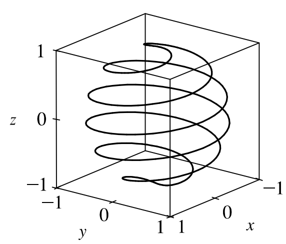
42. $\mathbf{r}(t) = \left\langle te^t, e^{-t}, t \right\rangle$



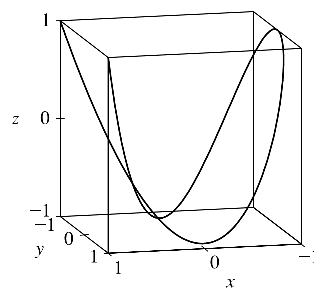
43. $\mathbf{r}(t) = \left\langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \right\rangle$



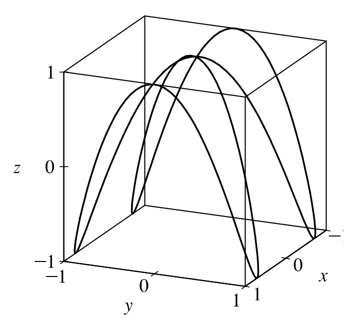
44. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$



45. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

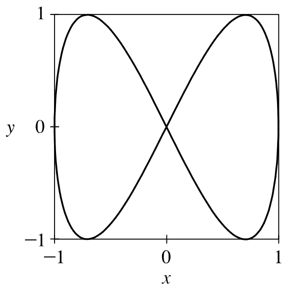


46. $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$.

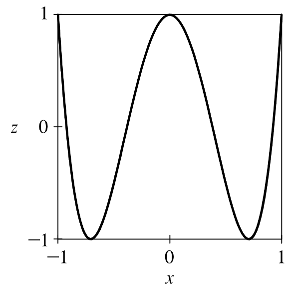


[continued]

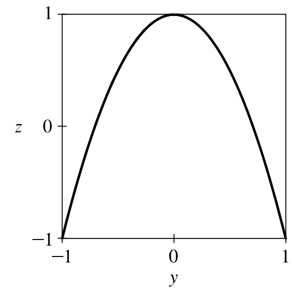
We graph the projections onto the coordinate planes.



xy-plane



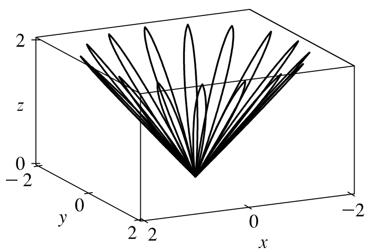
xz-plane



yz-plane

From the projection onto the xy-plane we see that from above the curve appears to be shaped like a "figure eight." The curve can be visualized as this shape wrapped around an almost parabolic cylindrical surface, the profile of which is visible in the projection onto the yz-plane.

47.

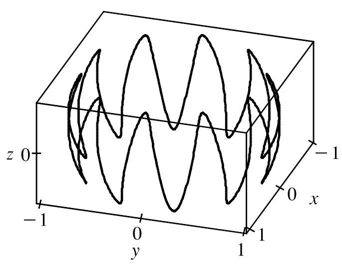


$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$$\begin{aligned} x^2 + y^2 &= (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t \\ &= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone } x^2 + y^2 = z^2. \end{aligned}$$

From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

48.



$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t,\ \ y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t,$

$z = 0.5\cos 10t$. At any point on the graph,

$$\begin{aligned} x^2 + y^2 + z^2 &= (1 - 0.25\cos^2 10t)\cos^2 t \\ &\quad + (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t \\ &= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1, \end{aligned}$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. We get the complete graph for $0 \le t \le 2\pi$.

49. If $t = -1$, then $x = 1,\ y = 4,\ z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9,\ y = -8,\ z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \Rightarrow t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the curve.

50. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. Since C also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x = 2\cos t,\ y = 2\sin t,$ $z = 2\sin(2t),\ 0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

51. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2+y^2} = 1+y \quad \Rightarrow \quad x^2+y^2 = 1+2y+y^2 \quad \Rightarrow$ $x^2 = 1+2y \quad \Rightarrow \quad y = \frac{1}{2}(x^2-1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2-1)$ and $z = 1+y = 1+\frac{1}{2}(t^2-1) = \frac{1}{2}(t^2+1)$. Thus a vector function representing C is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2-1)\,\mathbf{j} + \frac{1}{2}(t^2+1)\,\mathbf{k}$.

52. The projection of the curve C of intersection onto the xy-plane is the parabola $y = x^2$, $z = 0$. Then we can choose the parameter $x = t \quad \Rightarrow \quad y = t^2$. Since C also lies on the surface $z = 4x^2+y^2$, we have $z = 4x^2+y^2 = 4t^2+(t^2)^2$. Then parametric equations for C are $x = t,\ y = t^2,\ z = 4t^2+t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2+t^4)\,\mathbf{k}$.

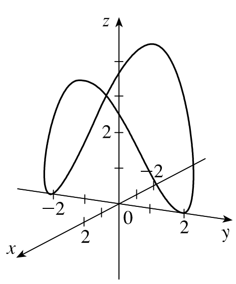
53. The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2 = 1$, $z = 0$, so we can write $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2-y^2$, we have $z = x^2-y^2 = \cos^2 t - \sin^2 t$ or $\cos 2t$. Thus parametric equations for C are $x = \cos t$, $y = \sin t$, $z = \cos 2t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}$, $0 \le t \le 2\pi$.

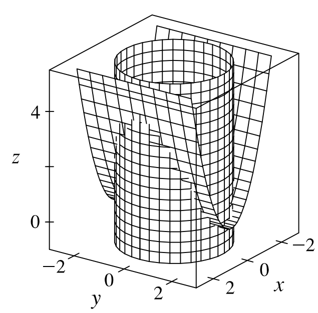
54. The projection of the curve C of intersection onto the xz-plane is the circle $x^2+z^2 = 1$, $y = 0$, so we can write $x = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$. C also lies on the surface $x^2+y^2+4z^2 = 4$, and since $y \ge 0$ we can write

$$y = \sqrt{4-x^2-4z^2} = \sqrt{4-\cos^2 t - 4\sin^2 t} = \sqrt{4-\cos^2 t - 4(1-\cos^2 t)} = \sqrt{3\cos^2 t} = \sqrt{3}\,|\cos t\,|$$

Thus parametric equations for C are $x = \cos t$, $y = \sqrt{3}\,|\cos t\,|$, $z = \sin t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sqrt{3}\,|\cos t\,|\,\mathbf{j} + \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$.

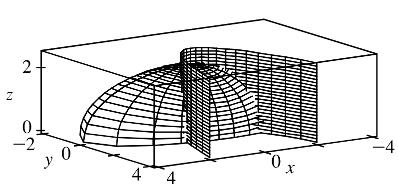
55.

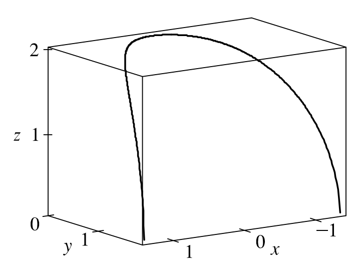




The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for C are $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \le t \le 2\pi$.

56.





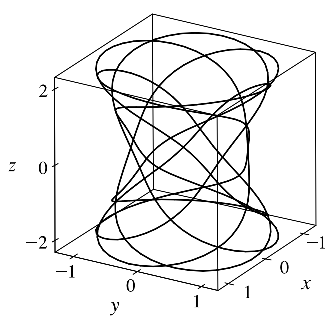
$x = t \quad \Rightarrow \quad y = t^2 \quad \Rightarrow \quad 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \quad \Rightarrow \quad z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$.

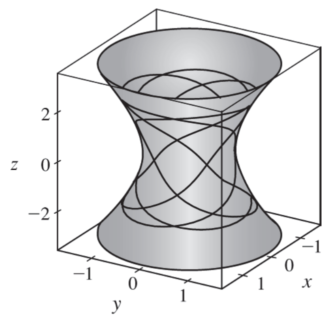
Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t,\ y = t^2,\ z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

57. For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives $t^2 = 4t - 3$, $7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \Leftrightarrow (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

58. The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$. Equating components gives $t = 1 + 2t$, $t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. The first equation gives $t = -1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Equating components, $t = 1 + 2s$, $t^2 = 1 + 6s$, and $t^3 = 1 + 14s$. Substituting the first equation into the second gives $(1 + 2s)^2 = 1 + 6s \Rightarrow 4s^2 - 2s = 0 \Rightarrow 2s(2s - 1) = 0 \Rightarrow s = 0$ or $s = \frac{1}{2}$. From the first equation, $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1, 1, 1)$ when $s = 0$ and $t = 1$, and at $(2, 4, 8)$ when $s = \frac{1}{2}$ and $t = 2$.

59. (a) We plot the parametric equations for $0 \le t \le 2\pi$ in the first figure. We get a better idea of the shape of the curve if we plot it simultaneously with the hyperboloid of one sheet from part (b), as shown in the second figure.





(b) Here $x = \frac{27}{26}\sin 8t - \frac{8}{39}\sin 18t$, $y = -\frac{27}{26}\cos 8t + \frac{8}{39}\cos 18t$, $z = \frac{144}{65}\sin 5t$.

For any point on the curve,

$$\begin{aligned}
x^2 + y^2 &= \left(\tfrac{27}{26}\sin 8t - \tfrac{8}{39}\sin 18t\right)^2 + \left(-\tfrac{27}{26}\cos 8t + \tfrac{8}{39}\cos 18t\right)^2 \\
&= \tfrac{27^2}{26^2}\sin^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\sin 8t\sin 18t + \tfrac{64}{39^2}\sin^2 18t \\
&\qquad + \tfrac{27^2}{26^2}\cos^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\cos 8t\cos 18t + \tfrac{64}{39^2}\cos^2 18t \\
&= \tfrac{27^2}{26^2}\left(\sin^2 8t + \cos^2 8t\right) + \tfrac{64}{39^2}\left(\sin^2 18t + \cos^2 18t\right) - \tfrac{72}{169}\left(\sin 8t\sin 18t + \cos 8t\cos 18t\right) \\
&= \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos\left(18t - 8t\right) = \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos 10t
\end{aligned}$$

using the trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ and $\cos(x - y) = \cos x\cos y + \sin x\sin y$. Also $z^2 = \frac{144^2}{65^2}\sin^2 5t$, and the identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ gives $z^2 = \frac{144^2}{65^2}\cdot\frac{1}{2}\left[1 - \cos(2\cdot 5t)\right] = \frac{144^2}{2\cdot 65^2} - \frac{144^2}{2\cdot 65^2}\cos 10t$.

[continued]

Then

$$144(x^2+y^2)-25z^2 = 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{72}{169}\cos 10t\right)-25\left(\tfrac{144^2}{2\cdot 65^2}-\tfrac{144^2}{2\cdot 65^2}\cos 10t\right)$$

$$= 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{25\cdot 144}{2\cdot 65^2}-\tfrac{72}{169}\cos 10t+\tfrac{25\cdot 144}{2\cdot 65^2}\cos 10t\right)$$

$$= 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{72}{169}-\tfrac{72}{169}\cos 10t+\tfrac{72}{169}\cos 10t\right) = 144\left(\tfrac{25}{36}\right) = 100$$

Thus the curve lies on the surface $144(x^2+y^2)-25z^2=100$ or $144x^2+144y^2-25z^2=100$, a hyperboloid of one sheet with axis the z-axis.

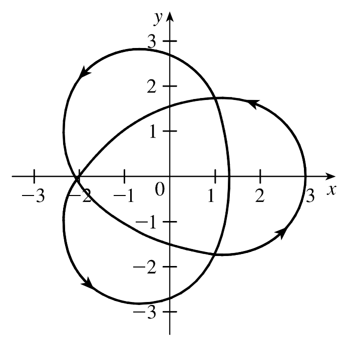
60. The projection of the curve onto the xy-plane is given by the parametric equations $x=(2+\cos 1.5t)\cos t$, $y=(2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

$$r^2 = x^2+y^2 = [(2+\cos 1.5t)\cos t]^2+[(2+\cos 1.5t)\sin t]^2$$

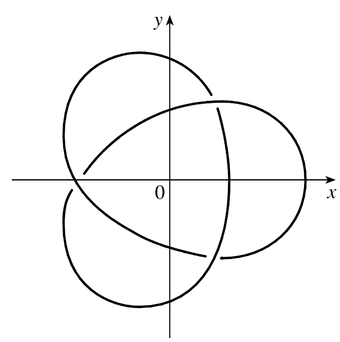
$$= (2+\cos 1.5t)^2(\cos^2 t+\sin^2 t) = (2+\cos 1.5t)^2 \quad\Rightarrow\quad r = 2+\cos 1.5t$$

Also, $\tan\theta = \dfrac{y}{x} = \dfrac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t} = \tan t \quad\Rightarrow\quad \theta = t.$

Thus the polar equation of the curve is $r = 2+\cos 1.5\theta$. At $\theta=0$, we have $r=3$, and r decreases to 1 as θ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}$, r increases to 3; r decreases to 1 again at $\theta=2\pi$, increases to 3 at $\theta=\frac{8\pi}{3}$, decreases to 1 at $\theta=\frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta=4\pi$. We sketch an approximate graph as shown in the figure.

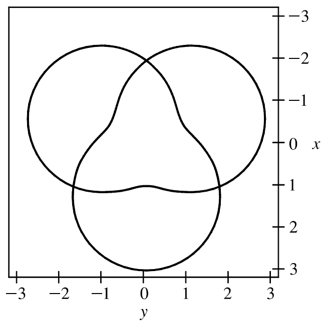


We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $0\le t\le 4\pi$. Since $z=\sin 1.5t$, z is maximized where $\sin 1.5t = 1 \quad\Rightarrow\quad 1.5t=\frac{\pi}{2}, \frac{5\pi}{2}$, or $\frac{9\pi}{2} \quad\Rightarrow$ $t=\frac{\pi}{3}, \frac{5\pi}{3}$, or 3π. z is minimized where $\sin 1.5t=-1 \quad\Rightarrow$ $1.5t=\frac{3\pi}{2}, \frac{7\pi}{2}$, or $\frac{11\pi}{2} \quad\Rightarrow\quad t=\pi, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \quad\Rightarrow\quad r=2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.

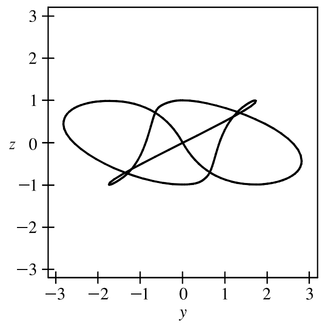


[continued]

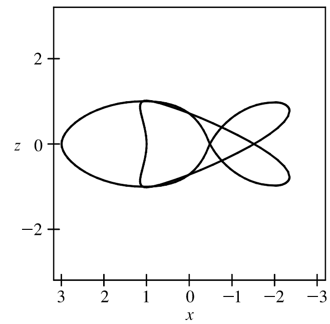
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

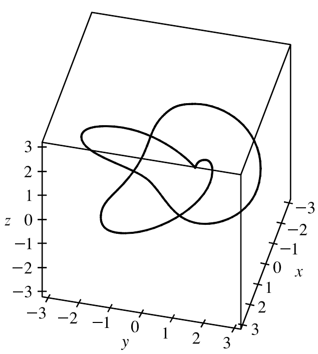


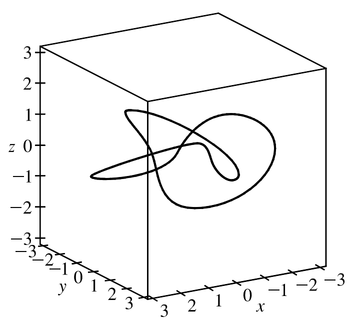
Front view

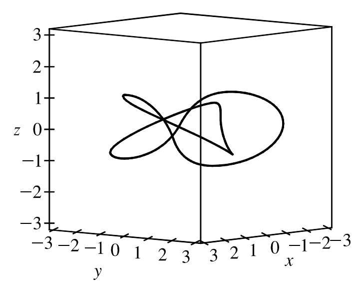


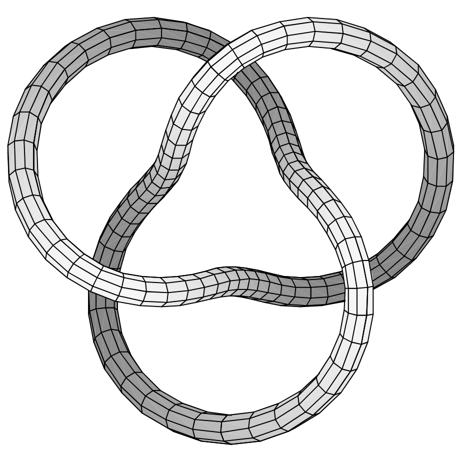
Side view

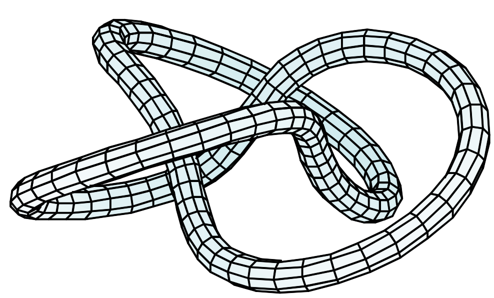
The top view graph shows a more accurate representation of the projection of the trefoil knot onto the xy-plane (the axes are rotated $90°$). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.











61. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim\limits_{t\to a}\mathbf{u}(t) + \lim\limits_{t\to a}\mathbf{v}(t) = \left\langle \lim\limits_{t\to a} u_1(t), \lim\limits_{t\to a} u_2(t), \lim\limits_{t\to a} u_3(t)\right\rangle + \left\langle \lim\limits_{t\to a} v_1(t), \lim\limits_{t\to a} v_2(t), \lim\limits_{t\to a} v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t) + \lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t) + \lim_{t\to a} v_1(t), \lim_{t\to a} u_2(t) + \lim_{t\to a} v_2(t), \lim_{t\to a} u_3(t) + \lim_{t\to a} v_3(t)\right\rangle \\
&= \left\langle \lim_{t\to a}\left[u_1(t)+v_1(t)\right], \lim_{t\to a}\left[u_2(t)+v_2(t)\right], \lim_{t\to a}\left[u_3(t)+v_3(t)\right]\right\rangle \\
&= \lim_{t\to a}\langle u_1(t)+v_1(t), u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle \qquad \text{[using (1) backward]} \\
&= \lim_{t\to a}\left[\mathbf{u}(t)+\mathbf{v}(t)\right]
\end{aligned}$$

(b)
$$\begin{aligned}
\lim_{t\to a} c\mathbf{u}(t) &= \lim_{t\to a}\langle cu_1(t), cu_2(t), cu_3(t)\rangle = \left\langle \lim_{t\to a} cu_1(t), \lim_{t\to a} cu_2(t), \lim_{t\to a} cu_3(t)\right\rangle \\
&= \left\langle c\lim_{t\to a} u_1(t), c\lim_{t\to a} u_2(t), c\lim_{t\to a} u_3(t)\right\rangle = c\left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \\
&= c\lim_{t\to a}\langle u_1(t), u_2(t), u_3(t)\rangle = c\lim_{t\to a}\mathbf{u}(t)
\end{aligned}$$

(c)
$$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \cdot \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle \\
&= \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_1(t)\right] + \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_2(t)\right] + \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_3(t)\right] \\
&= \lim_{t\to a} u_1(t)v_1(t) + \lim_{t\to a} u_2(t)v_2(t) + \lim_{t\to a} u_3(t)v_3(t) \\
&= \lim_{t\to a}\left[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right] = \lim_{t\to a}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]
\end{aligned}$$

(d)
$$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \times \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle \\
&= \Big\langle \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_3(t)\right] - \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_2(t)\right], \\
&\qquad \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_1(t)\right] - \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_3(t)\right], \\
&\qquad\qquad \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_2(t)\right] - \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_1(t)\right]\Big\rangle \\
&= \Big\langle \lim_{t\to a}\left[u_2(t)v_3(t)-u_3(t)v_2(t)\right], \lim_{t\to a}\left[u_3(t)v_1(t)-u_1(t)v_3(t)\right], \\
&\qquad \lim_{t\to a}\left[u_1(t)v_2(t)-u_2(t)v_1(t)\right]\Big\rangle \\
&= \lim_{t\to a}\langle u_2(t)v_3(t)-u_3(t)v_2(t), u_3(t)v_1(t)-u_1(t)v_3(t), u_1(t)v_2(t)-u_2(t)v_1(t)\rangle \\
&= \lim_{t\to a}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]
\end{aligned}$$

62. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. If $\lim\limits_{t\to a}\mathbf{r}(t) = \mathbf{b}$, then $\lim\limits_{t\to a}\mathbf{r}(t)$ exists, so by (1),

$\mathbf{b} = \lim\limits_{t\to a}\mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right\rangle$. By the definition of equal vectors we have $\lim\limits_{t\to a} f(t) = b_1$, $\lim\limits_{t\to a} g(t) = b_2$

and $\lim_{t\to a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\varepsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so that if $0 < |t-a| < \delta_1$ then $|f(t) - b_1| < \varepsilon/3$, if $0 < |t-a| < \delta_2$ then $|g(t) - b_2| < \varepsilon/3$, and if $0 < |t-a| < \delta_3$ then $|h(t) - b_3| < \varepsilon/3$. Letting $\delta =$ minimum of $\{\delta_1, \delta_2, \delta_3\}$, then if $0 < |t-a| < \delta$ we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. But

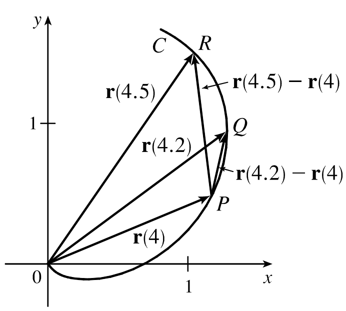
$$\begin{aligned}|\mathbf{r}(t) - \mathbf{b}| &= |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3\rangle| = \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \\ &\le \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2} = |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3|\end{aligned}$$

Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |t-a| < \delta$ then $|\mathbf{r}(t) - \mathbf{b}| \le |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \varepsilon$.

Conversely, suppose for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |t-a| < \delta$ then $|\mathbf{r}(t) - \mathbf{b}| < \varepsilon \quad \Leftrightarrow$ $|\langle f(t) - b_1, g(t) - b_2, h(t) - b_3\rangle| < \varepsilon \quad \Leftrightarrow \quad \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \varepsilon \quad \Leftrightarrow$ $[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \varepsilon^2$. But each term on the left side of the last inequality is positive, so if $0 < |t-a| < \delta$, then $[f(t) - b_1]^2 < \varepsilon^2$, $[g(t) - b_2]^2 < \varepsilon^2$ and $[h(t) - b_3]^2 < \varepsilon^2$ or, taking the square root of both sides in each of the above, $|f(t) - b_1| < \varepsilon$, $|g(t) - b_2| < \varepsilon$ and $|h(t) - b_3| < \varepsilon$. And by definition of limits of real-valued functions we have $\lim_{t\to a} f(t) = b_1$, $\lim_{t\to a} g(t) = b_2$, and $\lim_{t\to a} h(t) = b_3$. But by (1), $\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$, so $\lim_{t\to a} \mathbf{r}(t) = \langle b_1, b_2, b_3\rangle = \mathbf{b}$.

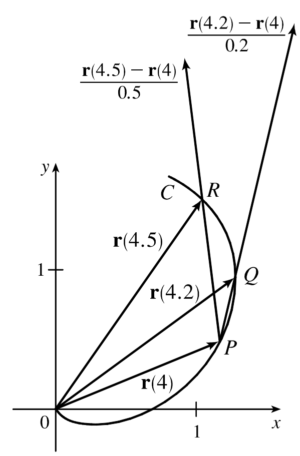
13.2 Derivatives and Integrals of Vector Functions

1. (a)



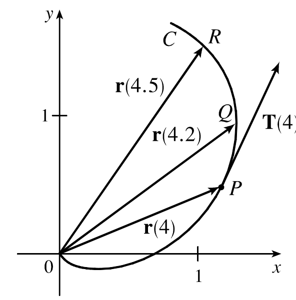
(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$.

$\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.

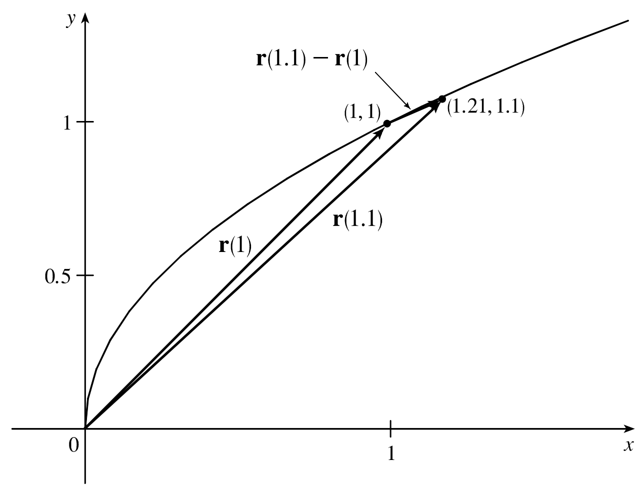


(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h\to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$. $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

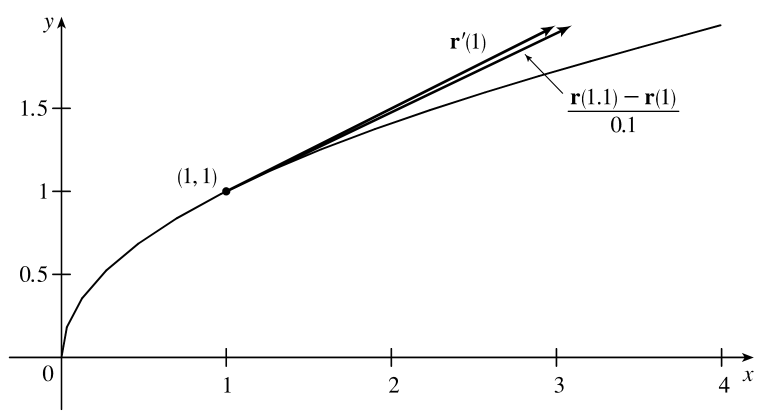


2. (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \le t \le 2$. Eliminating the parameter, we have $x = y^2$, $0 \le y \le 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.



(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10\,\langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$

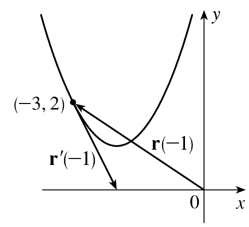


As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be

$\lim_{h\to 0} \dfrac{\mathbf{r}(1+h)-\mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$. Since h is close to 0, we would expect $\dfrac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

3. $\mathbf{r}(t) = \langle t-2, t^2+1\rangle$, $\mathbf{r}(-1) = \langle -3, 2\rangle$. Since $(x+2)^2 = t^2 = y-1 \Rightarrow y = (x+2)^2 + 1$, the curve is a parabola.

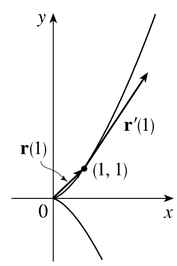
(a), (c)



(b) $\mathbf{r}'(t) = \langle 1, 2t\rangle$, $\mathbf{r}'(-1) = \langle 1, -2\rangle$

4. $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $\mathbf{r}(1) = \langle 1, 1\rangle$. Since $x = t^2 = (t^3)^{2/3} = y^{2/3}$, the curve is the graph of $x = y^{2/3}$.

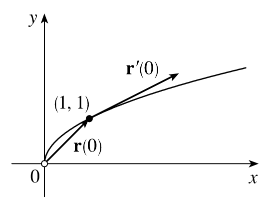
(a), (c)



(b) $\mathbf{r}'(t) = \langle 2t, 3t^2\rangle$, $\mathbf{r}'(1) = \langle 2, 3\rangle$

5. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$. Since $x = e^{2t} = (e^t)^2 = y^2$, the curve is part of a parabola. Note that here $x > 0$, $y > 0$.

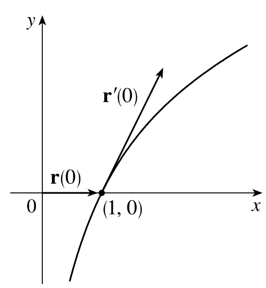
(a), (c)



(b) $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $\mathbf{r}'(0) = 2\,\mathbf{i} + \mathbf{j}$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j}$, $\mathbf{r}(0) = \mathbf{i}$. Since $x = e^t \Leftrightarrow t = \ln x$ and $y = 2t = 2\ln x$, the curve is the graph of $y = 2\ln x$.

(a), (c)

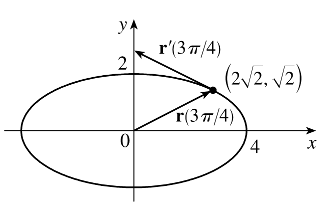


(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{r}'(0) = \mathbf{i} + 2\,\mathbf{j}$

7. $\mathbf{r}(t) = 4\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$, $\mathbf{r}(3\pi/4) = 4(\sqrt{2}/2)\,\mathbf{i} - 2(-\sqrt{2}/2)\,\mathbf{j} = 2\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$.

Here $(x/4)^2 + (y/2)^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

(a), (c)

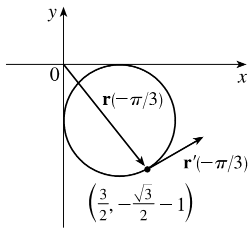


(b) $\mathbf{r}'(t) = 4\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $\mathbf{r}'(3\pi/4) = -2\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$.

8. $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}$, $\mathbf{r}(-\pi/3) = \left(\frac{1}{2} + 1\right)\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} = \frac{3}{2}\,\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} \approx 1.5\,\mathbf{i} - 1.87\,\mathbf{j}$.

Here $(x-1)^2 + (y+1)^2 = \cos^2 t + \sin^2 t = 1$, so the curve is a circle of radius 1 with center $(1, -1)$.

(a), (c)



(b) $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$,

$\mathbf{r}'(-\pi/3) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\,\mathbf{j} \approx 0.87\,\mathbf{i} + 0.5\,\mathbf{j}$

9. $\mathbf{r}(t) = \left\langle \sqrt{t-2}, 3,\ 1/t^2 \right\rangle \quad \Rightarrow$

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[\sqrt{t-2}\right], \frac{d}{dt}[3], \frac{d}{dt}\left[1/t^2\right] \right\rangle = \left\langle \tfrac{1}{2}(t-2)^{-1/2}, 0, -2t^{-3} \right\rangle = \left\langle \frac{1}{2\sqrt{t-2}}, 0, -\frac{2}{t^3} \right\rangle$$

10. $\mathbf{r}(t) = \left\langle e^{-t}, t - t^3, \ln t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle -e^{-t}, 1 - 3t^2, 1/t \right\rangle$

11. $\mathbf{r}(t) = t^2\ \mathbf{i} + \cos\left(t^2\right)\ \mathbf{j} + \sin^2 t\ \mathbf{k} \quad \Rightarrow$

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + \left[-\sin(t^2)\cdot 2t\right]\mathbf{j} + (2\sin t\cdot\cos t)\,\mathbf{k} = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$$

12. $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k} \quad \Rightarrow$

$$\mathbf{r}'(t) = \frac{0 - 1(1)}{(1+t)^2}\,\mathbf{i} + \frac{(1+t)\cdot 1 - t(1)}{(1+t)^2}\,\mathbf{j} + \frac{(1+t)\cdot 2t - t^2(1)}{(1+t)^2}\,\mathbf{k} = -\frac{1}{(1+t)^2}\,\mathbf{i} + \frac{1}{(1+t)^2}\,\mathbf{j} + \frac{t^2+2t}{(1+t)^2}\,\mathbf{k}$$

13. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [t\cdot\cos t + (\sin t)\cdot 1]\,\mathbf{i} + \left[e^t(-\sin t) + (\cos t)e^t\right]\mathbf{j} + [(\sin t)(-\sin t) + (\cos t)(\cos t)]\,\mathbf{k} \\ &= (t\cos t + \sin t)\,\mathbf{i} + e^t\,(\cos t - \sin t)\,\mathbf{j} + \left(\cos^2 t - \sin^2 t\right)\mathbf{k}\end{aligned}$$

14. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [2(\sin at)\cdot(\cos at)(a)]\,\mathbf{i} + \left[t\cdot e^{bt}(b) + e^{bt}\cdot 1\right]\mathbf{j} + [2(\cos ct)\cdot(-\sin ct)(c)]\,\mathbf{k} \\ &= 2a\sin at\cos at\,\mathbf{i} + e^{bt}\,(bt+1)\,\mathbf{j} - 2c\sin ct\cos ct\,\mathbf{k}\end{aligned}$$

15. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c} \quad \Rightarrow \quad \mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\mathbf{r}'(t)$, by Formula 5 of Theorem 3, we first expand $\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b} + t\,\mathbf{c}) = t(\mathbf{a}\times\mathbf{b}) + t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$.

17. $\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 2t - 2, 3, t^2 + t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(2) = \langle 2, 3, 6\rangle$.

So $|\mathbf{r}'(2)| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$ and $\mathbf{T}(2) = \dfrac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{1}{7}\langle 2, 3, 6\rangle = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$.

18. $\mathbf{r}(t) = \left\langle \tan^{-1} t, 2e^{2t}, 8te^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1/(1+t^2), 4e^{2t}, 8te^t + 8e^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 4, 8\rangle$.

So $|\mathbf{r}'(0)| = \sqrt{1^2 + 4^2 + 8^2} = \sqrt{81} = 9$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{9}\langle 1, 4, 8\rangle = \left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9} \right\rangle$.

19. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin 2t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. So

$|\mathbf{r}'(0)| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{5}(3\,\mathbf{j} + 4\,\mathbf{k}) = \frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$.

20. $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2\sin t\cos t\,\mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\tan t\sec^2 t\,\mathbf{k} \quad\Rightarrow$

$\mathbf{r}'\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\,\mathbf{i} - 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\,\mathbf{j} + 2\cdot 1\cdot(\sqrt{2})^2\,\mathbf{k} = \mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$. So $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$

and $\mathbf{T}\left(\frac{\pi}{4}\right) = \dfrac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \dfrac{1}{3\sqrt{2}}(\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) = \dfrac{1}{3\sqrt{2}}\,\mathbf{i} - \dfrac{1}{3\sqrt{2}}\,\mathbf{j} + \dfrac{4}{3\sqrt{2}}\,\mathbf{k}$.

21. The point $(2, -2, 4)$ corresponds to $t = 1$ [note that $4/t = 4$]. Then

$$\mathbf{r}(t) = \left\langle t^3 + 1, 3t - 5, 4/t\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 3t^2, 3, -4/t^2\right\rangle \quad\Rightarrow\quad \mathbf{r}'(1) = \langle 3, 3, -4\rangle$$

So
$$|\mathbf{r}'(1)| = \sqrt{3^2 + 3^2 + (-4)^2} = \sqrt{34}$$

and
$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{34}}\langle 3, 3, -4\rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{3}{\sqrt{34}}, -\frac{4}{\sqrt{34}}\right\rangle$$

22. The point $(0, 0, 1)$ corresponds to $t = 0$ [note that $5t = 0$]. Then

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + 5t\,\mathbf{j} + \cos t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \cos t\,\mathbf{i} + 5\,\mathbf{j} - \sin t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(0) = \mathbf{i} + 5\,\mathbf{j}$$

So
$$|\mathbf{r}'(0)| = \sqrt{1^2 + 5^2 + 0^2} = \sqrt{26}$$

and
$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{26}}(\mathbf{i} + 5\mathbf{j}) = \frac{1}{\sqrt{26}}\,\mathbf{i} + \frac{5}{\sqrt{26}}\,\mathbf{j}$$

23. $\mathbf{r}(t) = \left\langle t^4, t, t^2\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 4t^3, 1, 2t\right\rangle$. Then $\mathbf{r}'(1) = \langle 4, 1, 2\rangle$, $|\mathbf{r}'(1)| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$, and

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{21}}\langle 4, 1, 2\rangle = \left\langle \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right\rangle$$

$\mathbf{r}''(t) = \left\langle 12t^2, 0, 2\right\rangle$, so

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4t^3 & 1 & 2t\\ 12t^2 & 0 & 2\end{vmatrix} = \begin{vmatrix}1 & 2t\\ 0 & 2\end{vmatrix}\mathbf{i} - \begin{vmatrix}4t^3 & 2t\\ 12t^2 & 2\end{vmatrix}\mathbf{j} + \begin{vmatrix}4t^3 & 1\\ 12t^2 & 0\end{vmatrix}\mathbf{k}\\ &= (2-0)\,\mathbf{i} - (8t^3 - 24t^3)\,\mathbf{j} + (0 - 12t^2)\,\mathbf{k} = \left\langle 2, 16t^3, -12t^2\right\rangle\end{aligned}$$

24. $\mathbf{r}(t) = \left\langle e^{2t}, e^{-3t}, t\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 2e^{2t}, -3e^{-3t}, 1\right\rangle$. Then $\mathbf{r}'(0) = \langle 2, -3, 1\rangle$, $|\mathbf{r}'(0)| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$, and

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{14}}\langle 2, -3, 1\rangle = \left\langle \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right\rangle$$

$\mathbf{r}''(t) = \left\langle 4e^{2t}, 9e^{-3t}, 0\right\rangle \quad\Rightarrow\quad \mathbf{r}''(0) = \langle 4, 9, 0\rangle$. Then

$$\begin{aligned}\mathbf{r}'(0)\times\mathbf{r}''(0) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & 1\\ 4 & 9 & 0\end{vmatrix} = \begin{vmatrix}-3 & 1\\ 9 & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 1\\ 4 & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & -3\\ 4 & 9\end{vmatrix}\mathbf{k}\\ &= (0-9)\,\mathbf{i} - (0-4)\,\mathbf{j} + [18 - (-12)]\,\mathbf{k} = \langle -9, 4, 30\rangle\end{aligned}$$

25. The vector equation for the curve is $\mathbf{r}(t) = \left\langle t^2+1, 4\sqrt{t}, e^{t^2-t}\right\rangle$, so $\mathbf{r}'(t) = \left\langle 2t, 2/\sqrt{t}, (2t-1)e^{t^2-t}\right\rangle$. The point $(2,4,1)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2,2,1\rangle$. Thus, the tangent line goes through the point $(2,4,1)$ and is parallel to the vector $\langle 2,2,1\rangle$. Parametric equations are $x = 2+2t$, $y = 4+2t$, $z = 1+t$.

26. The vector equation for the curve is $\mathbf{r}(t) = \langle \ln(t+1), t\cos 2t, 2^t\rangle$, so $\mathbf{r}'(t) = \langle 1/(t+1), \cos 2t - 2t\sin 2t, 2^t\ln 2\rangle$. The point $(0,0,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 1,1,\ln 2\rangle$. Thus, the tangent line goes through the point $(0,0,1)$ and is parallel to the vector $\langle 1,1,\ln 2\rangle$. Parametric equations are $x = 0+1\cdot t = t$, $y = 0+1\cdot t = t$, $z = 1+(\ln 2)t$.

27. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, so

$$\begin{aligned}\mathbf{r}'(t) &= \langle e^{-t}(-\sin t)+(\cos t)(-e^{-t}), e^{-t}\cos t+(\sin t)(-e^{-t}), (-e^{-t})\rangle\\ &= \langle -e^{-t}(\cos t+\sin t), e^{-t}(\cos t-\sin t), -e^{-t}\rangle\end{aligned}$$

The point $(1,0,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0+\sin 0), e^0(\cos 0-\sin 0), -e^0\rangle = \langle -1,1,-1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1,1,-1\rangle$ and parametric equations are $x = 1+(-1)t = 1-t$, $y = 0+1\cdot t = t$, $z = 1+(-1)t = 1-t$.

28. The vector equation for the curve is $\mathbf{r}(t) = \langle \sqrt{t^2+3}, \ln(t^2+3), t\rangle$, so $\mathbf{r}'(t) = \langle t/\sqrt{t^2+3}, 2t/(t^2+3), 1\rangle$. At $(2,\ln 4,1)$, $t=1$ and $\mathbf{r}'(1) = \langle \frac{1}{2}, \frac{1}{2}, 1\rangle$. Thus, parametric equations of the tangent line are $x = 2+\frac{1}{2}t$, $y = \ln 4+\frac{1}{2}t$, $z = 1+t$.

29. First we parametrize the curve C of intersection. The projection of C onto the xy-plane is contained in the circle $x^2+y^2=25$, $z=0$, so we can write $x = 5\cos t$, $y = 5\sin t$. C also lies on the cylinder $y^2+z^2=20$, and $z\geq 0$ near the point $(3,4,2)$, so we can write $z = \sqrt{20-y^2} = \sqrt{20-25\sin^2 t}$. A vector equation then for C is $\mathbf{r}(t) = \left\langle 5\cos t, 5\sin t, \sqrt{20-25\sin^2 t}\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle -5\sin t, 5\cos t, \frac{1}{2}(20-25\sin^2 t)^{-1/2}(-50\sin t\cos t)\right\rangle$.

The point $(3,4,2)$ corresponds to $t = \cos^{-1}\left(\frac{3}{5}\right)$, so the tangent vector there is

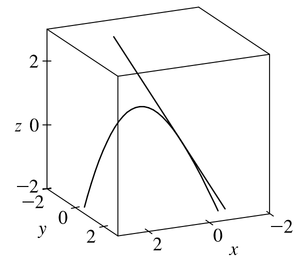
$\mathbf{r}'\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \left\langle -5\left(\frac{4}{5}\right), 5\left(\frac{3}{5}\right), \frac{1}{2}\left(20-25\left(\frac{4}{5}\right)^2\right)^{-1/2}\left(-50\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\right)\right\rangle = \langle -4,3,-6\rangle$.

The tangent line is parallel to this vector and passes through $(3,4,2)$, so a vector equation for the line is $\mathbf{r}(t) = (3-4t)\mathbf{i}+(4+3t)\mathbf{j}+(2-6t)\mathbf{k}$.

30. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t\rangle \Rightarrow \mathbf{r}'(t) = \langle -2\sin t, 2\cos t, e^t\rangle$. The tangent line to the curve is parallel to the plane when the curve's tangent vector is orthogonal to the plane's normal vector. Thus we require $\langle -2\sin t, 2\cos t, e^t\rangle\cdot\langle\sqrt{3},1,0\rangle = 0 \Rightarrow$ $-2\sqrt{3}\sin t+2\cos t+0 = 0 \Rightarrow \tan t = \frac{1}{\sqrt{3}} \Rightarrow t = \frac{\pi}{6}$ [since $0\leq t\leq\pi$].

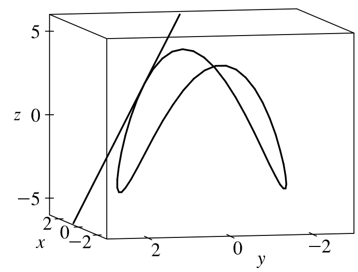
$\mathbf{r}\left(\frac{\pi}{6}\right) = \left\langle\sqrt{3}, 1, e^{\pi/6}\right\rangle$, so the point is $(\sqrt{3}, 1, e^{\pi/6})$.

31. $\mathbf{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle$. At $(0, 1, 0)$, $t = 0$ and $\mathbf{r}'(0) = \langle 1, -1, 2 \rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 1 - t$, $z = 2t$.

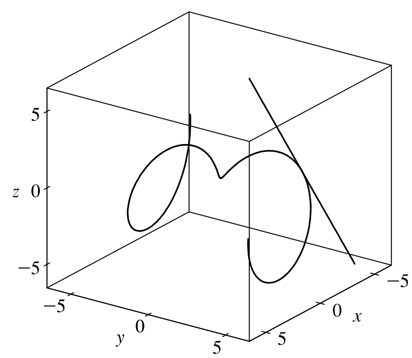


32. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t \rangle$,

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, -8\sin 2t \rangle$. At $(\sqrt{3}, 1, 2)$, $t = \frac{\pi}{6}$ and $\mathbf{r}'(\frac{\pi}{6}) = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$. Thus, parametric equations of the tangent line are $x = \sqrt{3} - t$, $y = 1 + \sqrt{3}\,t$, $z = 2 - 4\sqrt{3}\,t$.



33. $\mathbf{r}(t) = \langle t\cos t, t, t\sin t \rangle \Rightarrow \mathbf{r}'(t) = \langle \cos t - t\sin t, 1, t\cos t + \sin t \rangle$. At $(-\pi, \pi, 0)$, $t = \pi$ and $\mathbf{r}'(\pi) = \langle -1, 1, -\pi \rangle$. Thus, parametric equations of the tangent line are $x = -\pi - t$, $y = \pi + t$, $z = -\pi t$.



34. (a) The tangent line to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2\sin \pi t, \cos \pi t \rangle$ at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0 \rangle = \langle 0, 0, 1 \rangle$, and in the direction of the tangent vector, $\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0 \rangle = \langle \pi, 2\pi, 0 \rangle$. So an equation of the line is $\langle x, y, z \rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1 \rangle = \langle \pi u, 2\pi u, 1 \rangle$.

$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2} \right\rangle = \langle 1, 2, 0 \rangle$,

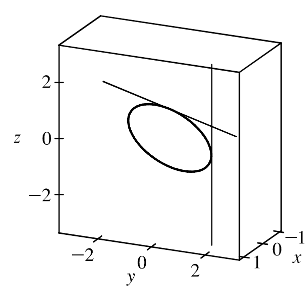
$\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2} \right\rangle = \langle 0, 0, -\pi \rangle$.

So the equation of the second line is

$\langle x, y, z \rangle = \mathbf{r}\left(\frac{1}{2}\right) + v\mathbf{r}'\left(\frac{1}{2}\right) = \langle 1, 2, 0 \rangle + v\,\langle 0, 0, -\pi \rangle = \langle 1, 2, -\pi v \rangle$.

The lines intersect where $\langle \pi u, 2\pi u, 1 \rangle = \langle 1, 2, -\pi v \rangle$, so the point of intersection is $(1, 2, 1)$.

(b)



35. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}_2'(0) = \langle 1, 2, 1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}} \langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

36. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle θ of intersection, we proceed as in Exercise 35. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}_1'(1) = \langle 1, -1, 2\rangle$ and $\mathbf{r}_2'(2) = \langle -1, 1, 4\rangle$. So

$\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

Note: In Exercise 35, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

37. $$\begin{aligned}\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt &= \left(\int_0^2 t\,dt\right)\mathbf{i} - \left(\int_0^2 t^3\,dt\right)\mathbf{j} + \left(\int_0^2 3t^5\,dt\right)\mathbf{k} \\ &= \left[\tfrac{1}{2}t^2\right]_0^2\mathbf{i} - \left[\tfrac{1}{4}t^4\right]_0^2\mathbf{j} + \left[\tfrac{1}{2}t^6\right]_0^2\mathbf{k} \\ &= \tfrac{1}{2}(4-0)\,\mathbf{i} - \tfrac{1}{4}(16-0)\,\mathbf{j} + \tfrac{1}{2}(64-0)\,\mathbf{k} = 2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}\end{aligned}$$

38. $$\begin{aligned}\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right)dt &= \left(\int_1^4 2t^{3/2}\,dt\right)\mathbf{i} + \left[\int_1^4 (t^{3/2} + t^{1/2})\,dt\right]\mathbf{k} \\ &= \left[\tfrac{4}{5}t^{5/2}\right]_1^4\mathbf{i} + \left[\tfrac{2}{5}t^{5/2} + \tfrac{2}{3}t^{3/2}\right]_1^4\mathbf{k} \\ &= \tfrac{4}{5}(4^{5/2} - 1)\,\mathbf{i} + \left(\tfrac{2}{5}(4)^{5/2} + \tfrac{2}{3}(4)^{3/2} - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} \\ &= \tfrac{4}{5}(31)\,\mathbf{i} + \left(\tfrac{2}{5}(32) + \tfrac{2}{3}(8) - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} = \tfrac{124}{5}\,\mathbf{i} + \tfrac{256}{15}\,\mathbf{k}\end{aligned}$$

39. $$\begin{aligned}\int_0^1 \left(\frac{1}{t+1}\,\mathbf{i} + \frac{1}{t^2+1}\,\mathbf{j} + \frac{t}{t^2+1}\,\mathbf{k}\right)dt &= \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t^2+1}\,dt\right)\mathbf{j} + \left(\int_0^1 \frac{t}{t^2+1}\,dt\right)\mathbf{k} \\ &= \left[\ln|t+1|\right]_0^1\mathbf{i} + \left[\tan^{-1}t\right]_0^1\mathbf{j} + \left[\tfrac{1}{2}\ln(t^2+1)\right]_0^1\mathbf{k} \\ &= (\ln 2 - \ln 1)\,\mathbf{i} + \left(\tfrac{\pi}{4} - 0\right)\mathbf{j} + \tfrac{1}{2}(\ln 2 - \ln 1)\,\mathbf{k} = \ln 2\,\mathbf{i} + \tfrac{\pi}{4}\,\mathbf{j} + \tfrac{1}{2}\ln 2\,\mathbf{k}\end{aligned}$$

40. $\int_0^{\pi/4}(\sec t\tan t\,\mathbf{i} + t\cos 2t\,\mathbf{j} + \sin^2 2t\cos 2t\,\mathbf{k})\,dt$

$$\begin{aligned}&= \left(\int_0^{\pi/4}\sec t\tan t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/4} t\cos 2t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/4}\sin^2 2t\cos 2t\,dt\right)\mathbf{k} \\ &= \left[\sec t\right]_0^{\pi/4}\mathbf{i} + \left(\left[\tfrac{1}{2}t\sin 2t\right]_0^{\pi/4} - \int_0^{\pi/4}\tfrac{1}{2}\sin 2t\,dt\right)\mathbf{j} + \left[\tfrac{1}{6}\sin^3 2t\right]_0^{\pi/4}\mathbf{k}\end{aligned}$$

[For the y-component, integrate by parts with $u = t$, $dv = \cos 2t\,dt$.]

$$\begin{aligned}&= \left(\sec\tfrac{\pi}{4} - \sec 0\right)\mathbf{i} + \left(\tfrac{\pi}{8}\sin\tfrac{\pi}{2} - 0 - \left[-\tfrac{1}{4}\cos 2t\right]_0^{\pi/4}\right)\mathbf{j} + \tfrac{1}{6}\left(\sin^3\tfrac{\pi}{2} - \sin^3 0\right)\mathbf{k} \\ &= (\sqrt{2} - 1)\,\mathbf{i} + \left(\tfrac{\pi}{8} + \tfrac{1}{4}\cos\tfrac{\pi}{2} - \tfrac{1}{4}\cos 0\right)\mathbf{j} + \tfrac{1}{6}(1-0)\,\mathbf{k} = (\sqrt{2}-1)\,\mathbf{i} + \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right)\mathbf{j} + \tfrac{1}{6}\,\mathbf{k}\end{aligned}$$

41. $$\begin{aligned}\int\left(\frac{1}{1+t^2}\,\mathbf{i} + te^{t^2}\,\mathbf{j} + \sqrt{t}\,\mathbf{k}\right)dt &= \left(\int\frac{1}{1+t^2}\,dt\right)\mathbf{i} + \left(\int te^{t^2}\,dt\right)\mathbf{j} + \left(\int\sqrt{t}\,dt\right)\mathbf{k} \\ &= \tan^{-1}t\,\mathbf{i} + \tfrac{1}{2}e^{t^2}\,\mathbf{j} + \tfrac{2}{3}t^{3/2}\,\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C}$ is a vector constant of integration.

42. $\displaystyle\int \left(t\cos t^2\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right) dt = \left(\int t\cos t^2\,dt\right)\mathbf{i} + \left(\int \frac{1}{t}\,dt\right)\mathbf{j} + \left(\int \sec^2 t\,dt\right)\mathbf{k}$

$$= \tfrac{1}{2}\sin t^2\,\mathbf{i} + \ln|t|\,\mathbf{j} + \tan t\,\mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration.

43. $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + \frac{2}{3}t^{3/2}\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.

But $\mathbf{i} + \mathbf{j} = \mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{2}{3}\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = -\frac{2}{3}\mathbf{k}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$.

44. $\mathbf{r}'(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + te^t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(t) = \frac{1}{2}t^2\,\mathbf{i} + e^t\,\mathbf{j} + \left(te^t - e^t\right)\,\mathbf{k} + \mathbf{C}$. But $\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C}$.

Thus $\mathbf{C} = \mathbf{i} + 2\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + e^t\,\mathbf{j} + \left(te^t - e^t + 2\right)\mathbf{k}$.

For Exercises 45–48, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

45.
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] &= \frac{d}{dt}\left\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\right\rangle \\ &= \left\langle \frac{d}{dt}\left[u_1(t) + v_1(t)\right], \frac{d}{dt}\left[u_2(t) + v_2(t)\right], \frac{d}{dt}\left[u_3(t) + v_3(t)\right]\right\rangle \\ &= \left\langle u_1'(t) + v_1'(t), u_2'(t) + v_2'(t), u_3'(t) + v_3'(t)\right\rangle \\ &= \left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle + \left\langle v_1'(t), v_2'(t), v_3'(t)\right\rangle = \mathbf{u}'(t) + \mathbf{v}'(t)\end{aligned}$$

46.
$$\begin{aligned}\frac{d}{dt}\left[f(t)\,\mathbf{u}(t)\right] &= \frac{d}{dt}\left\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\right\rangle \\ &= \left\langle \frac{d}{dt}\left[f(t)u_1(t)\right], \frac{d}{dt}\left[f(t)u_2(t)\right], \frac{d}{dt}\left[f(t)u_3(t)\right]\right\rangle \\ &= \left\langle f'(t)u_1(t) + f(t)u_1'(t), f'(t)u_2(t) + f(t)u_2'(t), f'(t)u_3(t) + f(t)u_3'(t)\right\rangle \\ &= f'(t)\left\langle u_1(t), u_2(t), u_3(t)\right\rangle + f(t)\left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)\end{aligned}$$

47.
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] &= \frac{d}{dt}\left\langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t), u_1(t)v_2(t) - u_2(t)v_1(t)\right\rangle \\ &= \langle u_2'v_3(t) + u_2(t)v_3'(t) - u_3'(t)v_2(t) - u_3(t)v_2'(t), \\ &\qquad u_3'(t)v_1(t) + u_3(t)v_1'(t) - u_1'(t)v_3(t) - u_1(t)v_3'(t), \\ &\qquad\quad u_1'(t)v_2(t) + u_1(t)v_2'(t) - u_2'(t)v_1(t) - u_2(t)v_1'(t)\rangle \\ &= \left\langle u_2'(t)v_3(t) - u_3'(t)v_2(t), u_3'(t)v_1(t) - u_1'(t)v_3(t), u_1'(t)v_2(t) - u_2'(t)v_1(t)\right\rangle \\ &\qquad + \left\langle u_2(t)v_3'(t) - u_3(t)v_2'(t), u_3(t)v_1'(t) - u_1(t)v_3'(t), u_1(t)v_2'(t) - u_2(t)v_1'(t)\right\rangle \\ &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)\end{aligned}$$

Alternate solution: Let $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h) - \mathbf{r}(t) &= \left[\mathbf{u}(t+h) \times \mathbf{v}(t+h)\right] - \left[\mathbf{u}(t) \times \mathbf{v}(t)\right] \\ &= \left[\mathbf{u}(t+h) \times \mathbf{v}(t+h)\right] - \left[\mathbf{u}(t) \times \mathbf{v}(t)\right] + \left[\mathbf{u}(t+h) \times \mathbf{v}(t)\right] - \left[\mathbf{u}(t+h) \times \mathbf{v}(t)\right] \\ &= \mathbf{u}(t+h) \times \left[\mathbf{v}(t+h) - \mathbf{v}(t)\right] + \left[\mathbf{u}(t+h) - \mathbf{u}(t)\right] \times \mathbf{v}(t)\end{aligned}$$

[continued]

(Be careful of the order of the cross product.) Dividing through by h and taking the limit as $h \to 0$ we have

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{u}(t+h) \times [\mathbf{v}(t+h) - \mathbf{v}(t)]}{h} + \lim_{h\to 0} \frac{[\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)}{h} = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

by Exercise 13.1.61(a) and Definition 1.

48. $\frac{d}{dt}[\mathbf{u}(f(t))] = \frac{d}{dt}\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\rangle = \left\langle \frac{d}{dt}[u_1(f(t))], \frac{d}{dt}[u_2(f(t))], \frac{d}{dt}[u_3(f(t))]\right\rangle$

$= \langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\rangle = f'(t)\,\mathbf{u}'(t)$

49. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$= \langle \cos t, -\sin t, 1\rangle \cdot \langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle \cdot \langle 1, -\sin t, \cos t\rangle$

$= t\cos t - \cos t\,\sin t + \sin t + \sin t - \cos t\,\sin t + t\cos t$

$= 2t\cos t + 2\sin t - 2\cos t\,\sin t$

50. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$= \langle \cos t, -\sin t, 1\rangle \times \langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle \times \langle 1, -\sin t, \cos t\rangle$

$= \left\langle -\sin^2 t - \cos t, t - \cos t\,\sin t, \cos^2 t + t\sin t\right\rangle$

$+ \left\langle \cos^2 t + t\sin t, t - \cos t\,\sin t, -\sin^2 t - \cos t\right\rangle$

$= \left\langle \cos^2 t - \sin^2 t - \cos t + t\sin t, 2t - 2\cos t\,\sin t, \cos^2 t - \sin^2 t - \cos t + t\sin t\right\rangle$

51. By Formula 4 of Theorem 3, $f'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$, and $\mathbf{v}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$, so

$f'(2) = \mathbf{u}'(2)\cdot\mathbf{v}(2) + \mathbf{u}(2)\cdot\mathbf{v}'(2) = \langle 3,0,4\rangle\cdot\langle 2,4,8\rangle + \langle 1,2,-1\rangle\cdot\langle 1,4,12\rangle = 6+0+32+1+8-12 = 35.$

52. By Formula 5 of Theorem 3, $\mathbf{r}'(t) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$, so

$$\mathbf{r}'(2) = \mathbf{u}'(2)\times\mathbf{v}(2) + \mathbf{u}(2)\times\mathbf{v}'(2) = \langle 3,0,4\rangle\times\langle 2,4,8\rangle + \langle 1,2,-1\rangle\times\langle 1,4,12\rangle$$

$$= \langle -16,-16,12\rangle + \langle 28,-13,2\rangle = \langle 12,-29,14\rangle$$

53. $\mathbf{r}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \quad\Rightarrow\quad \mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t$ by Formulas 1 and 3 of Theorem 3. Then

$\mathbf{r}(t)\times\mathbf{r}'(t) = (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t)$

$= (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t) + (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(\mathbf{b}\,\omega\cos\omega t)$

[by Property 3 of Theorem 12.4.11]

$= \mathbf{a}\cos\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{b}\sin\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{a}\cos\omega t\times\mathbf{b}\,\omega\cos\omega t + \mathbf{b}\sin\omega t\times\mathbf{b}\,\omega\cos\omega t$

[by Property 4]

$= (\cos\omega t)(-\omega\sin\omega t)(\mathbf{a}\times\mathbf{a}) + (\sin\omega t)(-\omega\sin\omega t)(\mathbf{b}\times\mathbf{a}) + (\cos\omega t)(\omega\cos\omega t)(\mathbf{a}\times\mathbf{b})$

$+ (\sin\omega t)(\omega\cos\omega t)(\mathbf{b}\times\mathbf{b})$ [by Property 2]

$= \mathbf{0} + \left(\omega\sin^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \left(\omega\cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \mathbf{0}$ [by Property 1 and Example 12.4.2]

$= \omega\left(\sin^2\omega t + \cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) = \omega(\mathbf{a}\times\mathbf{b}) = \omega\,\mathbf{a}\times\mathbf{b}$ [by Property 2]

54. From Exercise 53, $\mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t \quad\Rightarrow\quad \mathbf{r}''(t) = -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t$. Then

$$\begin{aligned}\mathbf{r}''(t) + \omega^2\mathbf{r}(t) &= \left(-\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t\right) + \omega^2\left(\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t\right)\\ &= -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t + \mathbf{a}\,\omega^2\cos\omega t + \mathbf{b}\,\omega^2\sin\omega t = \mathbf{0}\end{aligned}$$

55. $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (by Example 12.4.2).

Thus, $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

56.
$$\begin{aligned}\frac{d}{dt}\left(\mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\right) &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\frac{d}{dt}\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}'(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] - \mathbf{v}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{w}(t)\right] + \mathbf{w}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]\end{aligned}$$

57. $\dfrac{d}{dt}\left|\mathbf{r}(t)\right| = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{1/2} = \frac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\left[2\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] = \dfrac{1}{\left|\mathbf{r}(t)\right|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$

58. Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, by Formula 4 of Theorem 3 we have

$\dfrac{d}{dt}\left|\mathbf{r}(t)\right|^2 = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\left[\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] = 0$. This is true for all t, thus $\left|\mathbf{r}(t)\right|^2$, and so $\left|\mathbf{r}(t)\right|$ is a constant, and hence the curve lies on a sphere with center the origin.

59. Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$,

$$\begin{aligned}\mathbf{u}'(t) &= \mathbf{r}'(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right] + \mathbf{r}(t)\cdot\frac{d}{dt}\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\\ &= 0 + \mathbf{r}(t)\cdot\left[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)\right] && \left[\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\\ &= \mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right] && \left[\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}\right]\end{aligned}$$

60. The tangent vector $\mathbf{r}'(t)$ is defined as $\displaystyle\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$. Here we assume that this limit exists and $\mathbf{r}'(t)\neq\mathbf{0}$; then we know that this vector lies on the tangent line to the curve. As in Figure 1, let points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$. The vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ points from P to Q, so $\mathbf{r}(t+h)-\mathbf{r}(t) = \overrightarrow{PQ}$. If $h>0$ then $t<t+h$, so Q lies "ahead" of P on the curve. If h is sufficiently small (we can take h to be as small as we like since $h\to 0$) then $\overrightarrow{PQ}$ approximates the curve from P to Q and hence points approximately in the direction of the curve as t increases. Since h is positive, $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the same direction. If $h<0$, then $t>t+h$ so Q lies "behind" P on the curve. For h sufficiently small, $\overrightarrow{PQ}$ approximates the curve but points in the direction of decreasing t. However, h is negative, so $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the opposite direction, that is, in the direction of increasing t. In both cases, the difference quotient $\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the direction of increasing t. The tangent vector $\mathbf{r}'(t)$ is the limit of this difference quotient, so it must also point in the direction of increasing t.

13.3 Arc Length and Curvature

1. (a) $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = \langle 3-t, 2t, 4t+1\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = \langle -1, 2, 4\rangle$. Then

$$L = \int_1^3 \sqrt{(-1)^2 + 2^2 + 4^2}\,dt = \int_1^3 \sqrt{21}\,dt = \Big[t\sqrt{21}\Big]_1^3 = 3\sqrt{21} - \sqrt{21} = 2\sqrt{21}$$

(b) $\mathbf{r}(1) = \langle 2,2,5\rangle \quad\Rightarrow\quad P_1 = (2,2,5)$; $\mathbf{r}(3) = \langle 0,6,13\rangle \quad\Rightarrow\quad P_2 = (0,6,13)$. Then

$$|P_2P_1| = \sqrt{(0-2)^2 + (6-2)^2 + (13-5)^2} = \sqrt{84} = 2\sqrt{21}$$

2. (a) $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} = (t+2)\,\mathbf{i} - t\,\mathbf{j} + (3t-5)\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Then

$$L = \int_{-1}^2 \sqrt{1^2 + (-1)^2 + 3^2}\,dt = \int_{-1}^2 \sqrt{11}\,dt = \Big[t\sqrt{11}\Big]_{-1}^2 = 2\sqrt{11} + \sqrt{11} = 3\sqrt{11}$$

(b) $\mathbf{r}(-1) = \mathbf{i} + \mathbf{j} - 8\,\mathbf{k} \quad\Rightarrow\quad P_1 = (1,1,-8)$; $\mathbf{r}(2) = 4\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad P_2 = (4,-2,1)$. Then

$$|P_2P_1| = \sqrt{(4-1)^2 + (-2-1)^2 + [1-(-8)]^2} = \sqrt{99} = 3\sqrt{11}$$

3. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{1 + 9(\sin^2 t + \cos^2 t)} = \sqrt{10}.$

Then using Formula 3, we have $L = \int_{-5}^5 |\mathbf{r}'(t)|\,dt = \int_{-5}^5 \sqrt{10}\,dt = \sqrt{10}\,t\Big]_{-5}^5 = 10\sqrt{10}.$

4. $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 2, 2t, t^2\right\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$ for $0 \le t \le 1$. Then using Formula 3, we have

$L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (2+t^2)\,dt = 2t + \frac{1}{3}t^3\Big]_0^1 = \frac{7}{3}.$

5. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}.$

6. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{-\sin t}{\cos t}\,\mathbf{k} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \tan t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + (-\tan t)^2} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$. Since $\sec t > 0$ for $0 \le t \le \pi/4$, here we can say $|\mathbf{r}'(t)| = \sec t$. Then

$$\begin{aligned} L &= \int_0^{\pi/4} \sec t\,dt = \Big[\ln|\sec t + \tan t|\Big]_0^{\pi/4} = \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\ &= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln(\sqrt{2} + 1) \end{aligned}$$

7. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ [since $t \ge 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \frac{1}{18} \cdot \frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8).$

8. $\mathbf{r}(t) = t^2\,\mathbf{i} + 9t\,\mathbf{j} + 4t^{3/2}\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2t\,\mathbf{i} + 9\,\mathbf{j} + 6\sqrt{t}\,\mathbf{k} \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + 81 + 36t} = \sqrt{(2t+9)^2} = |2t+9| = 2t+9$ [since $2t + 9 \geq 0$ for $1 \leq t \leq 4$]. Then

$L = \int_1^4 |\mathbf{r}'(t)|\,dt = \int_1^4 (2t+9)\,dt = \Big[t^2 + 9t\Big]_1^4 = 52 - 10 = 42.$

9. $\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2} = \sqrt{4t^2 + 9t^4 + 16t^6}$, so

$L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\,dt \approx 18.6833.$

10. $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t}\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, -e^{-t}, (1-t)e^{-t}\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{1^2 + (-e^{-t})^2 + [(1-t)e^{-t}]^2} = \sqrt{1 + e^{-2t} + (1-t)^2 e^{-2t}} = \sqrt{1 + (2 - 2t + t^2)e^{-2t}}$, so

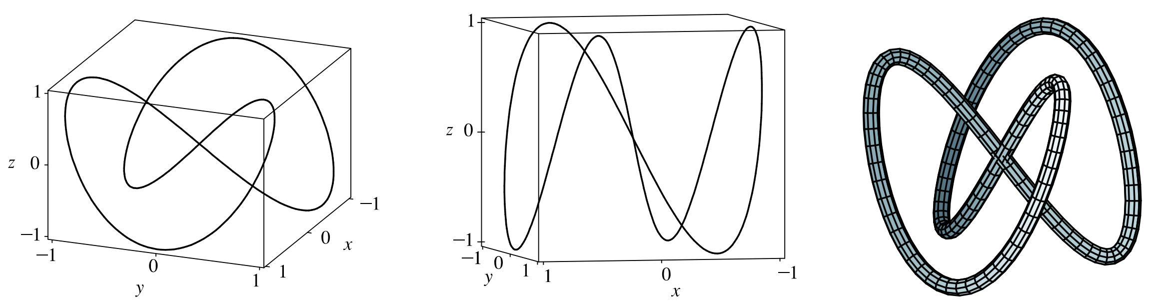
$L = \int_1^3 |\mathbf{r}'(t)|\,dt = \int_1^3 \sqrt{1 + (2 + 2t + t^2)e^{-2t}}\,dt \approx 2.0454.$

11. $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -\pi \sin \pi t, 2, 2\pi \cos 2\pi t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{\pi^2 \sin^2 \pi t + 4 + 4\pi^2 \cos^2 2\pi t}.$

The point $(1, 0, 0)$ corresponds to $t = 0$ and $(1, 4, 0)$ corresponds to $t = 2$, so the length is

$L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \sqrt{\pi^2 \sin^2 \pi t + 4 + 4\pi^2 \cos^2 2\pi t}\,dt \approx 10.3311.$

12. We plot two different views of the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. To help visualize the curve, we also include a plot showing a tube of radius 0.07 around the curve.



The complete curve is given by the parameter interval $[0, 2\pi]$ and we have $\mathbf{r}'(t) = \langle \cos t, 2\cos 2t, 3\cos 3t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + 4\cos^2 2t + 9\cos^2 3t}$, so $L = \int_0^{2\pi} |\mathbf{r}'(t)|\;dt = \int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t + 9\cos^2 3t}\,dt \approx 16.0264.$

13. The projection of the curve C onto the xy-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t \;\Rightarrow\; y = \frac{1}{2}t^2$. Since C also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for C are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$\begin{aligned} L &= \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\langle 1, t, \tfrac{1}{2}t^2\rangle\right|\,dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt \\ &= \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1 + \tfrac{1}{2}t^2)\,dt = \left[t + \tfrac{1}{6}t^3\right]_0^6 = 6 + 36 = 42 \end{aligned}$$

14. Let C be the curve of intersection. The projection of C onto the xy-plane is the ellipse $4x^2 + y^2 = 4$ or $x^2 + y^2/4 = 1$, $z = 0$. Then we can write $x = \cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$. Since C also lies on the plane $x + y + z = 2$, we have

$z = 2 - x - y = 2 - \cos t - 2\sin t$. Then parametric equations for C are $x = \cos t$, $y = 2\sin t$, $z = 2 - \cos t - 2\sin t$, $0 \le t \le 2\pi$, and the corresponding vector equation is $\mathbf{r}(t) = \langle \cos t, 2\sin t, 2 - \cos t - 2\sin t\rangle$. Differentiating gives $\mathbf{r}'(t) = \langle -\sin t, 2\cos t, \sin t - 2\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (2\cos t)^2 + (\sin t - 2\cos t)^2} = \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}$. The length of C is

$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}\,dt \approx 13.5191$.

15. (a) $\mathbf{r}(t) = (5 - t)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + 3t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = -\mathbf{i} + 4\,\mathbf{j} + 3\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{1 + 16 + 9} = \sqrt{26}$. The point $P\,(4, 1, 3)$ corresponds to $t = 1$, so the arc length function from P is

$s(t) = \int_1^t |\mathbf{r}'(u)|\,du = \int_1^t \sqrt{26}\,du = \sqrt{26}\,u\big|_1^t = \sqrt{26}\,(t - 1)$. Since $s = \sqrt{26}\,(t - 1)$, we have $t = \dfrac{s}{\sqrt{26}} + 1$.

Substituting for t in the original equation, the reparametrization of the curve with respect to arc length is

$$\begin{aligned}\mathbf{r}(t(s)) &= \left[5 - \left(\frac{s}{\sqrt{26}} + 1\right)\right]\mathbf{i} + \left[4\left(\frac{s}{\sqrt{26}} + 1\right) - 3\right]\mathbf{j} + 3\left(\frac{s}{\sqrt{26}} + 1\right)\mathbf{k} \\ &= \left(4 - \frac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\frac{4s}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\frac{3s}{\sqrt{26}} + 3\right)\mathbf{k}\end{aligned}$$

(b) The point 4 units along the curve from P has position vector

$\mathbf{r}(t(4)) = \left(4 - \dfrac{4}{\sqrt{26}}\right)\mathbf{i} + \left(\dfrac{4(4)}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\dfrac{3(4)}{\sqrt{26}} + 3\right)\mathbf{k}$, so the point is $\left(4 - \dfrac{4}{\sqrt{26}}, \dfrac{16}{\sqrt{26}} + 1, \dfrac{12}{\sqrt{26}} + 3\right)$.

16. (a) $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = e^t\,(\cos t + \sin t)\,\mathbf{i} + e^t\,(\cos t - \sin t)\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k}$ and

$$\begin{aligned}\frac{ds}{dt} &= |\mathbf{r}'(t)| = \sqrt{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2 + 2e^{2t}} \\ &= \sqrt{e^{2t}\left[2(\cos^2 t + \sin^2 t) + 2\cos t\sin t - 2\cos t\sin t + 2\right]} = \sqrt{4e^{2t}} = 2e^t\end{aligned}$$

The point $P\left(0, 1, \sqrt{2}\right)$ corresponds to $t = 0$, so the arc length function from P is

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2e^u\,du = 2e^u\big|_0^t = 2(e^t - 1)$. Since $s = 2(e^t - 1)$, we have $e^t = \dfrac{s}{2} + 1 \quad \Leftrightarrow$

$t = \ln\left(\frac{1}{2}s + 1\right)$. Substituting for t in the original equation, the reparametrization of the curve with respect to arc length is

$\mathbf{r}(t(s)) = \left(\frac{1}{2}s + 1\right)\sin\!\left(\ln\left(\frac{1}{2}s + 1\right)\right)\mathbf{i} + \left(\frac{1}{2}s + 1\right)\cos\!\left(\ln\left(\frac{1}{2}s + 1\right)\right)\mathbf{j} + \left(\frac{\sqrt{2}}{2}s + \sqrt{2}\right)\mathbf{k}$.

(b) The point 4 units along the curve from P has position vector

$\mathbf{r}(t(4)) = \left(\frac{1}{2}(4) + 1\right)\sin\!\left(\ln\left(\frac{1}{2}(4) + 1\right)\right)\mathbf{i} + \left(\frac{1}{2}(4) + 1\right)\cos\!\left(\ln\left(\frac{1}{2}(4) + 1\right)\right)\mathbf{j} + \left(\frac{\sqrt{2}}{2}(4) + \sqrt{2}\right)\mathbf{k}$, so the point is $\left(3\sin(\ln 3), 3\cos(\ln 3), 3\sqrt{2}\right)$.

17. Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0, 0, 3)$ corresponds to $t = 0$, so the arc length function beginning at $(0, 0, 3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $\quad s(t) = 5 \quad \Rightarrow \quad 5t = 5 \quad \Rightarrow \quad t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

18. $\mathbf{r}(t) = \left(\dfrac{2}{t^2+1} - 1\right)\mathbf{i} + \dfrac{2t}{t^2+1}\,\mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}} = \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}.$$

Since the initial point $(1, 0)$ corresponds to $t = 0$, the arc length function is

$s(t) = \displaystyle\int_0^t \left|\mathbf{r}'(u)\right| du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \quad \Rightarrow \quad t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{j}$$

$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j} = \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1, 0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

19. (a) $\mathbf{r}(t) = \left\langle t^2, \sin t - t\cos t, \cos t + t\sin t\right\rangle \quad \Rightarrow$

$\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t,\ -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\cos^2 t + \sin^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$ [since $t > 0$]. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \dfrac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\quad \mathbf{T}'(t) = \dfrac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \quad \Rightarrow$

$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \dfrac{1}{\sqrt{5}}$. Thus, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$.

20. (a) $\mathbf{r}(t) = \langle 5\sin t, t, 5\cos t\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 5\cos t, 1, -5\sin t\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{25\cos^2 t + 1 + 25\sin^2 t} = \sqrt{26}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{26}}\langle 5\cos t, 1, -5\sin t\rangle$. $\quad \mathbf{T}'(t) = \dfrac{1}{\sqrt{26}}\langle -5\sin t, 0, -5\cos t\rangle \quad \Rightarrow$

$|\mathbf{T}'(t)| = \dfrac{1}{\sqrt{26}}\sqrt{25\sin^2 t + 0^2 + 25\cos^2 t} = \dfrac{5}{\sqrt{26}}$. Thus,

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\sqrt{26}}{5}\cdot\dfrac{1}{\sqrt{26}}\langle -5\sin t, 0, -5\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{5/\sqrt{26}}{\sqrt{26}} = \dfrac{5}{26}$.

21. (a) $\mathbf{r}(t) = \left\langle t, t^2, 4\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 0\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 0}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1, 2t, 0\rangle$.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{1+4t^2}}\langle 0, 2, 0\rangle - \frac{4t}{(1+4t^2)^{3/2}}\langle 1, 2t, 0\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(1+4t^2)^{3/2}}\left[(1+4t^2)\langle 0, 2, 0\rangle - 4t\langle 1, 2t, 0\rangle\right] = \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2, 0\rangle$$

[continued]

$$|\mathbf{T}'(t)| = \frac{1}{(1+4t^2)^{3/2}}\sqrt{16t^2+4} = \frac{2}{1+4t^2}.\text{ Thus,}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1+4t^2}{2}\cdot\frac{1}{(1+4t^2)^{3/2}}\langle -4t,2,0\rangle = \frac{1}{\sqrt{1+4t^2}}\langle -2t,1,0\rangle.$$

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(1+4t^2)}{\sqrt{1+4t^2}} = \dfrac{2}{(1+4t^2)^{3/2}}$.

22. (a) $\mathbf{r}(t) = \left\langle t, t, \frac{1}{2}t^2\right\rangle \Rightarrow \mathbf{r}'(t) = \langle 1,1,t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+1+t^2} = \sqrt{2+t^2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2+t^2}}\langle 1,1,t\rangle$.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2+t^2}}\langle 0,0,1\rangle - \frac{t}{(2+t^2)^{3/2}}\langle 1,1,t\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(2+t^2)^{3/2}}\left[(2+t^2)\langle 0,0,1\rangle - t\langle 1,1,t\rangle\right] = \frac{1}{(2+t^2)^{3/2}}\langle -t,-t,2\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(2+t^2)^{3/2}}\sqrt{4+2t^2} = \frac{\sqrt{2}}{2+t^2}.\text{ Thus,}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2+t^2}{\sqrt{2}}\cdot\frac{1}{(2+t^2)^{3/2}}\langle -t,-t,2\rangle = \frac{1}{\sqrt{4+2t^2}}\langle -t,-t,2\rangle.$$

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}/(2+t^2)}{\sqrt{2+t^2}} = \dfrac{\sqrt{2}}{(2+t^2)^{3/2}}$.

23. (a) $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, t^2\right\rangle \Rightarrow \mathbf{r}'(t) = \langle 1,t,2t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+5t^2}}\langle 1,t,2t\rangle$.

$$\mathbf{T}'(t) = \frac{-5t}{(1+5t^2)^{3/2}}\langle 1,t,2t\rangle + \frac{1}{\sqrt{1+5t^2}}\langle 0,1,2\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(1+5t^2)^{3/2}}\left(\left\langle -5t,-5t^2,-10t^2\right\rangle + \left\langle 0,1+5t^2,2+10t^2\right\rangle\right) = \frac{1}{(1+5t^2)^{3/2}}\langle -5t,1,2\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+1+4} = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+5} = \frac{\sqrt{5}\sqrt{5t^2+1}}{(1+5t^2)^{3/2}} = \frac{\sqrt{5}}{1+5t^2}.$$

Thus, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1+5t^2}{\sqrt{5}}\cdot\dfrac{1}{(1+5t^2)^{3/2}}\langle -5t,1,2\rangle = \dfrac{1}{\sqrt{5+25t^2}}\langle -5t,1,2\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{5}/(1+5t^2)}{\sqrt{1+5t^2}} = \dfrac{\sqrt{5}}{(1+5t^2)^{3/2}}$.

24. (a) $\mathbf{r}(t) = \left\langle \sqrt{2}\,t, e^t, e^{-t}\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t+e^{-t}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t+e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle = \dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle \quad \left[\text{after multiplying by } \dfrac{e^t}{e^t}\right]$.

[continued]

$$\mathbf{T}'(t) = \frac{1}{e^{2t}+1}\left\langle\sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\left\langle\sqrt{2}\,e^t, e^{2t}, -1\right\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle\sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle\sqrt{2}\,e^t, e^{2t}, -1\right\rangle\right] = \frac{1}{(e^{2t}+1)^2}\left\langle\sqrt{2}\,e^t\left(1-e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t})+4e^{4t}+4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1+e^{2t}\right)^2} = \frac{\sqrt{2}\,e^t(1+e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}$$

Thus, $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\,\frac{1}{(e^{2t}+1)^2}\left\langle\sqrt{2}\,e^t(1-e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle$$

$$= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle\sqrt{2}\,e^t(1-e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle = \frac{1}{e^{2t}+1}\left\langle 1-e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\right\rangle$$

(b) By Formula 9, the curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\frac{1}{e^t+e^{-t}} = \frac{\sqrt{2}\,e^t}{e^{3t}+2e^t+e^{-t}} = \frac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1} = \frac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}.$$

25. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 3t^2\,\mathbf{j} + 2t\,\mathbf{k},\quad \mathbf{r}''(t) = 6t\,\mathbf{j} + 2\,\mathbf{k},\quad |\mathbf{r}'(t)| = \sqrt{0^2+(3t^2)^2+(2t)^2} = \sqrt{9t^4+4t^2},$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = -6t^2\,\mathbf{i},\quad |\mathbf{r}'(t)\times\mathbf{r}''(t)| = 6t^2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{6t^2}{\left(\sqrt{9t^4+4t^2}\right)^3} = \dfrac{6t^2}{(9t^4+4t^2)^{3/2}}.$

26. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + e^t\,\mathbf{k},\quad \mathbf{r}''(t) = 2\,\mathbf{j} + e^t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{1^2+(2t)^2+(e^t)^2} = \sqrt{1+4t^2+e^{2t}},\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = (2t-2)e^t\,\mathbf{i} - e^t\,\mathbf{j} + 2\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{[(2t-2)e^t]^2+(-e^t)^2+2^2} = \sqrt{(2t-2)^2e^{2t}+e^{2t}+4} = \sqrt{(4t^2-8t+5)e^{2t}+4}.$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{\left(\sqrt{1+4t^2+e^{2t}}\right)^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{(1+4t^2+e^{2t})^{3/2}}.$

27. $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2\sqrt{6}\,t\,\mathbf{i} + 2\,\mathbf{j} + 6t^2\,\mathbf{k},\quad \mathbf{r}''(t) = 2\sqrt{6}\,\mathbf{i} + 12t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = \sqrt{4(3t^2+1)^2} = 2(3t^2+1),$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = 24t\,\mathbf{i} - 12\sqrt{6}\,t^2\,\mathbf{j} - 4\sqrt{6}\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{96(3t^2+1)^2} = 4\sqrt{6}\,(3t^2+1).$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{4\sqrt{6}\,(3t^2+1)}{8(3t^2+1)^3} = \dfrac{\sqrt{6}}{2(3t^2+1)^2}.$

28. $\mathbf{r}(t) = \left\langle t^2, \ln t, t\ln t\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2t, 1/t, 1+\ln t\rangle,\quad \mathbf{r}''(t) = \left\langle 2, -1/t^2, 1/t\right\rangle.$ The point $(1,0,0)$ corresponds to $t=1$, and $\mathbf{r}'(1) = \langle 2,1,1\rangle,\quad |\mathbf{r}'(1)| = \sqrt{2^2+1^2+1^2} = \sqrt{6},\quad \mathbf{r}''(1) = \langle 2,-1,1\rangle,\quad \mathbf{r}'(1)\times\mathbf{r}''(1) = \langle 2,0,-4\rangle,$

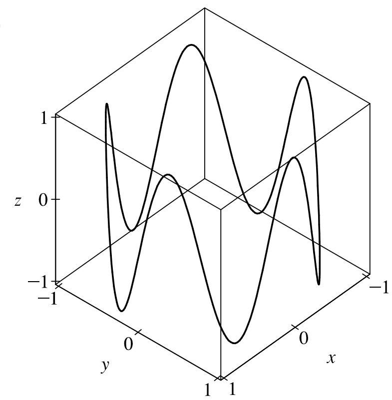
$|\mathbf{r}'(1)\times\mathbf{r}''(1)| = \sqrt{2^2+0^2+(-4)^2} = \sqrt{20} = 2\sqrt{5}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{2\sqrt{5}}{\left(\sqrt{6}\right)^3} = \dfrac{2\sqrt{5}}{6\sqrt{6}}$ or $\dfrac{\sqrt{30}}{18}.$

29. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The point $(1,1,1)$ corresponds to $t=1$, and $\mathbf{r}'(1) = \langle 1,2,3\rangle \;\Rightarrow$

$|\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0, 2, 6t\rangle \;\Rightarrow\; \mathbf{r}''(1) = \langle 0,2,6\rangle$. $\mathbf{r}'(1)\times\mathbf{r}''(1) = \langle 6,-6,2\rangle$, so

$|\mathbf{r}'(1)\times\mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

30.



Note that we get the complete curve for $0 \le t < 2\pi$.

$\mathbf{r}(t) = \langle \cos t, \sin t, \sin 5t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -\sin t, \cos t, 5\cos 5t\rangle$,

$\mathbf{r}''(t) = \langle -\cos t, -\sin t, -25\sin 5t\rangle$. The point $(1,0,0)$ corresponds to $t=0$, and $\mathbf{r}'(0) = \langle 0,1,5\rangle \;\Rightarrow$

$|\mathbf{r}'(0)| = \sqrt{0^2+1^2+5^2} = \sqrt{26}$, $\mathbf{r}''(0) = \langle -1,0,0\rangle$,

$\mathbf{r}'(0)\times\mathbf{r}''(0) = \langle 0,-5,1\rangle \;\Rightarrow$

$|\mathbf{r}'(0)\times\mathbf{r}''(0)| = \sqrt{0^2+(-5)^2+1^2} = \sqrt{26}$. The curvature at the point $(1,0,0)$ is $\kappa(0) = \dfrac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{\sqrt{26}}{\left(\sqrt{26}\right)^3} = \dfrac{1}{26}$.

31. $f(x) = x^4 \;\Rightarrow\; f'(x) = 4x^3 \;\Rightarrow\; f''(x) = 12x^2$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|12x^2|}{[1+(4x^3)^2]^{3/2}} = \frac{12x^2}{(1+16x^6)^{3/2}}.$$

32. $f(x) = \tan x \;\Rightarrow\; f'(x) = \sec^2 x \;\Rightarrow\; f''(x) = 2\sec x\cdot\sec x\,\tan x = 2\sec^2 x\,\tan x$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|2\sec^2 x\,\tan x|}{[1+(\sec^2 x)^2]^{3/2}} = \frac{2\sec^2 x\,|\tan x|}{(1+\sec^4 x)^{3/2}}.$$

33. $f(x) = xe^x \;\Rightarrow\; f'(x) = xe^x + e^x \;\Rightarrow\; f''(x) = xe^x + 2e^x$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|xe^x+2e^x|}{[1+(xe^x+e^x)^2]^{3/2}} = \frac{e^x\,|x+2|}{[1+(xe^x+e^x)^2]^{3/2}}.$$

34. $y = \ln x \;\Rightarrow\; y' = \dfrac{1}{x} \;\Rightarrow\; y'' = -\dfrac{1}{x^2}$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|y''(x)|}{\left[1+(y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right|\frac{1}{(1+1/x^2)^{3/2}} = \frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2+1)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}} = \frac{x}{(x^2+1)^{3/2}} \quad [\text{since } x>0].$$

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2+1)^{3/2} - x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{\left[(x^2+1)^{3/2}\right]^2} = \frac{(x^2+1)^{1/2}[(x^2+1)-3x^2]}{(x^2+1)^3} = \frac{1-2x^2}{(x^2+1)^{5/2}};$$

$\kappa'(x) = 0 \;\Rightarrow\; 1-2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$.

Since $\displaystyle\lim_{x\to\infty}\frac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x\to\infty$.

35. Since $y = y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \dfrac{e^x}{(1 + e^{2x})^{3/2}} = e^x(1 + e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = e^x(1 + e^{2x})^{-3/2} + e^x\left(-\tfrac{3}{2}\right)(1 + e^{2x})^{-5/2}(2e^{2x}) = e^x\frac{1 + e^{2x} - 3e^{2x}}{(1 + e^{2x})^{5/2}} = e^x\frac{1 - 2e^{2x}}{(1 + e^{2x})^{5/2}}.$$

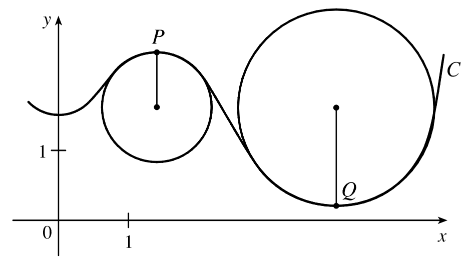
$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} e^x(1 + e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

36. We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2$, $a > 0$. Then by Formula 11, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1 + (2ax)^2]^{3/2}} = \dfrac{2a}{(1 + 4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

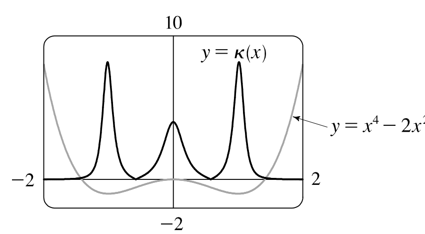
37. (a) C appears to be changing direction more quickly at P than Q, so we would expect the curvature to be greater at P.

(b) First we sketch approximate osculating circles at P and Q. Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \Rightarrow \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.



38. $y = x^4 - 2x^2 \quad\Rightarrow\quad y' = 4x^3 - 4x \quad\Rightarrow\quad y'' = 12x^2 - 4$.

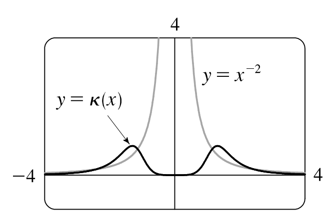
$\kappa(x) = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{\left|12x^2 - 4\right|}{\left[1 + (4x^3 - 4x)^2\right]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of $y = x^4 - 2x^2$ appears to be bending most sharply at the origin and near $x = \pm 1$.



39. $y = x^{-2} \quad\Rightarrow\quad y' = -2x^{-3} \quad\Rightarrow\quad y'' = 6x^{-4}$.

$\kappa(x) = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{\left|6x^{-4}\right|}{\left[1 + (-2x^{-3})^2\right]^{3/2}} = \dfrac{6}{x^4\left(1 + 4x^{-6}\right)^{3/2}}$.

The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^{-2}$ increases asymptotically at the origin from both directions, and so its graph has very little bend there. [Note that $\kappa(0)$ is undefined.]

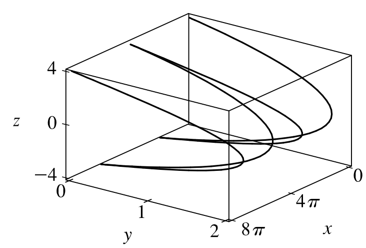


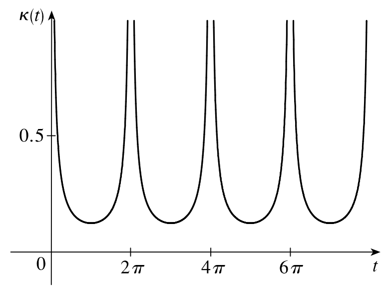
40. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4\cos(t/2)\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1 - \cos t, \sin t, -2\sin(t/2)\rangle \quad \Rightarrow$

$\mathbf{r}''(t) = \langle \sin t, \cos t, -\cos(t/2)\rangle$. Using a CAS, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -2\sin^3(t/2), -\sin(t/2)\sin t, \cos t - 1\right\rangle$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{3 - 4\cos t + \cos 2t}$ or $2\sqrt{2}\sin^2(t/2)$, and $|\mathbf{r}'(t)| = 2\sqrt{1 - \cos t}$ or $2\sqrt{2}\,|\sin(t/2)|$.

(To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command. Here loading the `RealDomain` package will give simpler results. In Mathematica, use `Cross[a,b]`.) Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{3 - 4\cos t + \cos 2t}}{8\,(1 - \cos t)^{3/2}}$ or $\dfrac{1}{4\sqrt{2 - 2\cos t}}$ or $\dfrac{1}{8\,|\sin(t/2)|}$. We plot the space curve and its curvature function for $0 \le t \le 8\pi$ below.





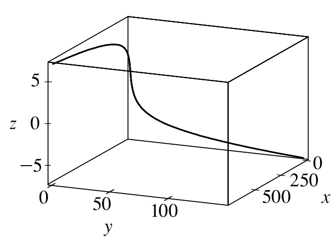
The asymptotes in the graph of $\kappa(t)$ correspond to the sharp cusps we see in the graph of $\mathbf{r}(t)$. The space curve bends most sharply as it approaches these cusps (mostly in the x-direction) and bends most gradually between these, near its intersections with the xy-plane, where $t = \pi + 2n\pi$ (n an integer). (The bending we see in the z-direction on the curve near these points is deceiving; most of the curvature occurs in the x-direction.) The curvature graph has local minima at these values of t.

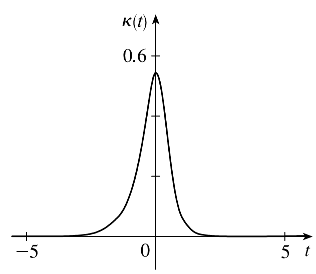
41. $\mathbf{r}(t) = \left\langle te^t, e^{-t}, \sqrt{2}t\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle (t+1)e^t, -e^{-t}, \sqrt{2}\right\rangle \quad \Rightarrow \quad \mathbf{r}''(t) = \left\langle (t+2)e^t, e^{-t}, 0\right\rangle$.

Then $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -\sqrt{2}e^{-t}, \sqrt{2}(t+2)e^t, 2t+3\right\rangle$, $\quad |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2e^{-2t} + 2(t+2)^2e^{2t} + (2t+3)^2}$,

$|\mathbf{r}'(t)| = \sqrt{(t+1)^2e^{2t} + e^{-2t} + 2}$, and $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{2e^{-2t} + 2(t+2)^2e^{2t} + (2t+3)^2}}{\left[(t+1)^2e^{2t} + e^{-2t} + 2\right]^{3/2}}$.

We plot the space curve and its curvature function for $-5 \le t \le 5$ below.



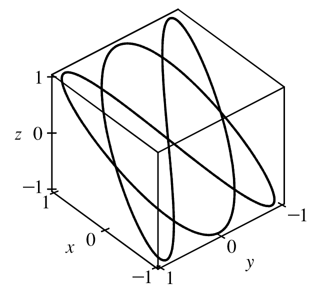


From the graph of $\kappa(t)$ we see that curvature is maximized for $t = 0$, so the curve bends most sharply at the point $(0, 1, 0)$. The curve bends more gradually as we move away from this point, becoming almost linear. This is reflected in the curvature graph, where $\kappa(t)$ becomes nearly 0 as $|t|$ increases.

42. Notice that the curve a is highest for the same x-values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.

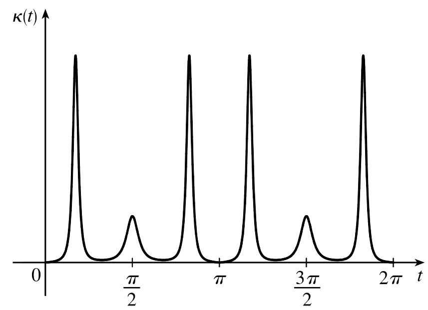
43. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y = f(x)$ rather than the graph of curvature, and b is the graph of $y = \kappa(x)$.

44. (a) The complete curve for $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$ is given by $0 \leq t \leq 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



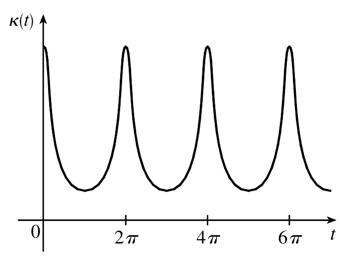
(b) Using a CAS, we find (after simplifying)

$\kappa(t) = \dfrac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{3/2}}$. (To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command; in Mathematica, use `Cross[a,b]`.) The graph shows 6 local (or absolute) maximum points for $0 \leq t \leq 2\pi$, as observed in part (a).



45. $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \right\rangle$. Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$.

(To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command; in Mathematica, use `Cross[a,b]`.) Curvature is largest at integer multiples of 2π.



46. Here $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, $\quad \mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$, $\quad \mathbf{r}''(t) = \langle f''(t), g''(t) \rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = [(f'(t))^2 + (g'(t))^2]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)\,g''(t) - f''(t)\,g'(t) \rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$. Thus, by Theorem 10,

$\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$.

47. $x = t^2 \quad \Rightarrow \quad \dot{x} = 2t \quad \Rightarrow \quad \ddot{x} = 2, \quad y = t^3 \quad \Rightarrow \quad \dot{y} = 3t^2 \quad \Rightarrow \quad \ddot{y} = 6t$.

Then $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \dfrac{|(2t)(6t) - (3t^2)(2)|}{[(2t)^2 + (3t^2)^2]^{3/2}} = \dfrac{|12t^2 - 6t^2|}{(4t^2 + 9t^4)^{3/2}} = \dfrac{6t^2}{(4t^2 + 9t^4)^{3/2}}$.

48. $x = a\cos\omega t \quad\Rightarrow\quad \dot{x} = -a\omega\sin\omega t \quad\Rightarrow\quad \ddot{x} = -a\omega^2\cos\omega t$,

$y = b\sin\omega t \quad\Rightarrow\quad \dot{y} = b\omega\cos\omega t \quad\Rightarrow\quad \ddot{y} = -b\omega^2\sin\omega t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{\left|(-a\omega\sin\omega t)(-b\omega^2\sin\omega t) - (b\omega\cos\omega t)(-a\omega^2\cos\omega t)\right|}{[(-a\omega\sin\omega t)^2 + (b\omega\cos\omega t)^2]^{3/2}}$$

$$= \frac{\left|ab\omega^3\sin^2\omega t + ab\omega^3\cos^2\omega t\right|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}} = \frac{\left|ab\omega^3\right|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}}$$

49. $x = e^t\cos t \quad\Rightarrow\quad \dot{x} = e^t(\cos t - \sin t) \quad\Rightarrow\quad \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t\sin t$,

$y = e^t\sin t \quad\Rightarrow\quad \dot{y} = e^t(\cos t + \sin t) \quad\Rightarrow\quad \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{\left|e^t(\cos t - \sin t)(2e^t\cos t) - e^t(\cos t + \sin t)(-2e^t\sin t)\right|}{\left([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2\right)^{3/2}}$$

$$= \frac{\left|2e^{2t}(\cos^2 t - \sin t\cos t + \sin t\cos t + \sin^2 t)\right|}{\left[e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\cos t\sin t + \sin^2 t)\right]^{3/2}} = \frac{\left|2e^{2t}(1)\right|}{\left[e^{2t}(1+1)\right]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}$$

50. $f(x) = e^{cx}$, $\quad f'(x) = ce^{cx}$, $\quad f''(x) = c^2e^{cx}$. Using Formula 11 we have

$\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{\left|c^2e^{cx}\right|}{[1 + (ce^{cx})^2]^{3/2}} = \dfrac{c^2e^{cx}}{(1 + c^2e^{2cx})^{3/2}}$ so the curvature at $x = 0$ is

$\kappa(0) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. To determine the maximum value for $\kappa(0)$, let $f(c) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. Then

$f'(c) = \dfrac{2c\cdot(1 + c^2)^{3/2} - c^2\cdot\frac{3}{2}(1 + c^2)^{1/2}(2c)}{[(1 + c^2)^{3/2}]^2} = \dfrac{(1 + c^2)^{1/2}\left[2c(1 + c^2) - 3c^3\right]}{(1 + c^2)^3} = \dfrac{(2c - c^3)}{(1 + c^2)^{5/2}}$. We have a critical

number when $2c - c^3 = 0 \quad\Rightarrow\quad c(2 - c^2) = 0 \quad\Rightarrow\quad c = 0$ or $c = \pm\sqrt{2}$. $\quad f'(c)$ is positive for $c < -\sqrt{2}$, $0 < c < \sqrt{2}$

and negative elsewhere, so f achieves its maximum value when $c = \sqrt{2}$ or $-\sqrt{2}$. In either case, $\kappa(0) = \dfrac{2}{3^{3/2}}$, so the members

of the family with the largest value of $\kappa(0)$ are $f(x) = e^{\sqrt{2}x}$ and $f(x) = e^{-\sqrt{2}x}$.

51. $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t\right\rangle$. $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$.

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\left\langle 2t, 2t^2, 1\right\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\left\langle 2t, 2t^2, 1\right\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{T}'(t) = -4t(2t^2 + 1)^{-2}\left\langle 2t, 2t^2, 1\right\rangle + (2t^2 + 1)^{-1}\langle 2, 4t, 0\rangle \qquad$ [by Formula 3 of Theorem 13.2.3]

$\quad = (2t^2 + 1)^{-2}\left\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\right\rangle = 2(2t^2 + 1)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} = \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

52. $(1, 0, 0)$ corresponds to $t = 0$. $\quad\mathbf{r}(t) = \langle\cos t, \sin t, \ln\cos t\rangle$, and in Exercise 6 we found that $\mathbf{r}'(t) = \langle -\sin t, \cos t, -\tan t\rangle$

and $|\mathbf{r}'(t)| = |\sec t|$. Here we can assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and then $\sec t > 0 \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sec t$.

[continued]

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle -\sin t, \cos t, -\tan t\rangle}{\sec t} = \left\langle -\sin t\cos t, \cos^2 t, -\sin t\right\rangle$ and $\mathbf{T}(0) = \langle 0, 1, 0\rangle$.

$\mathbf{T}'(t) = \langle -[(\sin t)(-\sin t) + (\cos t)(\cos t)], 2(\cos t)(-\sin t), -\cos t\rangle = \left\langle \sin^2 t - \cos^2 t, -2\sin t\cos t, -\cos t\right\rangle$, so

$\mathbf{N}(0) = \dfrac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \dfrac{\langle -1, 0, -1\rangle}{\sqrt{1+0+1}} = \dfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle$.

Finally, $\mathbf{B}(0) = \mathbf{T}(0)\times\mathbf{N}(0) = \langle 0, 1, 0\rangle \times \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$.

53. $\mathbf{r}(t) = \langle \sin 2t, -\cos 2t, 4t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2\cos 2t, 2\sin 2t, 4\rangle$. The point $(0, 1, 2\pi)$ corresponds to $t = \pi/2$, and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -2, 0, 4\rangle$. An equation for the normal plane is $-2(x-0) + 0(y-1) + 4(z-2\pi) = 0$ or $-2x + 4z = 8\pi$ or $x - 2z = -4\pi$.

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2\cos 2t, 2\sin 2t, 4\rangle}{\sqrt{4\cos^2 2t + 4\sin^2 2t + 16}} = \dfrac{1}{2\sqrt{5}}\langle 2\cos 2t, 2\sin 2t, 4\rangle = \dfrac{1}{\sqrt{5}}\langle \cos 2t, \sin 2t, 2\rangle \;\Rightarrow$

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\sin 2t, 2\cos 2t, 0\rangle \;\Rightarrow\; |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{4\sin^2 2t + 4\cos^2 2t} = \frac{2}{\sqrt{5}}$, and

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin 2t, \cos 2t, 0\rangle$. Then $\mathbf{T}(\pi/2) = \frac{1}{\sqrt{5}}\langle -1, 0, 2\rangle$, $\mathbf{N}(\pi/2) = \langle 0, -1, 0\rangle$, and

$\mathbf{B}(\pi/2) = \mathbf{T}(\pi/2)\times\mathbf{N}(\pi/2) = \frac{1}{\sqrt{5}}\langle 2, 0, 1\rangle$. Since $\mathbf{B}(\pi/2)$ is normal to the osculating plane, so is $\langle 2, 0, 1\rangle$, and an equation of the plane is $2(x-0) + 0(y-1) + 1(z-2\pi) = 0$ or $2x + z = 2\pi$.

54. $\mathbf{r}(t) = \left\langle \ln t, 2t, t^2\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1/t, 2, 2t\rangle$. The point $(0, 2, 1)$ corresponds to $t = 1$, and the normal plane there has normal vector $\mathbf{r}'(1) = \langle 1, 2, 2\rangle$. An equation for the normal plane is $1(x-0) + 2(y-2) + 2(z-1) = 0$ or $x + 2y + 2z = 6$.

$|\mathbf{r}'(t)| = \sqrt{1/t^2 + 4 + 4t^2} = \sqrt{[(1/t) + 2t]^2} = (1/t) + 2t$ [since $t > 0$] and then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 1/t, 2, 2t\rangle}{(1/t) + 2t} = \dfrac{1}{1+2t^2}\left\langle 1, 2t, 2t^2\right\rangle$ $\left[\text{after multiplying by } \dfrac{t}{t}\right]$. By Formula 3 of Theorem 13.2.3,

$$\begin{aligned}\mathbf{T}'(t) &= -\frac{4t}{(1+2t^2)^2}\left\langle 1, 2t, 2t^2\right\rangle + \frac{1}{1+2t^2}\langle 0, 2, 4t\rangle \\ &= \frac{1}{(1+2t^2)^2}\left\langle -4t, -8t^2 + 2(1+2t^2), -8t^3 + 4t(1+2t^2)\right\rangle = \frac{1}{(1+2t^2)^2}\left\langle -4t, 2-4t^2, 4t\right\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(1+2t^2)^2}\sqrt{16t^2 + (2-4t^2)^2 + 16t^2} = \frac{1}{(1+2t^2)^2}\sqrt{16t^2 + 4 + 16t^4} \\ &= \frac{1}{(1+2t^2)^2}\cdot 2\sqrt{(1+2t^2)^2} = \frac{2}{1+2t^2}\end{aligned}$$

and $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1}{2(1+2t^2)}\left\langle -4t, 2-4t^2, 4t\right\rangle = \dfrac{1}{1+2t^2}\left\langle -2t, 1-2t^2, 2t\right\rangle$.

Thus $\mathbf{T}(1) = \frac{1}{3}\langle 1, 2, 2\rangle$, $\mathbf{N}(1) = \frac{1}{3}\langle -2, -1, 2\rangle$, and $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \frac{1}{9}\langle 6, -6, 3\rangle$ is normal to the osculating plane.

[continued]

We can take the parallel vector $\langle 2, -2, 1\rangle$ as a normal vector for the plane, so an equation is

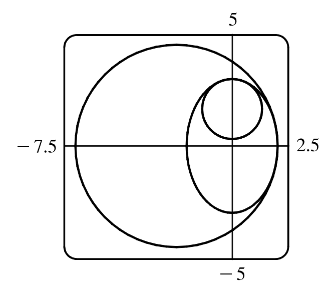
$2(x-0) - 2(y-2) + 1(z-1) = 0$ or $2x - 2y + z = -3$.

Note: Since $\mathbf{r}'(1)$ is parallel to $\mathbf{T}(1)$ and $\mathbf{T}'(1)$ is parallel to $\mathbf{N}(1)$, we could have taken $\mathbf{r}'(1) \times \mathbf{T}'(1)$ as a normal vector for the plane.

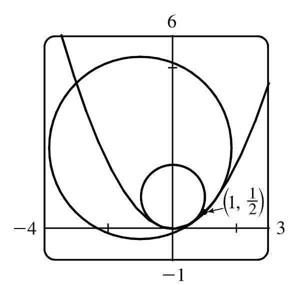
55. The ellipse is given by the parametric equations $x = 2\cos t$, $y = 3\sin t$, so using the result from Exercise 46,

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{(4\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}.$$

At $(2, 0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $\left(-\frac{5}{2}, 0\right)$. Its equation is therefore $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$. At $(0, 3)$, $t = \frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0, \frac{5}{3}\right)$. Hence its equation is $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$.



56. $y = \frac{1}{2}x^2 \quad\Rightarrow\quad y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1+x^2)^{3/2}}$. So the curvature at $(0, 0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y-1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1+1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope -1. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2}\right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$. So the equation of the circle is $(x+1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.



57. Here $\mathbf{r}(t) = \left\langle t^3, 3t, t^4\right\rangle$, and $\mathbf{r}'(t) = \left\langle 3t^2, 3, 4t^3\right\rangle$ is normal to the normal plane for any t. The given plane has normal vector $\langle 6, 6, -8\rangle$, and the planes are parallel when their normal vectors are parallel. Thus we need to find a value for t where $\left\langle 3t^2, 3, 4t^3\right\rangle = k\,\langle 6, 6, -8\rangle$ for some $k \neq 0$. From the y-component we see that $k = \frac{1}{2}$, and $\left\langle 3t^2, 3, 4t^3\right\rangle = \frac{1}{2}\langle 6, 6, -8\rangle = \langle 3, 3, -4\rangle$ for $t = -1$. Thus the planes are parallel at the point $(-1, -3, 1)$.

58. To find the osculating plane, we first calculate the unit tangent and normal vectors.

In Maple, we use the `VectorCalculus` package and set `r:=<t^3,3*t,t^4>;`. After differentiating, the `Normalize` command converts the tangent vector to the unit tangent vector: `T:=Normalize(diff(r,t));`. After simplifying, we find that $\mathbf{T}(t) = \dfrac{\left\langle 3t^2, 3, 4t^3\right\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. We use a similar procedure to compute the unit normal vector,

`N:=Normalize(diff(T,t));`. After simplifying, we have $\mathbf{N}(t) = \dfrac{\left\langle -t(8t^6-9), -3t^3(3+8t^2), 6t^2(t^4+3)\right\rangle}{\sqrt{t^2(4t^6+36t^2+9)(16t^6+9t^4+9)}}$. Then we use the command `B:=CrossProduct(T,N);`. After simplification, we find that $\mathbf{B}(t) = \dfrac{\left\langle 6t^2, -2t^4, -3t\right\rangle}{\sqrt{t^2(4t^6+36t^2+9)}}$.

In Mathematica, we define the vector function `r={t^3,3*t,t^4}` and use the command `Dt` to differentiate. We find $\mathbf{T}(t)$ by dividing the result by its magnitude, computed using the `Norm` command. (You may wish to include the option `Element[t,Reals]` to obtain simpler expressions.) $\mathbf{N}(t)$ is found similarly, and we use `Cross[T,N]` to find $\mathbf{B}(t)$.

Now $\mathbf{B}(t)$ is parallel to $\left\langle 6t^2, -2t^4, -3t\right\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some $t \neq 0$ [since $\mathbf{B}(0) = \mathbf{0}$], then $\left\langle 6t^2, -2t^4, -3t\right\rangle = k\,\langle 1,1,1\rangle$ for some value of k. But then $6t^2 = -2t^4 = -3t$ which has no solution for $t \neq 0$. So there is no such osculating plane.

59. First we parametrize the curve of intersection. We can choose $y = t$; then $x = y^2 = t^2$ and $z = x^2 = t^4$, and the curve is given by $\mathbf{r}(t) = \left\langle t^2, t, t^4\right\rangle$. $\mathbf{r}'(t) = \left\langle 2t, 1, 4t^3\right\rangle$ and the point $(1,1,1)$ corresponds to $t = 1$, so $\mathbf{r}'(1) = \langle 2,1,4\rangle$ is a normal vector for the normal plane. Thus an equation of the normal plane is

$2(x-1) + 1(y-1) + 4(z-1) = 0$ or $2x + y + 4z = 7$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2+1+16t^6}}\left\langle 2t, 1, 4t^3\right\rangle$ and

$\mathbf{T}'(t) = -\frac{1}{2}(4t^2+1+16t^6)^{-3/2}(8t+96t^5)\left\langle 2t,1,4t^3\right\rangle + (4t^2+1+16t^6)^{-1/2}\left\langle 2,0,12t^2\right\rangle$. A normal vector for the osculating plane is $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1)$, but $\mathbf{r}'(1) = \langle 2,1,4\rangle$ is parallel to $\mathbf{T}(1)$ and

$\mathbf{T}'(1) = -\frac{1}{2}(21)^{-3/2}(104)\langle 2,1,4\rangle + (21)^{-1/2}\langle 2,0,12\rangle = \frac{2}{21\sqrt{21}}\langle -31,-26,22\rangle$ is parallel to $\mathbf{N}(1)$ as is $\langle -31,-26,22\rangle$, so $\langle 2,1,4\rangle \times \langle -31,-26,22\rangle = \langle 126,-168,-21\rangle$ is normal to the osculating plane. Thus an equation for the osculating plane is $126(x-1) - 168(y-1) - 21(z-1) = 0$ or $6x - 8y - z = -3$.

60. $\mathbf{r}(t) = \left\langle t+2, 1-t, \frac{1}{2}t^2\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1,-1,t\rangle$, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2+t^2}}\langle 1,-1,t\rangle$,

$$\begin{aligned}\mathbf{T}'(t) &= -\tfrac{1}{2}(2+t^2)^{-3/2}(2t)\langle 1,-1,t\rangle + (2+t^2)^{-1/2}\langle 0,0,1\rangle\\ &= -(2+t^2)^{-3/2}\left[t\langle 1,-1,t\rangle - (2+t^2)\langle 0,0,1\rangle\right] = \frac{-1}{(2+t^2)^{3/2}}\langle t,-t,-2\rangle\end{aligned}$$

A normal vector for the osculating plane is $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$, but $\mathbf{r}'(t) = \langle 1,-1,t\rangle$ is parallel to $\mathbf{T}(t)$ and $\langle t,-t,-2\rangle$ is parallel to $\mathbf{T}'(t)$ and hence parallel to $\mathbf{N}(t)$, so $\langle 1,-1,t\rangle \times \langle t,-t,-2\rangle = \left\langle t^2+2, t^2+2, 0\right\rangle$ is normal to the osculating plane for any t. All such vectors are parallel to $\langle 1,1,0\rangle$, so at any point $\left(t+2, 1-t, \frac{1}{2}t^2\right)$ on the curve, an equation for the osculating plane is $1[x-(t+2)] + 1[y-(1-t)] + 0\left(z-\frac{1}{2}t^2\right) = 0$ or $x + y = 3$. Because the osculating plane at every point on the curve is the same, we can conclude that the curve itself lies in that same plane. In fact, we can easily verify that the parametric equations of the curve satisfy $x + y = 3$.

61. $\mathbf{r}(t) = \left\langle e^t \cos t, e^t \sin t, e^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \right\rangle$ so

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} \\ &= \sqrt{e^{2t}\left[2(\cos^2 t + \sin^2 t) - 2\cos t \sin t + 2\cos t \sin t + 1\right]} = \sqrt{3e^{2t}} = \sqrt{3}\, e^t \end{aligned}$$

and $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{3}\, e^t} \left\langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \right\rangle = \dfrac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1\rangle$. The vector $\mathbf{k} = \langle 0, 0, 1\rangle$ is parallel to the z-axis, so for any t, the angle α between $\mathbf{T}(t)$ and the z-axis is given by

$$\cos\alpha = \frac{\mathbf{T}(t)\cdot\mathbf{k}}{|\mathbf{T}(t)|\;|\mathbf{k}|} = \frac{\frac{1}{\sqrt{3}}\langle \cos t - \sin t, \cos t + \sin t, 1\rangle \cdot \langle 0,0,1\rangle}{\frac{1}{\sqrt{3}}\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1}\;\sqrt{1}} = \frac{1}{\sqrt{2(\cos^2 t + \sin^2 t) + 1}} = \frac{1}{\sqrt{3}}.$$

Thus the angle is constant; specifically, $\alpha = \cos^{-1}(1/\sqrt{3}) \approx 54.7\,^\circ$.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{(1/\sqrt{3}\,)\,\langle -\sin t - \cos t, -\sin t + \cos t, 0\rangle}{(1/\sqrt{3}\,)\sqrt{2\left(\sin^2 t + \cos^2 t\right)}} = \frac{1}{\sqrt{2}}\langle -\sin t - \cos t, -\sin t + \cos t, 0\rangle,$$

and the angle β made with the z-axis is given by $\cos\beta = \dfrac{\mathbf{N}(t)\cdot\mathbf{k}}{|\mathbf{N}(t)|\;|\mathbf{k}|} = 0$, so $\beta = 90\,^\circ$.

$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{6}}\langle \sin t - \cos t, -\sin t - \cos t, 2\rangle$ and the angle γ made with the z-axis is given by

$$\cos\gamma = \frac{\mathbf{B}(t)\cdot\mathbf{k}}{|\mathbf{B}(t)|\;|\mathbf{k}|} = \frac{\frac{1}{\sqrt{6}}\langle \sin t - \cos t, -\sin t - \cos t, 2\rangle \cdot\langle 0,0,1\rangle}{\frac{1}{\sqrt{6}}\sqrt{(\sin t - \cos t)^2 + (-\sin t - \cos t)^2 + 4}\;\sqrt{1}} = \frac{2}{\sqrt{6}} \text{ or equivalently } \frac{\sqrt{6}}{3}.$$

Again the angle is constant; specifically, $\gamma = \cos^{-1}(2/\sqrt{6}) \approx 35.3\,^\circ$.

62. If vectors $\mathbf{T}$ and $\mathbf{B}$ lie in the rectifying plane then $\mathbf{N}$ is a normal vector for the plane, as it is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$. The point $\left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$ corresponds to $t = \pi/4$, so we can take $\mathbf{T}'(\pi/4)$ as a normal vector for the plane [since it is parallel to $\mathbf{N}(\pi/4)$]. $\quad \mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$ and

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+\sec^4 t}}\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right)$.

By Formula 3 of Theorem 13.2.3,

$$\mathbf{T}'(t) = -\frac{2\sec^4 t\tan t}{(1+\sec^4 t)^{3/2}}\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right) + \frac{1}{\sqrt{1+\sec^4 t}}\left(-\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2\sec^2 t\tan t\,\mathbf{k}\right) \text{ and}$$

$$\begin{aligned} \mathbf{T}'(\pi/4) &= -\frac{2(\sqrt{2}\,)^4(1)}{[1+(\sqrt{2}\,)^4]^{3/2}}\left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + (\sqrt{2}\,)^2\,\mathbf{k}\right) + \frac{1}{\sqrt{1+(\sqrt{2}\,)^4}}\left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2(\sqrt{2}\,)^2(1)\,\mathbf{k}\right) \\ &= -\frac{8}{5\sqrt{5}}\left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2\,\mathbf{k}\right) + \frac{1}{\sqrt{5}}\left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 4\,\mathbf{k}\right) = -\frac{13\sqrt{2}}{10\sqrt{5}}\,\mathbf{i} + \frac{3\sqrt{2}}{10\sqrt{5}}\,\mathbf{j} + \frac{4}{5\sqrt{5}}\,\mathbf{k} \end{aligned}$$

We can take the parallel vector $-13\sqrt{2}\,\mathbf{i} + 3\sqrt{2}\,\mathbf{j} + 8\,\mathbf{k}$ as a normal for the plane, so an equation for the plane is

$$-13\sqrt{2}\left(x - \tfrac{\sqrt{2}}{2}\right) + 3\sqrt{2}\left(y - \tfrac{\sqrt{2}}{2}\right) + 8\,(z-1) = 0 \;\text{ or }\; -13\sqrt{2}\,x + 3\sqrt{2}\,y + 8z = -2 \;\text{ or }\; 13x - 3y - 4\sqrt{2}\,z = \sqrt{2}.$$

63. $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

64. For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then

$\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

65. (a) $|\mathbf{B}| = 1 \;\Rightarrow\; \mathbf{B}\cdot\mathbf{B} = 1 \;\Rightarrow\; \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \;\Rightarrow\; 2\,\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \;\Rightarrow\; \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$.

This shows that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$. Alternatively, note that this is a direct result of Theorem 13.2.4.

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \;\Rightarrow$

$$\begin{aligned}
\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{|\mathbf{r}'(t)|} && \text{[by Formula 7]}\\
&= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\,\frac{1}{|\mathbf{r}'(t)|} && \text{[by Formula 5 of Theorem 13.2.3]}\\
&= \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|}\\
&= [\mathbf{0} + (\mathbf{T}\times\mathbf{N}')]\,\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|} && [\mathbf{a}\times c\mathbf{a} = \mathbf{0}]\\
\Rightarrow \frac{d\mathbf{B}}{ds} &\perp \mathbf{T} && \text{[by Theorem 12.4.8]}
\end{aligned}$$

(c) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \;\Rightarrow\; \mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. Since $\mathbf{T}\perp\mathbf{N}$, $\mathbf{B}$, $\mathbf{T}$, and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$.

66. We need to find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ in terms of t. $\mathbf{r}(t) = \langle \sin t, 3t, \cos t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle \cos t, 3, -\sin t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + 3^2 + \sin^2 t} = \sqrt{10}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{10}}\langle \cos t, 3, -\sin t\rangle \;\Rightarrow$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{10}}\langle -\sin t, 0, -\cos t\rangle$, and $|\mathbf{T}'(t)| = \dfrac{1}{\sqrt{10}}\sqrt{\sin^2 t + 0^2 + \cos^2 t} = \dfrac{1}{\sqrt{10}}$.

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\sqrt{10}}{1}\cdot\dfrac{1}{\sqrt{10}}\langle -\sin t, 0, -\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

Then

$$\begin{aligned}
\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) &= \frac{1}{\sqrt{10}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos t & 3 & -\sin t\\ -\sin t & 0 & -\cos t\end{vmatrix}\\
&= \frac{1}{\sqrt{10}}\left[-3\cos t\,\mathbf{i} - (-\cos^2 t - \sin^2 t)\,\mathbf{j} + 3\sin t\,\mathbf{k}\right] = \frac{1}{\sqrt{10}}\langle -3\cos t, 1, 3\sin t\rangle
\end{aligned}$$

$$\mathbf{B}'(t) = \frac{1}{\sqrt{10}}\langle 3\sin t, 0, 3\cos t\rangle \;\Rightarrow\; \mathbf{B}'(\pi/2) = \frac{1}{\sqrt{10}}\langle 3, 0, 0\rangle \quad\text{and}\quad \mathbf{N}(\pi/2) = \langle -1, 0, 0\rangle$$

Thus, the torsion is $\tau(t) = -\dfrac{\mathbf{B}'(\pi/2)\cdot\mathbf{N}(\pi/2)}{|\mathbf{r}'(\pi/2)|} = -\dfrac{-3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$.

67. We need to find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ in terms of t. $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, 2t, t \right\rangle \Rightarrow \mathbf{r}'(t) = \langle t, 2, 1 \rangle \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{t^2 + 2^2 + 1^2} = \sqrt{t^2 + 5}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{t^2 + 5}} \langle t, 2, 1 \rangle \Rightarrow$

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{t^2+5}} \langle 1, 0, 0 \rangle - \frac{t}{(t^2+5)^{3/2}} \langle t, 2, 1 \rangle = \frac{1}{(t^2+5)^{3/2}} \left[(t^2+5) \langle 1, 0, 0 \rangle - t \langle t, 2, 1 \rangle \right] \\ &= \frac{1}{(t^2+5)^{3/2}} \langle 5, -2t, -t \rangle, \text{ and}\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(t^2+5)^{3/2}} \sqrt{25 + 5t^2} = \frac{\sqrt{5}}{t^2+5}.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{t^2+5}{\sqrt{5}} \cdot \frac{1}{(t^2+5)^{3/2}} \langle 5, -2t, -t \rangle = \frac{1}{\sqrt{5}\sqrt{t^2+5}} \langle 5, -2t, -t \rangle.$$

Then $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \dfrac{1}{\sqrt{t^2+5}} \cdot \dfrac{1}{\sqrt{5}\sqrt{t^2+5}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2 & 1 \\ 5 & -2t & -t \end{vmatrix}$

$$= \frac{1}{\sqrt{5}(t^2+5)} \left[(-2t+2t)\mathbf{i} - (-t^2-5)\mathbf{j} + (-2t^2-10)\mathbf{k} \right] = \left\langle 0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

Since $\mathbf{B}(t)$ is constant, $\mathbf{B}'(t) = \mathbf{0}$, and $\mathbf{B}'(1) \cdot \mathbf{N}(1) = 0$. Thus, the torsion is $\tau(1) = -\dfrac{\mathbf{B}'(1) \cdot \mathbf{N}(1)}{|\mathbf{r}'(1)|} = -\dfrac{0}{\sqrt{6}} = 0$.

68. $\mathbf{r} = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle \Rightarrow \mathbf{r}' = \left\langle 1, t, t^2 \right\rangle \Rightarrow \mathbf{r}'' = \langle 0, 1, 2t \rangle \Rightarrow \mathbf{r}''' = \langle 0, 0, 2 \rangle$.

$\mathbf{r}' \times \mathbf{r}'' = \left\langle t^2, -2t, 1 \right\rangle$. By Theorem 15, the torsion is $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \dfrac{\left\langle t^2, -2t, 1 \right\rangle \cdot \langle 0, 0, 2 \rangle}{\left(\sqrt{(t^2)^2 + (-2t)^2 + 1^2} \right)^2} = \dfrac{2}{t^4 + 4t^2 + 1}$.

The torsion at $t = 0$ is $\tau(0) = \frac{2}{1} = 2$.

69. $\mathbf{r} = \left\langle e^t, e^{-t}, t \right\rangle \Rightarrow \mathbf{r}' = \left\langle e^t, -e^{-t}, 1 \right\rangle \Rightarrow \mathbf{r}'' = \left\langle e^t, e^{-t}, 0 \right\rangle \Rightarrow \mathbf{r}''' = \left\langle e^t, -e^{-t}, 0 \right\rangle$.

$\mathbf{r}' \times \mathbf{r}'' = \left\langle -e^{-t}, e^t, 2 \right\rangle$. By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle -e^{-t}, e^t, 2 \right\rangle \cdot \left\langle e^t, -e^{-t}, 0 \right\rangle}{\left(\sqrt{(-e^{-t})^2 + (e^t)^2 + 2^2} \right)^2} = \frac{-1-1+0}{e^{-2t} + e^{2t} + 4} = \frac{-2}{e^{2t} + e^{-2t} + 4}$$

The torsion at $t = 0$ is $\tau(0) = \frac{-2}{1+1+4} = -\frac{1}{3}$.

70. $\mathbf{r} = \langle \cos t, \sin t, \sin t \rangle \Rightarrow \mathbf{r}' = \langle -\sin t, \cos t, \cos t \rangle \Rightarrow \mathbf{r}'' = \langle -\cos t, -\sin t, -\sin t \rangle \Rightarrow$

$\mathbf{r}''' = \langle \sin t, -\cos t, -\cos t \rangle$. $\mathbf{r}' \times \mathbf{r}'' = \langle 0, -1, 1 \rangle$. By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle 0, -1, 1 \rangle \cdot \langle \sin t, -\cos t, -\cos t \rangle}{\left(\sqrt{0^2 + (-1)^2 + 1^2} \right)^2} = \frac{0 + \cos t - \cos t}{1+1} = 0$$

The torsion at $t = 0$, or any value of t, is 0.

71. $\mathbf{N} = \mathbf{B} \times \mathbf{T} \quad \Rightarrow$

$$\begin{aligned}\frac{d\mathbf{N}}{ds} &= \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} && \text{[by Formula 5 of Theorem 13.2.3]}\\ &= -\tau\mathbf{N}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{N} && \text{[by Formulas 3 and 1]}\\ &= -\tau(\mathbf{N}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{N}) && \text{[by Property 2 of Theorem 12.4.11]}\end{aligned}$$

But $\mathbf{B}\times\mathbf{N} = \mathbf{B}\times(\mathbf{B}\times\mathbf{T}) = (\mathbf{B}\cdot\mathbf{T})\,\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\,\mathbf{T}$ [by Property 6 ofTheorem 12.4.11]

$$= 0 - \mathbf{T} = -\mathbf{T} \quad \Rightarrow$$

$d\mathbf{N}/ds = \tau(\mathbf{T}\times\mathbf{N}) - \kappa\mathbf{T} = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$.

72. (a) $\mathbf{r}' = s'\,\mathbf{T} \quad\Rightarrow\quad \mathbf{r}'' = s''\,\mathbf{T} + s'\,\mathbf{T}' = s''\,\mathbf{T} + s'\,\dfrac{d\mathbf{T}}{ds}s' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Frenet-Serret formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}'\times\mathbf{r}'' &= (s'\,\mathbf{T})\times[s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]\\ &= [(s'\,\mathbf{T})\times(s''\,\mathbf{T})] + \left[(s'\mathbf{T})\times(\kappa(s')^2\,\mathbf{N})\right] && \text{[by Property 3 of Theorem 12.4.11]}\\ &= (s's'')(\mathbf{T}\times\mathbf{T}) + \kappa(s')^3(\mathbf{T}\times\mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\mathbf{r}''' &= [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]' = s'''\,\mathbf{T} + s''\,\mathbf{T}' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\mathbf{N}'\\ &= s'''\,\mathbf{T} + s''\frac{d\mathbf{T}}{ds}\,s' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\frac{d\mathbf{N}}{ds}\,s'\\ &= s'''\,\mathbf{T} + s''s'\kappa\,\mathbf{N} + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^3(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) && \text{[by Formulas 1 and 2]}\\ &= [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B}\cdot\mathbf{T} = 0$, $\mathbf{B}\cdot\mathbf{N} = 0$, and $\mathbf{B}\cdot\mathbf{B} = 1$, we get

$$\frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = \frac{\kappa(s')^3\,\mathbf{B}\cdot\left\{[s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\right\}}{|\kappa(s')^3\,\mathbf{B}|^2} = \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau.$$

73. First we find the quantities required to compute κ:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \quad\Rightarrow\quad \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \quad\Rightarrow\quad \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2 + b^2}$$

$$(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t) = (ab\sin t)(a\sin t) + (-ab\cos t)(-a\cos t) + (a^2)(0) = a^2b$$

[continued]

By Theorem 10, $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{a\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}\right)^3} = \dfrac{a}{a^2+b^2}$, which is a constant.

By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle ab\sin t, -ab\cos t, a^2\right\rangle \cdot \langle a\sin t, -a\cos t, 0\rangle}{\left(\sqrt{a^2b^2+a^4}\right)^2} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}\text{, which is also a constant.}$$

74. $\mathbf{r} = \langle \sinh t, \cosh t, t\rangle \quad\Rightarrow\quad \mathbf{r}' = \langle \cosh t, \sinh t, 1\rangle$, $\mathbf{r}'' = \langle \sinh t, \cosh t, 0\rangle$, $\mathbf{r}''' = \langle \cosh t, \sinh t, 0\rangle \quad\Rightarrow$

$\mathbf{r}' \times \mathbf{r}'' = \left\langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t\right\rangle = \langle -\cosh t, \sinh t, 1\rangle \quad\Rightarrow$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1\rangle|}{|\langle \cosh t, \sinh t, 1\rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{\left(\cosh^2 t + \sinh^2 t + 1\right)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1\rangle \cdot \langle \cosh t, \sinh t, 0\rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2\cosh^2 t} = \frac{-1}{2\cosh^2 t}$$

So at the point $(0,1,0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

75. (a) At any point P on C where $\kappa(t) \neq 0$, the circle of curvature of C at P has center a distance $1/\kappa(t)$ from P in the direction of the unit normal vector $\mathbf{N}(t)$. The position vector of P is $\mathbf{r}(t)$, so we get a position vector for the center of curvature by adding $\dfrac{1}{\kappa(t)}\mathbf{N}(t)$ to $\mathbf{r}(t)$: $\mathbf{r}_e(t) = \mathbf{r}(t) + \dfrac{1}{\kappa(t)}\mathbf{N}(t)$, $\kappa(t) \neq 0$.

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\dfrac{\sin t}{\sqrt{2}}\,\mathbf{i} + \dfrac{\cos t}{\sqrt{2}}\,\mathbf{j} + \dfrac{1}{\sqrt{2}}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{T}'(t) = -\dfrac{\cos t}{\sqrt{2}}\,\mathbf{i} - \dfrac{\sin t}{\sqrt{2}}\,\mathbf{j} \quad\Rightarrow\quad |\mathbf{T}'(t)| = \dfrac{1}{\sqrt{2}}$.

So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ and $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{2}}{\sqrt{2}} = \dfrac{1}{2}$. Thus,

$$\mathbf{r}_e(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} + 2(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + t\,\mathbf{k}.$$

(c) The parabola $y = x^2$ can be parameterized $x = t$, $y = t^2$, which gives the corresponding vector equation

$\mathbf{r}(t) = \left\langle t, t^2\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 2t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{1+4t^2}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle \quad\Rightarrow$

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{1+4t^2}}\langle 0, 2\rangle - \frac{4t}{(1+4t^2)^{3/2}}\langle 1, 2t\rangle = \frac{1}{(1+4t^2)^{3/2}}\left[(1+4t^2)\langle 0,2\rangle - 4t\langle 1, 2t\rangle\right]\\ &= \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle \quad\Rightarrow\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+4t^2)^{3/2}}\sqrt{16t^2+4} = \frac{2}{1+4t^2}.$$

So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1+4t^2}{2} \cdot \dfrac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle = \dfrac{1}{\sqrt{1+4t^2}}\langle -2t, 1\rangle$ and

$\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(1+4t^2)}{\sqrt{1+4t^2}} = \dfrac{2}{(1+4t^2)^{3/2}}$. Thus,

$$\mathbf{r}_e(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \langle t, t^2 \rangle + \frac{(1+4t^2)^{3/2}}{2} \cdot \frac{1}{\sqrt{1+4t^2}} \langle -2t, 1 \rangle$$
$$= \langle t, t^2 \rangle + (1+4t^2)\left\langle -t, \tfrac{1}{2} \right\rangle = \left\langle -4t^3, \tfrac{1}{2} + 3t^2 \right\rangle$$

To obtain a function form of the answer, note that $x = -4t^3 \quad \Rightarrow \quad t = (-x/4)^{1/3}$, so

$y_e = \frac{1}{2} + 3t^2 = \frac{1}{2} + 3[(-x/4)^{1/3}]^2 = \frac{1}{2} + 3(x/4)^{2/3}$.

76. (a) If C is planar, then it lies in a plane that we can express in the form $ax + by + cz = d$, where a, b, c, and d are not all zero. (See Equation 12.5.8.) For any t, the point $(x(t), y(t), z(t))$ lies on the curve and hence on the plane, so the equation $ax(t) + by(t) + cz(t) = d$ must be satisfied.

Conversely, if there exist scalars a, b, c, and d, not all zero, such that $ax(t) + by(t) + cz(t) = d$ for all t, then each point $(x(t), y(t), z(t))$ on C satisfies the equation $ax + by + cz = d$. By Exercise 12.5.83, this equation represents a plane, and hence C is planar.

(b) By part (a) there exist scalars a, b, c, and d, not all zero, such that for all t, $ax(t) + by(t) + cz(t) = d \quad \Rightarrow$ $\langle a, b, c \rangle \cdot \mathbf{r}(t) = d$. In addition, for all t, $ax'(t) + by'(t) + cz'(t) = 0 \quad \Rightarrow \quad \langle a, b, c \rangle \cdot \mathbf{r}'(t) = 0 \quad \Rightarrow \quad \mathbf{r}'$ is perpendicular to the normal vector $\mathbf{n} = \langle a, b, c \rangle$ of the plane containing $C \quad \Rightarrow \quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is also perpendicular to $\mathbf{n}$.

Similarly, $\langle a, b, c \rangle \cdot \mathbf{r}''(t) = 0 \quad \Rightarrow \quad \mathbf{r}''$ is perpendicular to $\mathbf{n} \quad \Rightarrow$

$\mathbf{T}'(t) \cdot \mathbf{n} = \dfrac{d}{dt}\left(\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) \cdot \mathbf{n} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \cdot \mathbf{n} + \dfrac{d}{dt}\left(\dfrac{1}{|\mathbf{r}'(t)|}\right)\mathbf{r}'(t) \cdot \mathbf{n} = 0 + 0 = 0$. So then $\mathbf{N} \cdot \mathbf{n} = \dfrac{\mathbf{T}'(t) \cdot \mathbf{n}}{|\mathbf{T}'(t)|} = 0$, and hence $\mathbf{N}$ is perpendicular to $\mathbf{n}$. So, since $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, it follows that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is parallel to $\mathbf{n}$ and hence is normal to the plane containing C.

(c) By part (b), $\mathbf{B}$ is normal to the plane containing C and so $\mathbf{B}(t) = \dfrac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}}$ for all t, that is, $\mathbf{B}$ is a constant vector.

Therefore, $\mathbf{B}'(t) = \mathbf{0}$ and $\tau(t) = -\dfrac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\dfrac{0}{|\mathbf{r}'(t)|} = 0$ for all t.

(d) The projection of the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$ in the xy-plane is the curve $\mathbf{r}(t) = \langle t, 2t, 0 \rangle$, which is the line with corresponding parametric equations $x = t$, $y = 2t$, $z = 0$. Therefore, the equation of the plane containing $\mathbf{r}(t)$ is $y = 2x$, or $2x - y = 0$, with normal vector $\mathbf{n} = \langle 2, -1, 0 \rangle$. By part (b), $\mathbf{B} = \frac{1}{\sqrt{5}} \langle 2, -1, 0 \rangle$.

77. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$L = \int_0^{2.9\times10^8\times2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2.9\times10^8\times2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\,dt = \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\,t\,\Big]_0^{2.9\times10^8\times2\pi}$$
$$= 2.9 \times 10^8 \times 2\pi \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2} \approx 2.07 \times 10^{10} \text{ Å — more than two meters!}$$

78. (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$.

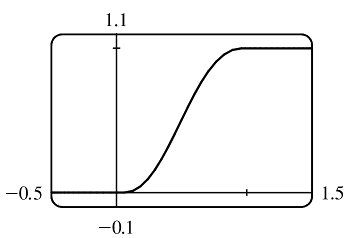
For F' to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \dfrac{|F''(x)|}{\left(1 + [F'(x)]^2\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$.

Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$\begin{array}{llll} P(0) = 0 \;\Rightarrow\; f = 0 & \textbf{(1)} & P(1) = 1 \;\Rightarrow\; a + b + c + d + e + f = 1 & \textbf{(2)} \\ P'(0) = 0 \;\Rightarrow\; e = 0 & \textbf{(3)} & P'(1) = 0 \;\Rightarrow\; 5a + 4b + 3c + 2d + e = 0 & \textbf{(4)} \\ P''(0) = 0 \;\Rightarrow\; d = 0 & \textbf{(5)} & P''(1) = 0 \;\Rightarrow\; 20a + 12b + 6c + 2d = 0 & \textbf{(6)} \end{array}$$

From **(1)**, **(3)**, and **(5)**, we have $d = e = f = 0$. Thus **(2)**, **(4)** and **(6)** become **(7)** $a + b + c = 1$, **(8)** $5a + 4b + 3c = 0$, and **(9)** $10a + 6b + 3c = 0$. Subtracting **(8)** from **(9)** gives **(10)** $5a + 2b = 0$. Multiplying **(7)** by 3 and subtracting from **(8)** gives **(11)** $2a + b = -3$. Multiplying **(11)** by 2 and subtracting from **(10)** gives $a = 6$. By **(10)**, $b = -15$. By **(7)**, $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

(b)



13.4 Motion in Space: Velocity and Acceleration

1. (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y\,(t)\;\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over the time interval $[0, 1]$ is

$$\mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$$

Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$:

$\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is

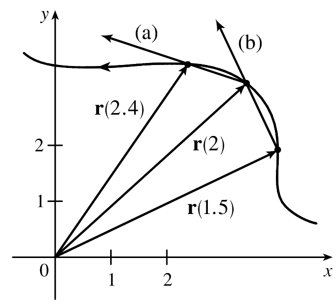
$|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58.$

2. (a) The average velocity over $2 \le t \le 2.4$ is

$\dfrac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5\,[\mathbf{r}(2.4) - \mathbf{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\mathbf{r}(2.4) - \mathbf{r}(2)]$.

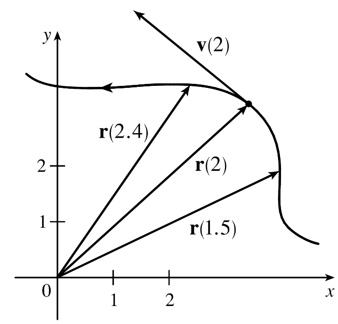
(b) The average velocity over $1.5 \le t \le 2$ is

$\dfrac{\mathbf{r}(2) - \mathbf{r}(1.5)}{2 - 1.5} = 2[\mathbf{r}(2) - \mathbf{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2) - \mathbf{r}(1.5)]$.



(c) Using Equation 2 we have $\mathbf{v}(2) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(2 + h) - \mathbf{r}(2)}{h}$.

(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing t. Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \le t \le 2.4$ and $1.5 \le t \le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8 + 2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.



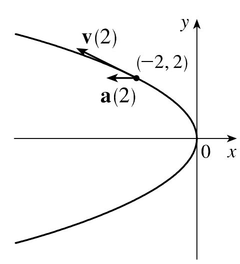
3. $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle \quad \Rightarrow$ At $t = 2$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -t, 1 \rangle$ $\quad$ $\mathbf{v}(2) = \langle -2, 1 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -1, 0 \rangle$ $\quad$ $\mathbf{a}(2) = \langle -1, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$

Notice that $x = -\frac{1}{2}y^2$, so the path is a parabola.



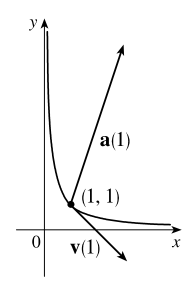
4. $\mathbf{r}(t) = \left\langle t^2, 1/t^2 \right\rangle \quad \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t, -2/t^3 \right\rangle$ $\quad$ $\mathbf{v}(1) = \langle 2, -2 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 2, 6/t^4 \right\rangle$ $\quad$ $\mathbf{a}(1) = \langle 2, 6 \rangle$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 4/t^6} = 2\sqrt{t^2 + 1/t^6}$

Notice that $y = 1/x$ and $x > 0$, so the path is part of the hyperbola $y = 1/x$.



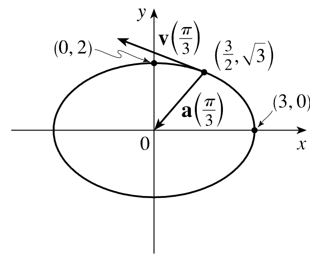
5. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} \quad\Rightarrow$ At $t = \pi/3$:

$\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ $\mathbf{v}\left(\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$ $\mathbf{a}\left(\frac{\pi}{3}\right) = -\frac{3}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t}$$
$$= \sqrt{4 + 5\sin^2 t}$$

Notice that $x^2/9 + y^2/4 = \sin^2 t + \cos^2 t = 1$, so the path is an ellipse.



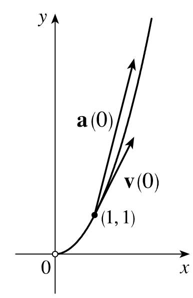
6. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} \quad\Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}$ $\mathbf{v}(0) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(t) = e^t\,\mathbf{i} + 4e^{2t}\,\mathbf{j}$ $\mathbf{a}(0) = \mathbf{i} + 4\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{e^{2t} + 4e^{4t}} = e^t\sqrt{1 + 4e^{2t}}$$

Notice that $y = e^{2t} = \left(e^t\right)^2 = x^2$, so the particle travels along a parabola, but $x = e^t$, so $x > 0$.



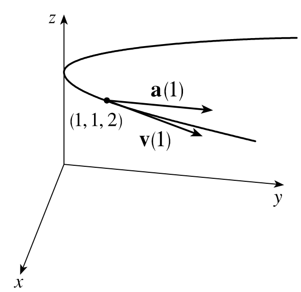
7. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k} \quad\Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$ $\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(t) = 2\,\mathbf{j}$ $\mathbf{a}(1) = 2\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$$

Here $x = t,\ y = t^2 \quad\Rightarrow\quad y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.



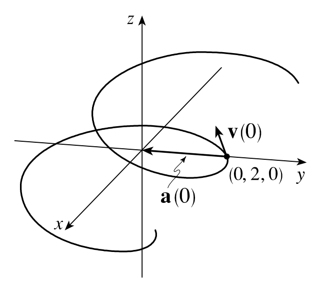
8. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k} \quad\Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = \mathbf{i} - 2\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$ $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$

$\mathbf{a}(t) = -2\cos t\,\mathbf{j} - \sin t\,\mathbf{k}$ $\mathbf{a}(0) = -2\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{1 + 4\sin^2 t + \cos^2 t} = \sqrt{2 + 3\sin^2 t}$$

Since $y^2/4 + z^2 = 1$, $x = t$, the path of the particle is an elliptical helix about the x-axis.



9. $\mathbf{r}(t) = \left\langle t^2 + t, t^2 - t, t^3\right\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t + 1, 2t - 1, 3t^2\right\rangle, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 6t\rangle,$

$|\mathbf{v}(t)| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2} = \sqrt{9t^4 + 8t^2 + 2}.$

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t\rangle, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t\rangle,$

$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}.$

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k},$

$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$

12. $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + 2\,\mathbf{j} + (1/t)\,\mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} - (1/t^2)\,\mathbf{k},$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 4 + (1/t^2)} = \sqrt{[2t + (1/t)]^2} = |2t + (1/t)|.$

13. $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + te^t\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}
\mathbf{v}(t) = \mathbf{r}'(t) &= [e^t(-\sin t) + (\cos t)e^t]\,\mathbf{i} + [e^t\cos t + (\sin t)e^t]\,\mathbf{j} + (te^t + e^t)\,\mathbf{k} \\
&= e^t[(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} + (t+1)\,\mathbf{k}]
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}(t) = \mathbf{v}'(t) &= [e^t(-\sin t - \cos t) + (\cos t - \sin t)e^t]\,\mathbf{i} + [e^t(\cos t - \sin t) + (\sin t + \cos t)e^t]\,\mathbf{j} \\
&\qquad + [e^t \cdot 1 + (t+1)e^t]\,\mathbf{k} \\
&= e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t+2)\,\mathbf{k}]
\end{aligned}$$

$$\begin{aligned}
|\mathbf{v}(t)| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}(t+1)^2} \\
&= \sqrt{e^{2t}}\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1} \\
&= e^t\sqrt{t^2 + 2t + 3}
\end{aligned}$$

14. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, \cos t - (-t\sin t + \cos t), -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle,$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, t\cos t + \sin t, -t\sin t + \cos t\rangle,$

$|\mathbf{v}(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}\,t \quad$ [since $t \geq 0$].

15. $\mathbf{a}(t) = 2\,\mathbf{i} + 2t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (2\,\mathbf{i} + 2t\,\mathbf{k})\,dt = 2t\,\mathbf{i} + t^2\,\mathbf{k} + \mathbf{C}$. Then $\mathbf{v}(0) = \mathbf{C}$ but we were given that $\mathbf{v}(0) = 3\,\mathbf{i} - \mathbf{j}$, so $\mathbf{C} = 3\,\mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{k} + 3\,\mathbf{i} - \mathbf{j} = (2t+3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left[(2t+3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}\right] dt = (t^2 + 3t)\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{D}$. Here $\mathbf{r}(0) = \mathbf{D}$ and we were given that $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, so $\mathbf{D} = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} + (1 - t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$.

16. $\mathbf{a}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k})\,dt = -\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t^2\,\mathbf{k} + \mathbf{C}$.

Then $\mathbf{v}(0) = -\mathbf{i} + \mathbf{C}$ but we were given that $\mathbf{v}(0) = -\mathbf{k}$, so $-\mathbf{i} + \mathbf{C} = -\mathbf{k} \quad \Rightarrow \quad \mathbf{C} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v}(t) = (1 - \cos t)\,\mathbf{i} + 2\sin t\,\mathbf{j} + (3t^2 - 1)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left[(1 - \cos t)\,\mathbf{i} + 2\sin t\,\mathbf{j} + (3t^2 - 1)\,\mathbf{k}\right] dt = (t - \sin t)\,\mathbf{i} - 2\cos t\,\mathbf{j} + (t^3 - t)\,\mathbf{k} + \mathbf{D}$. Here $\mathbf{r}(0) = -2\,\mathbf{j} + \mathbf{D}$ and we were given that $\mathbf{r}(0) = \mathbf{j} - 4\,\mathbf{k}$, so $\mathbf{D} = 3\,\mathbf{j} - 4\,\mathbf{k}$ and $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (3 - 2\cos t)\,\mathbf{j} + (t^3 - t - 4)\,\mathbf{k}$.

17. (a) $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}(t) = \int (2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt = t^2\,\mathbf{i} - \cos t\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k} + \mathbf{C}$

and $\mathbf{i} = \mathbf{v}\,(0) = -\mathbf{j} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} + \mathbf{j}$

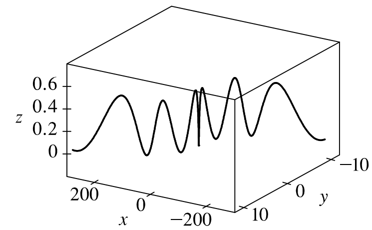
and $\mathbf{v}(t) = \left(t^2 + 1\right)\mathbf{i} + (1 - \cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}$.

$\mathbf{r}(t) = \int [\left(t^2 + 1\right)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}]dt$

$= \left(\frac{1}{3}t^3 + t\right)\,\mathbf{i} + (t - \sin t)\,\mathbf{j} - \frac{1}{4}\cos 2t\,\mathbf{k} + \mathbf{D}$

But $\mathbf{j} = \mathbf{r}\,(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{D}$, so $\mathbf{D} = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)



18. (a) $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}\,(t) = \int \left(t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}\right)dt = \frac{1}{2}t^2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k} + \mathbf{C}$

and $\mathbf{k} = \mathbf{v}\,(0) = \mathbf{j} - \mathbf{k} + \mathbf{C}$, so $\mathbf{C} = -\mathbf{j} + 2\,\mathbf{k}$

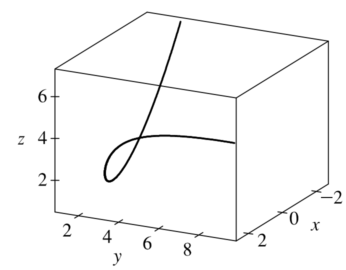
and $\mathbf{v}(t) = \frac{1}{2}t^2\,\mathbf{i} + \left(e^t - 1\right)\,\mathbf{j} + \left(2 - e^{-t}\right)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \left[\frac{1}{2}t^2\,\mathbf{i} + (e^t - 1)\,\mathbf{j} + (2 - e^{-t})\,\mathbf{k}\right]dt$

$= \frac{1}{6}t^3\,\mathbf{i} + (e^t - t)\,\mathbf{j} + (e^{-t} + 2t)\,\mathbf{k} + \mathbf{D}$

But $\mathbf{j} + \mathbf{k} = \mathbf{r}(0) = \mathbf{j} + \mathbf{k} + \mathbf{D}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = \frac{1}{6}t^3\,\mathbf{i} + (e^t - t)\,\mathbf{j} + (e^{-t} + 2t)\,\mathbf{k}$.

(b)



19. $\mathbf{r}(t) = \left\langle t^2, 5t, t^2 - 16t\right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$, $|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$

and $\dfrac{d}{dt}\,|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is,

$16t - 64 = 0$ or $t = 4$. Since $\dfrac{d}{dt}\,|\mathbf{v}(t)| < 0$ for $t < 4$ and $\dfrac{d}{dt}\,|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

20. Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law, $\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

21. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

22. The argument here is the same as that in the proof of Theorem 13.2.4 with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

23. $|\mathbf{v}(0)| = 200$ m/s and, since the angle of elevation is 60°, a unit vector in the direction of the velocity is $(\cos 60^\circ)\mathbf{i} + (\sin 60^\circ)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}$. Thus $\mathbf{v}(0) = 200\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) = 100\,\mathbf{i} + 100\,\sqrt{3}\,\mathbf{j}$ and if we set up the axes so that the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = m\mathbf{a}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s^2. Thus $\mathbf{a}(t) = -9.8\,\mathbf{j}$ and, integrating, we have $\mathbf{v}(t) = -9.8t\,\mathbf{j} + \mathbf{C}$. But

$100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j} = \mathbf{v}(0) = \mathbf{C}$, so $\mathbf{v}(t) = 100\,\mathbf{i} + \left(100\sqrt{3} - 9.8t\right)\mathbf{j}$ and then (integrating again)

$\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j} + \mathbf{D}$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{D}$. Thus the position function of the projectile is

$\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j}$.

(a) Parametric equations for the projectile are $x(t) = 100t$, $y(t) = 100\sqrt{3}\,t - 4.9t^2$. The projectile reaches the ground when

$y(t) = 0$ (and $t > 0$) $\Rightarrow$ $100\sqrt{3}\,t - 4.9t^2 = t\left(100\sqrt{3} - 4.9t\right) = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{4.9} \approx 35.3$ s. So the range is

$x\left(\frac{100\sqrt{3}}{4.9}\right) = 100\left(\frac{100\sqrt{3}}{4.9}\right) \approx 3535$ m.

(b) The maximum height is reached when $y(t)$ has a critical number (or equivalently, when the vertical component of velocity is 0): $y'(t) = 0$ $\Rightarrow$ $100\sqrt{3} - 9.8t = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{9.8} \approx 17.7$ s. Thus the maximum height is

$y\left(\frac{100\sqrt{3}}{9.8}\right) = 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1531$ m.

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3}}{4.9}$ s. Thus, the velocity at impact is

$\mathbf{v}\left(\frac{100\sqrt{3}}{4.9}\right) = 100\,\mathbf{i} + \left[100\sqrt{3} - 9.8\left(\frac{100\sqrt{3}}{4.9}\right)\right]\mathbf{j} = 100\,\mathbf{i} - 100\sqrt{3}\,\mathbf{j}$ and the speed is

$\left|\mathbf{v}\left(\frac{100\sqrt{3}}{4.9}\right)\right| = \sqrt{10{,}000 + 30{,}000} = 200\ \text{m/s}$.

24. As in Exercise 23, $\mathbf{v}(t) = 100\,\mathbf{i} + \left(100\sqrt{3} - 9.8t\right)\mathbf{j}$ and $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j} + \mathbf{D}$.

But $\mathbf{r}(0) = 100\,\mathbf{j}$, so $\mathbf{D} = 100\,\mathbf{j}$ and $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100 + 100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j}$.

(a) $y = 0$ $\Rightarrow$ $100 + 100\sqrt{3}\,t - 4.9t^2 = 0$ or $4.9t^2 - 100\sqrt{3}\,t - 100 = 0$. From the quadratic formula we have

$$t = \frac{100\sqrt{3} \pm \sqrt{(-100\sqrt{3}\,)^2 - 4(4.9)(-100)}}{2(4.9)} = \frac{100\sqrt{3} \pm \sqrt{31{,}960}}{9.8}.$$ Taking the positive t-value gives

$t = \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8} \approx 35.9$ s. Thus the range is $x = 100 \cdot \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8} \approx 3592$ m.

(b) The maximum height is attained when $\dfrac{dy}{dt} = 0$ $\Rightarrow$ $100\sqrt{3} - 9.8t = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{9.8} \approx 17.7$ s and the

maximum height is $100 + 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1631$ m.

Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 100 m higher, the maximum height reached is 100 m higher than that found in Exercise 23, that is, $1531\text{ m} + 100\text{ m} = 1631$ m.

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}$ s. Thus the velocity at impact is

$\mathbf{v}\left(\frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}\right) = 100\,\mathbf{i} + \left[100\sqrt{3} - 9.8\left(\frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}\right)\right]\mathbf{j} = 100\,\mathbf{i} - \sqrt{31{,}960}\,\mathbf{j}$ and the speed is

$|\mathbf{v}| = \sqrt{10{,}000 + 31{,}960} = \sqrt{41{,}960} \approx 205\ \text{m/s}$.

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. The ball lands when $y = 0$ (and $t > 0$) $\Rightarrow$ $t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = x = \frac{1}{2}v_0\sqrt{2}\,\dfrac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

26. Let α be the angle of elevation. Here $v_0 = 400$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$. We want $\dfrac{400^2 \sin 2\alpha}{g} = 3000$ $\Rightarrow$ $\sin 2\alpha = \dfrac{3000g}{400^2} \approx 0.1838$ $\Rightarrow$ $2\alpha \approx \sin^{-1}(0.1838) \approx 10.6°$ or $2\alpha \approx 180° - 10.6° = 169.4°$. Thus two angles of elevation are $\alpha \approx 5.3°$ and $\alpha \approx 84.7°$.

27. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 36°)t\,\mathbf{i} + \left[(v_0 \sin 36°)t - \frac{1}{2}gt^2\right]\mathbf{j}$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = (v_0 \cos 36°)\,\mathbf{i} + [(v_0 \sin 36°) - gt]\,\mathbf{j}$. The shell reaches its maximum height when the vertical component of velocity is zero, so $(v_0 \sin 36°) - gt = 0$ $\Rightarrow$ $t = \dfrac{v_0 \sin 36°}{g}$. The vertical height of the shell at that time is 1600 ft, so

$$(v_0 \sin 36°)\left(\frac{v_0 \sin 36°}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin 36°}{g}\right)^2 = 1600 \quad\Rightarrow\quad \left(\frac{v_0^2 \sin^2 36°}{g}\right) - \frac{1}{2}\left(\frac{v_0^2 \sin^2 36°}{g}\right) = 1600 \quad\Rightarrow$$

$$\frac{v_0^2 \sin^2 36°}{2g} = 1600 \quad\Rightarrow\quad v_0^2 = \frac{1600(2g)}{\sin^2 36°} \quad\Rightarrow\quad v_0 = \sqrt{\frac{3200g}{\sin^2 36°}} \approx \frac{\sqrt{3200(32)}}{\sin 36°} \approx 544 \text{ ft/s}.$$

28. Here $v_0 = 115$ ft/s, the angle of elevation is $\alpha = 50°$, and if we place the origin at home plate, then $\mathbf{r}(0) = 3\,\mathbf{j}$. As in Example 5, we have $\mathbf{r}\,(t) = -\frac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{D}$ where $\mathbf{D} = \mathbf{r}(0) = 3\,\mathbf{j}$ and $\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$, so $\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3\right]\mathbf{j}$. Thus, parametric equations for the trajectory of the ball are $x = (v_0 \cos\alpha)t,\ y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3$. The ball reaches the fence when $x = 400$ $\Rightarrow$ $(v_0 \cos\alpha)t = 400$ $\Rightarrow$ $t = \dfrac{400}{v_0 \cos\alpha} = \dfrac{400}{115 \cos 50°} \approx 5.41$ s. At this time, the height of the ball is $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3 \approx (115 \sin 50°)(5.41) - \frac{1}{2}(32)(5.41)^2 + 3 \approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

29. Place the catapult at the origin and assume the catapult is 100 meters from the city, so the city lies between $(100, 0)$ and $(600, 0)$. The initial speed is $v_0 = 80$ m/s and let θ be the angle the catapult is set at. As in Example 5, the trajectory of the catapulted rock is given by $\mathbf{r}\,(t) = (80 \cos\theta)t\,\mathbf{i} + \left[(80 \sin\theta)t - 4.9t^2\right]\mathbf{j}$. The top of the near city wall is at $(100, 15)$, which the rock will hit when $(80 \cos\theta)\,t = 100$ $\Rightarrow$ $t = \dfrac{5}{4\cos\theta}$ and $(80 \sin\theta)t - 4.9t^2 = 15$ $\Rightarrow$ $80 \sin\theta \cdot \dfrac{5}{4\cos\theta} - 4.9\left(\dfrac{5}{4\cos\theta}\right)^2 = 15$ $\Rightarrow$ $100 \tan\theta - 7.65625 \sec^2\theta = 15$. Replacing $\sec^2\theta$ with $\tan^2\theta + 1$ gives $7.65625 \tan^2\theta - 100 \tan\theta + 22.65625 = 0$. Using the quadratic formula, we have $\tan\theta \approx 0.230635, 12.8306$ $\Rightarrow$ $\theta \approx 13.0°, 85.5°$. So for $13.0° < \theta < 85.5°$, the rock will land beyond the near city wall. The base of the far wall is located at $(600, 0)$ which the rock hits if $(80 \cos\theta)t = 600$ $\Rightarrow$ $t = \dfrac{15}{2\cos\theta}$ and $(80 \sin\theta)t - 4.9t^2 = 0$ $\Rightarrow$

$80\sin\theta \cdot \dfrac{15}{2\cos\theta} - 4.9\left(\dfrac{15}{2\cos\theta}\right)^2 = 0 \quad\Rightarrow\quad 600\tan\theta - 275.625\sec^2\theta = 0 \quad\Rightarrow$

$275.625\tan^2\theta - 600\tan\theta + 275.625 = 0$. Solutions are $\tan\theta \approx 0.658678, 1.51819 \quad\Rightarrow\quad \theta \approx 33.4^\circ, 56.6^\circ$. Thus the rock lands beyond the enclosed city ground for $33.4^\circ < \theta < 56.6^\circ$, and the angles that allow the rock to land on city ground are $13.0^\circ < \theta < 33.4^\circ$, $56.6^\circ < \theta < 85.5^\circ$. If you consider that the rock can hit the far wall and bounce back into the city, we calculate the angles that cause the rock to hit the top of the wall at $(600, 15)$: $(80\cos\theta)t = 600 \quad\Rightarrow\quad t = \dfrac{15}{2\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 15 \quad\Rightarrow\quad 600\tan\theta - 275.625\sec^2\theta = 15 \quad\Rightarrow\quad 275.625\tan^2\theta - 600\tan\theta + 290.625 = 0$. Solutions are $\tan\theta \approx 0.727506, 1.44936 \quad\Rightarrow\quad \theta \approx 36.0^\circ, 55.4^\circ$, so the catapult should be set with angle θ where $13.0^\circ < \theta < 36.0^\circ$, $55.4^\circ < \theta < 85.5^\circ$.

30. If we place the projectile at the origin then, as in Example 5, $\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$ and $\mathbf{v}(t) = (v_0\cos\alpha)\,\mathbf{i} + [(v_0\sin\alpha) - gt]\,\mathbf{j}$. The maximum height is reached when the vertical component of velocity is zero, so $(v_0\sin\alpha) - gt = 0 \quad\Rightarrow\quad t = \dfrac{v_0\sin\alpha}{g}$, and the corresponding height is the vertical component of the position function:

$$(v_0\sin\alpha)t - \tfrac{1}{2}gt^2 = (v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0\sin\alpha}{g}\right)^2 = \frac{1}{2g}v_0^2\sin^2\alpha$$

Half that time is $t = \dfrac{v_0\sin\alpha}{2g}$, when the height of the projectile is

$$\begin{aligned}(v_0\sin\alpha)t - \tfrac{1}{2}gt^2 &= (v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{2g}\right) - \tfrac{1}{2}g\left(\frac{v_0\sin\alpha}{2g}\right)^2 \\ &= \frac{1}{2g}v_0^2\sin^2\alpha - \frac{1}{8g}v_0^2\sin^2\alpha = \frac{3}{8g}v_0^2\sin^2\alpha = \frac{3}{4}\left(\frac{1}{2g}v_0^2\sin^2\alpha\right)\end{aligned}$$

or three-quarters of the maximum height.

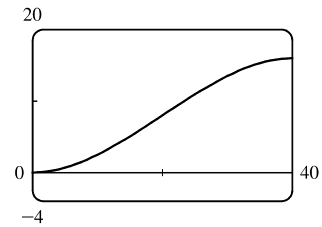
31. Here $\mathbf{a}(t) = -4\,\mathbf{j} - 32\,\mathbf{k}$ so $\mathbf{v}(t) = -4t\,\mathbf{j} - 32t\,\mathbf{k} + \mathbf{v}_0 = -4t\,\mathbf{j} - 32t\,\mathbf{k} + 50\,\mathbf{i} + 80\,\mathbf{k} = 50\,\mathbf{i} - 4t\,\mathbf{j} + (80 - 32t)\,\mathbf{k}$ and $\mathbf{r}(t) = 50t\,\mathbf{i} - 2t^2\,\mathbf{j} + (80t - 16t^2)\,\mathbf{k}$ (note that $\mathbf{r}_0 = \mathbf{0}$). The ball lands when the z-component of $\mathbf{r}(t)$ is zero and $t > 0$: $80t - 16t^2 = 16t(5 - t) = 0 \quad\Rightarrow\quad t = 5$. The position of the ball then is $\mathbf{r}(5) = 50(5)\,\mathbf{i} - 2(5)^2\,\mathbf{j} + [80(5) - 16(5)^2]\,\mathbf{k} = 250\,\mathbf{i} - 50\,\mathbf{j}$ or equivalently the point $(250, -50, 0)$. This is a distance of $\sqrt{250^2 + (-50)^2 + 0^2} = \sqrt{65{,}000} \approx 255$ ft from the origin at an angle of $\tan^{-1}\left(\frac{50}{250}\right) \approx 11.3^\circ$ from the eastern direction toward the south. The speed of the ball is $|\mathbf{v}(5)| = |50\,\mathbf{i} - 20\,\mathbf{j} - 80\,\mathbf{k}| = \sqrt{50^2 + (-20)^2 + (-80)^2} = \sqrt{9300} \approx 96.4$ ft/s.

32. Place the ball at the origin and consider $\mathbf{j}$ to be pointing in the northward direction with $\mathbf{i}$ pointing east and $\mathbf{k}$ pointing upward. Force = mass × acceleration $\quad\Rightarrow\quad$ acceleration = force/mass, so the wind applies a constant acceleration of $4\text{ N}/0.8\text{ kg} = 5\text{ m/s}^2$ in the easterly direction. Combined with the acceleration due to gravity, the acceleration acting on the ball is $\mathbf{a}(t) = 5\,\mathbf{i} - 9.8\,\mathbf{k}$. Then $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = 5t\,\mathbf{i} - 9.8t\,\mathbf{k} + \mathbf{C}$ where $\mathbf{C}$ is a constant vector.

We know $\mathbf{v}(0) = \mathbf{C} = -30\cos 30^\circ\,\mathbf{j} + 30\sin 30^\circ\,\mathbf{k} = -15\sqrt{3}\,\mathbf{j} + 15\,\mathbf{k} \quad\Rightarrow\quad \mathbf{C} = -15\sqrt{3}\,\mathbf{j} + 15\,\mathbf{k}$ and

$\mathbf{v}(t) = 5t\,\mathbf{i} - 15\sqrt{3}\,\mathbf{j} + (15 - 9.8t)\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = 2.5t^2\,\mathbf{i} - 15\sqrt{3}\,t\,\mathbf{j} + \left(15t - 4.9t^2\right)\mathbf{k} + \mathbf{D}$ but $\mathbf{r}(0) = \mathbf{D} = \mathbf{0}$ so $\mathbf{r}(t) = 2.5t^2\,\mathbf{i} - 15\sqrt{3}\,t\,\mathbf{j} + \left(15t - 4.9t^2\right)\mathbf{k}$. The ball lands when $15t - 4.9t^2 = 0 \;\Rightarrow\; t = 0$, $t = 15/4.9 \approx 3.0612$ s, so the ball lands at approximately $\mathbf{r}(3.0612) \approx 23.43\,\mathbf{i} - 79.53\,\mathbf{j}$ which is 82.9 m away in the direction S 16.4°E. Its speed is approximately $|\mathbf{v}(3.0612)| \approx \left|15.306\,\mathbf{i} - 15\sqrt{3}\,\mathbf{j} - 15\,\mathbf{k}\right| \approx 33.68$ m/s.

33. (a) After t seconds, the boat will be $5t$ meters west of point A. The velocity of the water at that location is $\frac{3}{400}(5t)(40 - 5t)\,\mathbf{j}$. The velocity of the boat in still water is $5\,\mathbf{i}$, so the resultant velocity of the boat is $\mathbf{v}(t) = 5\,\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\,\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{2}t - \frac{3}{16}t^2\right)\mathbf{j}$. Integrating, we obtain $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$. If we place the origin at A (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \;\Rightarrow\; \mathbf{C} = \mathbf{0}$ and we have $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$.



The boat reaches the east bank when $5t = 40$, that is, when $t = 8$ s, and it is located at $\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\,\mathbf{i} + 16\,\mathbf{j}$. Thus the boat is 16 m downstream.

(b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40 - 5(\cos\alpha)t]\;\mathbf{j}$. The resultant velocity of the boat is given by

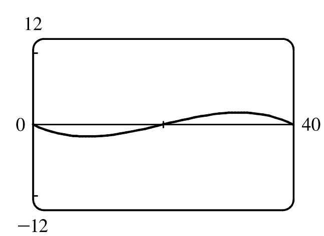
$\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + \frac{3}{400}(5t\cos\alpha)(40 - 5t\cos\alpha)\right]\mathbf{j} = (5\cos\alpha)\,\mathbf{i} + \left(5\sin\alpha + \frac{3}{2}t\cos\alpha - \frac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}$.

Integrating, $\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left(5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed the origin at A). The boat will reach the east bank when $5t\cos\alpha = 40 \;\Rightarrow\; t = \dfrac{40}{5\cos\alpha} = \dfrac{8}{\cos\alpha}$.

In order to land at point $B(40, 0)$ we need $5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha = 0 \;\Rightarrow$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha + \frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha - \frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha = 0 \;\Rightarrow\; \frac{1}{\cos\alpha}(40\sin\alpha + 48 - 32) = 0 \;\Rightarrow$$

$40\sin\alpha + 16 = 0 \;\Rightarrow\; \sin\alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6°$, so the boat should head 23.6° south of east (upstream). The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B.



34. As in Exercise 33(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $3\sin(\pi x/40)\,\mathbf{j} = 3\sin[\pi\cdot 5(\cos\alpha)t/40]\,\mathbf{j} = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat

then is given by $\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j} + \mathbf{C}.$$

If we place the origin at A then $\mathbf{r}(0) = \mathbf{0} \quad\Rightarrow\quad -\dfrac{24}{\pi\cos\alpha}\mathbf{j} + \mathbf{C} = \mathbf{0} \quad\Rightarrow\quad \mathbf{C} = \dfrac{24}{\pi\cos\alpha}\mathbf{j}$ and

$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha}\right]\mathbf{j}$. The boat will reach the east bank when

$5t\cos\alpha = 40 \quad\Rightarrow\quad t = \dfrac{8}{\cos\alpha}$. In order to land at point $B(40, 0)$ we need

$$5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha} = 0 \quad\Rightarrow$$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0 \quad\Rightarrow\quad \frac{1}{\cos\alpha}\left(40\sin\alpha - \frac{24}{\pi}\cos\pi + \frac{24}{\pi}\right) = 0 \quad\Rightarrow$$

$40\sin\alpha + \dfrac{48}{\pi} = 0 \quad\Rightarrow\quad \sin\alpha = -\dfrac{6}{5\pi}$. Thus $\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5^\circ$, so the boat should head 22.5° south of east.

35. If $\mathbf{r}'(t) = \mathbf{c}\times\mathbf{r}(t)$, then $\mathbf{r}'(t)$ is perpendicular to both $\mathbf{c}$ and $\mathbf{r}(t)$. Remember that $\mathbf{r}'(t)$ points in the direction of motion, so if $\mathbf{r}'(t)$ is always perpendicular to $\mathbf{c}$, the path of the particle must lie in a plane perpendicular to $\mathbf{c}$. But $\mathbf{r}'(t)$ is also perpendicular to the position vector $\mathbf{r}(t)$ which confines the path to a sphere centered at the origin. Considering both restrictions, the path must be contained in a circle that lies in a plane perpendicular to $\mathbf{c}$, and the circle is centered on a line through the origin in the direction of $\mathbf{c}$.

36. (a) From Equation 7 we have $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$. If a particle moves along a straight line, then $\kappa = 0$ [see Section 13.3], so the acceleration vector becomes $\mathbf{a} = v'\mathbf{T}$. Because the acceleration vector is a scalar multiple of the unit tangent vector, it is parallel to the tangent vector.

(b) If the speed of the particle is constant, then $v' = 0$ and Equation 7 gives $\mathbf{a} = \kappa v^2\mathbf{N}$. Thus the acceleration vector is parallel to the unit normal vector (which is perpendicular to the tangent vector and points in the direction that the curve is turning).

37. $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + t^3\,\mathbf{j} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} = t\sqrt{4+9t^2}$ [since $t \geq 0$], $\quad \mathbf{r}''(t) = 2\,\mathbf{i} + 6t\,\mathbf{j}, \quad \mathbf{r}'(t)\times\mathbf{r}''(t) = 6t^2\,\mathbf{k}$.

Then Equation 9 gives $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{(2t)(2) + (3t^2)(6t)}{t\sqrt{4+9t^2}} = \dfrac{4t + 18t^3}{t\sqrt{4+9t^2}} = \dfrac{4 + 18t^2}{\sqrt{4+9t^2}}$

$$\left[\begin{array}{l} \text{or by Equation 8,} \quad a_T = v' = \dfrac{d}{dt}\left[t\sqrt{4+9t^2}\,\right] = t\cdot\tfrac{1}{2}\left(4+9t^2\right)^{-1/2}(18t) + \left(4+9t^2\right)^{1/2}\cdot 1 \\ \qquad\qquad = \left(4+9t^2\right)^{-1/2}\left(9t^2 + 4 + 9t^2\right) = (4+18t^2)/\sqrt{4+9t^2} \end{array}\right]$$

and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{6t^2}{t\sqrt{4+9t^2}} = \dfrac{6t}{\sqrt{4+9t^2}}$.

38. $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \left(\frac{2}{3}t^3 - 2t\right)\mathbf{j} \quad\Rightarrow\quad \mathbf{r}'(t) = 4t\,\mathbf{i} + (2t^2 - 2)\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{16t^2 + (2t^2 - 2)^2} = \sqrt{4t^4 + 8t^2 + 4} = \sqrt{4\,(t^2+1)^2} = 2(t^2+1)$,

$\mathbf{r}''(t) = 4\,\mathbf{i} + 4t\,\mathbf{j}$, $\quad \mathbf{r}'(t) \times \mathbf{r}''(t) = (8t^2 + 8)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(4t)(4) + (2t^2-2)(4t)}{2(t^2+1)} = \frac{8t(t^2+1)}{2(t^2+1)} = 4t \quad \left[\text{or by Equation 8, } a_T = v' = \frac{d}{dt}\left[2(t^2+1)\right] = 4t\right]$$

and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{8(t^2+1)}{2(t^2+1)} = 4$.

39. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,

$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\,\cos t - \sin t\,\cos t}{\sqrt{2}} = 0$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$.

40. $\mathbf{r}(t) = t\,\mathbf{i} + 2e^t\,\mathbf{j} + e^{2t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \mathbf{i} + 2e^t\,\mathbf{j} + 2e^{2t}\,\mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{1 + 4e^{2t} + 4e^{4t}} = \sqrt{(1+2e^{2t})^2} = 1 + 2e^{2t}$,

$\mathbf{r}''(t) = 2e^t\,\mathbf{j} + 4e^{2t}\,\mathbf{k}$, $\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = 4e^{3t}\,\mathbf{i} - 4e^{2t}\,\mathbf{j} + 2e^t\,\mathbf{k}$,

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{16e^{6t} + 16e^{4t} + 4e^{2t}} = \sqrt{4e^{2t}(2e^{2t}+1)^2} = 2e^t(2e^{2t}+1)$. Then

$a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4e^{2t} + 8e^{4t}}{1+2e^{2t}} = \dfrac{4e^{2t}(1+2e^{2t})}{1+2e^{2t}} = 4e^{2t}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2e^t(2e^{2t}+1)}{1+2e^{2t}} = 2e^t$.

41. $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t^2 + 3t)\,\mathbf{j} + 4\sqrt{t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = (1/t)\,\mathbf{i} + (2t+3)\,\mathbf{j} + (2/\sqrt{t}\,)\,\mathbf{k} \quad\Rightarrow$

$\mathbf{r}''(t) = (-1/t^2)\,\mathbf{i} + 2\,\mathbf{j} - (1/t^{3/2})\,\mathbf{k}$. The point $(0,4,4)$ corresponds to $t = 1$, where

$\mathbf{r}'(1) = \mathbf{i} + 5\,\mathbf{j} + 2\,\mathbf{k}$, $\quad \mathbf{r}''(1) = -\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, and $\quad \mathbf{r}'(1)\times\mathbf{r}''(1) = -9\,\mathbf{i} - \mathbf{j} + 7\,\mathbf{k}$. Thus at the point $(0,4,4)$,

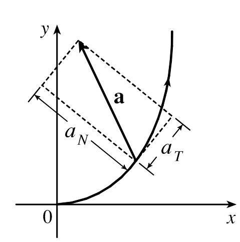
$a_T = \dfrac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \dfrac{-1 + 10 - 2}{\sqrt{1 + 25 + 4}} = \dfrac{7}{\sqrt{30}}$ and $a_N = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \dfrac{\sqrt{81 + 1 + 49}}{\sqrt{30}} = \sqrt{\dfrac{131}{30}}$.

42. $\mathbf{r}(t) = t^{-1}\,\mathbf{i} + t^{-2}\,\mathbf{j} + t^{-3}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -t^{-2}\,\mathbf{i} - 2t^{-3}\,\mathbf{j} - 3t^{-4}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}''(t) = 2t^{-3}\,\mathbf{i} + 6t^{-4}\,\mathbf{j} + 12t^{-5}\,\mathbf{k}$. The point $(1,1,1)$ corresponds to $t = 1$, where $\mathbf{r}'(1) = -\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$, $\quad \mathbf{r}''(1) = 2\,\mathbf{i} + 6\,\mathbf{j} + 12\,\mathbf{k}$, and

$\mathbf{r}'(1)\times\mathbf{r}''(1) = -6\,\mathbf{i} + 6\,\mathbf{j} - 2\,\mathbf{k}$. Thus at the point $(1,1,1)$, $a_T = \dfrac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \dfrac{-2 - 12 - 36}{\sqrt{1 + 4 + 9}} = -\dfrac{50}{\sqrt{14}}$ and

$a_N = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \dfrac{\sqrt{36 + 36 + 4}}{\sqrt{14}} = \sqrt{\dfrac{76}{14}} = \sqrt{\dfrac{38}{7}}$.

43. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5\ \text{cm/s}^2$ and $a_N \approx 9.0\ \text{cm/s}^2$.



44. $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t) \quad \Rightarrow$

$$\begin{aligned} \mathbf{L}'(t) &= m[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}\,(t) \times \mathbf{v}'(t)] \qquad \text{[by Formula 5 of Theorem 13.2.3]} \\ &= m[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] = m[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}\,(t)] = \boldsymbol{\tau}(t) \end{aligned}$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all t, and so $\mathbf{L}(t)$ is constant.

45. If the engines are turned off at time t, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a t such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9 \rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\,\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \dfrac{s}{t}, 7 - \dfrac{4}{t^2+1} + \dfrac{8st}{(t^2+1)^2} \right\rangle \quad \Rightarrow \quad 3 + t + s = 6 \quad \Rightarrow \quad s = 3 - t,$

so $7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \quad \Leftrightarrow \quad \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \quad \Leftrightarrow \quad t^4 + 8t^2 - 12t + 3 = 0.$

It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

46. (a) $m\,\dfrac{d\mathbf{v}}{dt} = \dfrac{dm}{dt}\,\mathbf{v}_e \quad \Leftrightarrow \quad \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{m}\,\dfrac{dm}{dt}\,\mathbf{v}_e$. Integrating both sides of this equation with respect to t gives

$$\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e \int_0^t \frac{1}{m}\,\frac{dm}{du}\,du \quad \Rightarrow \quad \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m} \quad \text{[Substitution Rule]} \quad \Rightarrow$$

$$\mathbf{v}(t) - \mathbf{v}(0) = \mathbf{v}_e \ln \frac{m(t)}{m(0)} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{v}(0) - \mathbf{v}_e \ln \frac{m(0)}{m(t)}.$$

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left| -\ln\!\left(\dfrac{m(0)}{m(t)}\right) \mathbf{v}_e \right| \quad \Rightarrow$

$2\,|\mathbf{v}_e| = \ln\!\left(\dfrac{m(0)}{m\,(t)}\right) |\mathbf{v}_e|$. $\left[\textit{Note: } m(0) > m(t) \text{ so that } \ln\!\left(\dfrac{m(0)}{m(t)}\right) > 0\right] \quad \Rightarrow \quad m(t) = e^{-2}m(0).$

Thus $\dfrac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.

APPLIED PROJECT Kepler's Laws

1. With $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$ and $\mathbf{h} = \alpha\,\mathbf{k}$ where $\alpha > 0$,

(a) $\mathbf{h} = \mathbf{r} \times \mathbf{r}' = [(r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}] \times \left[\left(r'\cos\theta - r\sin\theta\,\dfrac{d\theta}{dt}\right)\mathbf{i} + \left(r'\sin\theta + r\cos\theta\,\dfrac{d\theta}{dt}\right)\mathbf{j}\right]$

$$= \left[rr'\cos\theta\sin\theta + r^2\cos^2\theta\,\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\,\frac{d\theta}{dt}\right]\mathbf{k} = r^2\,\frac{d\theta}{dt}\,\mathbf{k}$$

(b) Since $\mathbf{h} = \alpha\,\mathbf{k}$, $\alpha > 0$, $\alpha = |\mathbf{h}|$. But by part (a), $\alpha = |\mathbf{h}| = r^2\,(d\theta/dt)$.

(c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta} |\mathbf{r}|^2\,d\theta = \frac{1}{2}\int_{t_0}^{t} r^2\,(d\theta/dt)\,dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{r^2}{2}\,\frac{d\theta}{dt}.$$

(d) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt} = \dfrac{h}{2} =$ constant since $\mathbf{h}$ is a constant vector and $h = |\mathbf{h}|$.

2. (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But $A(T) =$ area of the ellipse $= \pi ab$ and $A(T) = \frac{1}{2}hT$, so $T = 2\pi ab/h$.

(b) $h^2/(GM) = ed$ where e is the eccentricity of the ellipse. But $a = ed/(1-e^2)$ or $ed = a(1-e^2)$ and $1 - e^2 = b^2/a^2$. Hence $h^2/(GM) = ed = b^2/a$.

(c) $T^2 = \dfrac{4\pi a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \dfrac{a}{GMb^2} = \dfrac{4\pi^2}{GM}a^3$.

3. From Problem 2, $T^2 = \dfrac{4\pi^2}{GM}a^3$. $T \approx 365.25 \text{ days} \times 24 \cdot 60^2 \dfrac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7$ seconds. Therefore

$a^3 = \dfrac{GMT^2}{4\pi^2} \approx \dfrac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(3.1558 \times 10^7)^2}{4\pi^2} \approx 3.348 \times 10^{33}\text{ m}^3 \quad\Rightarrow\quad a \approx 1.496 \times 10^{11}$ m. Thus, the length of the major axis of the earth's orbit (that is, $2a$) is approximately $2.99 \times 10^{11}\text{ m} = 2.99 \times 10^8$ km.

4. We can adapt the equation $T^2 = \dfrac{4\pi^2}{GM}a^3$ from Problem 2(c) with the earth at the center of the system, so T is the period of the satellite's orbit about the earth, M is the mass of the earth, and a is the length of the semimajor axis of the satellite's orbit (measured from the earth's center). Since we want the satellite to remain fixed above a particular point on the earth's equator, T must coincide with the period of the earth's own rotation, so $T = 24\text{ h} = 86{,}400$ s. The mass of the earth is

$M = 5.98 \times 10^{24}$ kg, so $a = \left(\dfrac{T^2 GM}{4\pi^2}\right)^{1/3} \approx \left[\dfrac{(86{,}400)^2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{4\pi^2}\right]^{1/3} \approx 4.23 \times 10^7$ m. If we assume a circular orbit, the radius of the orbit is a, and since the radius of the earth is 6.37×10^6 m, the required altitude above the earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7$ m, or 35,900 km.

13 Review

TRUE-FALSE QUIZ

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

2. True. Parametric equations for the curve are $x = 0$, $y = t^2$, $z = 4t$, and since $t = z/4$ we have $y = t^2 = (z/4)^2$ or $y = \frac{1}{16}z^2$, $x = 0$. This is an equation of a parabola in the yz-plane.

3. False. The vector function $\mathbf{r}(t) = \langle 2t, 3-t, 0\rangle$ represents a line, but the line does not pass through the origin; the x-component is 0 only for $t = 0$, which corresponds to the point $(0, 3, 0)$, not $(0, 0, 0)$.

4. True. See Theorem 13.2.2.

5. False. By Formula 5 of Theorem 13.2.3, $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \quad \Rightarrow \quad \dfrac{d}{dt}\,|\mathbf{r}(t)| = 0$, but

$|\mathbf{r}'(t)| = |\langle -\sin t, \cos t \rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.

7. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.

8. False. The binormal vector, by the definition given in Section 13.3, is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -\,[\mathbf{N}(t) \times \mathbf{T}(t)]$.

9. True. At an inflection point where f is twice continuously differentiable we must have $f''(x) = 0$, and by Equation 13.3.11, the curvature is 0 there.

10. True. From Equation 13.3.9 , $\kappa(t) = 0 \quad \Leftrightarrow \quad |\mathbf{T}'(t)| = 0 \quad \Leftrightarrow \quad \mathbf{T}'(t) = \mathbf{0}$ for all t. But then $\mathbf{T}(t) = \mathbf{C}$, a constant vector, which is true only for a straight line.

11. False. If $\mathbf{r}(t)$ is the position of a moving particle at time t and $|\mathbf{r}(t)| = 1$ then the particle lies on the unit circle or the unit sphere, but this does not mean that the speed $|\mathbf{r}'(t)|$ must be constant. As a counterexample, let $\mathbf{r}(t) = \left\langle t, \sqrt{1-t^2} \right\rangle$, then $\mathbf{r}'(t) = \left\langle 1, -t/\sqrt{1-t^2} \right\rangle$ and $|\mathbf{r}(t)| = \sqrt{t^2 + 1 - t^2} = 1$ but $|\mathbf{r}'(t)| = \sqrt{1 + t^2/(1-t^2)} = 1/\sqrt{1-t^2}$ which is not constant.

12. True. See Theorem 13.2.4.

13. True. See the discussion preceding Example 8 in Section 13.3.

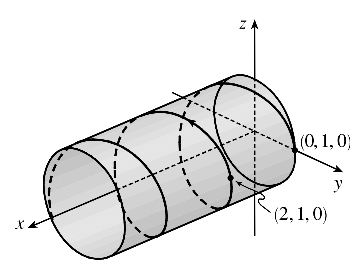
14. False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1 \rangle$ for all t, while $\mathbf{r}_2'(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

15. True. The projection in the xz-plane is given by $\mathbf{r}(t) = \langle \cos 2t, 0, \sin 2t \rangle$. Since $x^2 + z^2 = (\cos 2t)^2 + (\sin 2t)^2 = 1$, the projection onto the xz-plane is a circle with radius 1.

16. True. Note that the direction vector for both lines is $\mathbf{d} = \langle 1, 2, 1 \rangle$, so the lines are parallel. Further, the point $(0, 0, 1)$ is contained on both lines for $t = 0$ and $t = 1$, respectively. The lines are parallel and have a point in common; therefore, they are the same line.

EXERCISES

1. (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the x-axis. Since $x = t$, the curve is a helix.



(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

2. (a) The expressions $\sqrt{2-t}$, $(e^t-1)/t$, and $\ln(t+1)$ are all defined when $2-t \geq 0 \quad \Rightarrow \quad t \leq 2, t \neq 0$, and $t+1>0 \quad \Rightarrow \quad t>-1$. Thus the domain of $\mathbf{r}$ is $(-1,0) \cup (0,2]$.

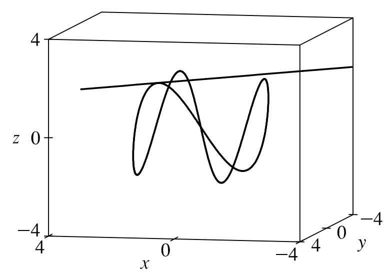
(b) $\lim\limits_{t \to 0} \mathbf{r}(t) = \left\langle \lim\limits_{t \to 0} \sqrt{2-t}, \lim\limits_{t \to 0} \dfrac{e^t-1}{t}, \lim\limits_{t \to 0} \ln(t+1) \right\rangle = \left\langle \sqrt{2-0}, \lim\limits_{t \to 0} \dfrac{e^t}{1}, \ln(0+1) \right\rangle$

$= \left\langle \sqrt{2}, 1, 0 \right\rangle$ [using l'Hospital's Rule in the y-component]

(c) $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\sqrt{2-t}, \dfrac{d}{dt}\dfrac{e^t-1}{t}, \dfrac{d}{dt}\ln(t+1) \right\rangle = \left\langle -\dfrac{1}{2\sqrt{2-t}}, \dfrac{te^t-e^t+1}{t^2}, \dfrac{1}{t+1} \right\rangle$

3. The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2=16, z=0$. So we can write $x=4\cos t,\ y=4\sin t,\ 0 \leq t \leq 2\pi$. From the equation of the plane, we have $z=5-x=5-4\cos t$, so parametric equations for C are $x=4\cos t,\ y=4\sin t,\ z=5-4\cos t, 0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t)=4\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}+(5-4\cos t)\,\mathbf{k}, 0 \leq t \leq 2\pi$.

4. The curve is given by $\mathbf{r}(t)=\langle 2\sin t, 2\sin 2t, 2\sin 3t\rangle$, so $\mathbf{r}'(t)=\langle 2\cos t, 4\cos 2t, 6\cos 3t\rangle$. The point $\left(1,\sqrt{3},2\right)$ corresponds to $t=\frac{\pi}{6}$ (or $\frac{\pi}{6}+2k\pi$, k an integer), so the tangent vector there is $\mathbf{r}'(\frac{\pi}{6})=\left\langle\sqrt{3},2,0\right\rangle$. Then the tangent line has direction vector $\left\langle\sqrt{3},2,0\right\rangle$ and includes the point $\left(1,\sqrt{3},2\right)$, so parametric equations are $x=1+\sqrt{3}\,t, y=\sqrt{3}+2t, z=2$.



5. $\int_0^1 (t^2\,\mathbf{i}+t\cos\pi t\,\mathbf{j}+\sin\pi t\,\mathbf{k})\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i}+\left(\int_0^1 t\cos\pi t\,dt\right)\mathbf{j}+\left(\int_0^1 \sin\pi t\,dt\right)\mathbf{k}$

$= \left[\frac{1}{3}t^3\right]_0^1\mathbf{i}+\left(\frac{t}{\pi}\sin\pi t\right]_0^1-\int_0^1\frac{1}{\pi}\sin\pi t\,dt\Big)\mathbf{j}+\left[-\frac{1}{\pi}\cos\pi t\right]_0^1\mathbf{k}$

$= \frac{1}{3}\,\mathbf{i}+\left[\frac{1}{\pi^2}\cos\pi t\right]_0^1\mathbf{j}+\frac{2}{\pi}\,\mathbf{k}=\frac{1}{3}\,\mathbf{i}-\frac{2}{\pi^2}\,\mathbf{j}+\frac{2}{\pi}\,\mathbf{k}$

where we integrated by parts in the y-component.

6. (a) C intersects the xz-plane where $y=0 \quad \Rightarrow \quad 2t-1=0 \quad \Rightarrow \quad t=\frac{1}{2}$, so the point is $\left(2-\left(\frac{1}{2}\right)^3, 0, \ln\frac{1}{2}\right)=\left(\frac{15}{8},0,-\ln 2\right)$.

(b) The curve is given by $\mathbf{r}(t)=\left\langle 2-t^3, 2t-1, \ln t\right\rangle$, so $\mathbf{r}'(t)=\left\langle -3t^2, 2, 1/t\right\rangle$. The point $(1,1,0)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1)=\langle -3,2,1\rangle$. Then the tangent line has direction vector $\langle -3,2,1\rangle$ and includes the point $(1,1,0)$, so parametric equations are $x=1-3t, y=1+2t, z=t$.

(c) The normal plane has normal vector $\mathbf{r}'(1)=\langle -3,2,1\rangle$ and equation $-3(x-1)+2(y-1)+z=0$ or $3x-2y-z=1$.

7. $\mathbf{r}(t)=\left\langle t^2,t^3,t^4\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t)=\left\langle 2t,3t^2,4t^3\right\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)|=\sqrt{4t^2+9t^4+16t^6}$ and $L=\int_0^3|\mathbf{r}'(t)|\,dt=\int_0^3\sqrt{4t^2+9t^4+16t^6}\,dt$. Using Simpson's Rule with $f(t)=\sqrt{4t^2+9t^4+16t^6}$ and $n=6$ we

have $\Delta t = \frac{3-0}{6} = \frac{1}{2}$ and

$$\begin{aligned} L &\approx \tfrac{\Delta t}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right] \\ &= \tfrac{1}{6}\Bigg[\sqrt{0+0+0} + 4\cdot\sqrt{4\left(\tfrac{1}{2}\right)^2 + 9\left(\tfrac{1}{2}\right)^4 + 16\left(\tfrac{1}{2}\right)^6} + 2\cdot\sqrt{4(1)^2 + 9(1)^4 + 16(1)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{3}{2}\right)^2 + 9\left(\tfrac{3}{2}\right)^4 + 16\left(\tfrac{3}{2}\right)^6} + 2\cdot\sqrt{4(2)^2 + 9(2)^4 + 16(2)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{5}{2}\right)^2 + 9\left(\tfrac{5}{2}\right)^4 + 16\left(\tfrac{5}{2}\right)^6} + \sqrt{4(3)^2 + 9(3)^4 + 16(3)^6}\,\Bigg] \\ &\approx 86.631 \end{aligned}$$

8. $\mathbf{r}(t) = \left\langle 2t^{3/2}, \cos 2t, \sin 2t\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 3t^{1/2}, -2\sin 2t, 2\cos 2t\right\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t+4}$.

Thus, $L = \int_0^1 \sqrt{9t+4}\,dt = \int_4^{13} \frac{1}{9}u^{1/2}\,du = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{2}{27}(13^{3/2} - 8)$.

9. $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ and $\mathbf{r}_2(t) = (1+t)\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. The angle θ of intersection of the two curves is the angle between their respective tangents at the point of intersection. For both curves, the point $(1,0,0)$ corresponds to $t=0$.

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad \mathbf{r}_1'(0) = \mathbf{j} + \mathbf{k}$, and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}_2'(0) = \mathbf{i}$.

$\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0) = (\mathbf{j}+\mathbf{k})\cdot\mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = 90^\circ$.

10. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + e^t\cos t\,\mathbf{k}$. The parametric value corresponding to the point $(1,0,1)$ is $t=0$.

$\mathbf{r}'(t) = e^t\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t(\cos t - \sin t)\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = e^t\sqrt{1^2 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3}\,e^t$

and $s(t) = \int_0^t e^u\sqrt{3}\,du = \sqrt{3}(e^t - 1) \quad\Rightarrow\quad 1 + \frac{1}{\sqrt{3}}s = e^t \quad\Rightarrow\quad t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right)$.

Therefore, $\mathbf{r}(t(s)) = \left(1+\frac{1}{\sqrt{3}}s\right)\mathbf{i} + \left(1+\frac{1}{\sqrt{3}}s\right)\sin\ln\left(1+\frac{1}{\sqrt{3}}s\right)\mathbf{j} + \left(1+\frac{1}{\sqrt{3}}s\right)\cos\ln\left(1+\frac{1}{\sqrt{3}}s\right)\mathbf{k}$.

11. (a) $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\rangle$,

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t + 4\sin^2 t\cos^2 t} \\ &= \sqrt{\sin^2 t\cos^2 t\,(9\sin^2 t + 9\cos^2 t + 4)} = \sqrt{13}\,\sin t\cos t \quad [\text{since } 0\le t\le \pi/2 \quad\Rightarrow\quad \sin t, \cos t \ge 0] \end{aligned}$$

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{13}\,\sin t\cos t}\left\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\right\rangle = \frac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle$.

(b) $\mathbf{T}'(t) = \frac{1}{\sqrt{13}}\langle 3\cos t, 3\sin t, 0\rangle$, $\quad|\mathbf{T}'(t)| = \frac{1}{\sqrt{13}}\sqrt{9\cos^2 t + 9\sin^2 t + 0} = \frac{3}{\sqrt{13}}$, and

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{3}\langle 3\cos t, 3\sin t, 0\rangle = \langle \cos t, \sin t, 0\rangle$.

(c) $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle \times \langle \cos t, \sin t, 0\rangle = \frac{1}{\sqrt{13}}\langle -2\sin t, 2\cos t, 3\rangle$

(d) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{3/\sqrt{13}}{\sqrt{13}\,\sin t\cos t} = \dfrac{3}{13\,\sin t\cos t}$ or $\frac{3}{13}\csc t\sec t$

(e) $\tau(t) = -\dfrac{\mathbf{B}'(t)\cdot\mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\dfrac{\frac{1}{\sqrt{13}}\langle -2\cos t, -2\sin t, 0\rangle\cdot\langle\cos t, \sin t, 0\rangle}{\sqrt{13}\,\sin t\cos t} = -\dfrac{-2\cos^2 t - 2\sin^2 t + 0}{13\sin t\cos t}$

$= \dfrac{2}{13\,\sin t\cos t}$ or $\frac{2}{13}\csc t\sec t$

12. See the instructions for Exercises 13.3.46–49. $x = 3\cos t,\ y = 4\sin t \ \Rightarrow\ \dot{x} = -3\sin t,\ \dot{y} = 4\cos t \ \Rightarrow$ $\ddot{x} = -3\cos t,\ \ddot{y} = -4\sin t$.

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{|(-3\sin t)(-4\sin t) - (4\cos t)(-3\cos t)|}{(9\sin^2 t + 16\cos^2 t)^{3/2}} = \frac{12}{(9\sin^2 t + 16\ \cos^2 t)^{3/2}}.$$

At $(3, 0)$, $t = 0$ and $\kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}$. At $(0, 4)$, $t = \frac{\pi}{2}$ and $\kappa\left(\frac{\pi}{2}\right) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}$.

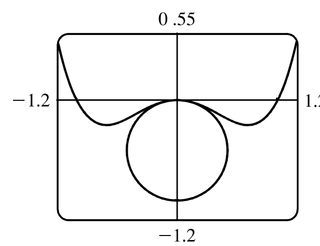
13. $y = x^4,\ y' = 4x^3,\ y'' = 12x^2$, and $\kappa(x) = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{\left|12x^2\right|}{(1 + 16x^6)^{3/2}}$, so $\kappa(1) = \dfrac{12}{17^{3/2}}$.

14. $y = x^4 - x^2,\ y' = 4x^3 - 2x,\ y'' = 12x^2 - 2$, and

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|12x^2 - 2\right|}{[1 + (4x^3 - 2x)^2]^{3/2}} \ \Rightarrow\ \kappa(0) = 2.$$

So the osculating circle has radius $\frac{1}{2}$ and center $\left(0, -\frac{1}{2}\right)$.

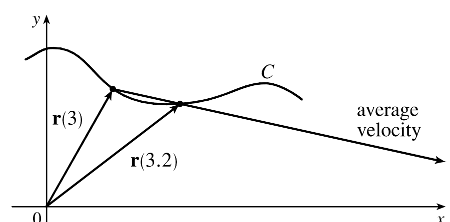
Thus, its equation is $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.



15. $\mathbf{r}(t) = \langle\sin 2t, t, \cos 2t\rangle \ \Rightarrow\ \mathbf{r}'(t) = \langle 2\cos 2t, 1, -2\sin 2t\rangle \ \Rightarrow\ \mathbf{T}(t) = \frac{1}{\sqrt{5}}\langle 2\cos 2t, 1, -2\sin 2t\rangle \ \Rightarrow$ $\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -4\sin 2t, 0, -4\cos 2t\rangle \ \Rightarrow\ \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t\rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1\rangle$ and $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle$. So a normal to the osculating plane is $\langle -1, 2, 0\rangle$ and an equation is $-1(x - 0) + 2(y - \pi) + 0(z - 1) = 0$ or $x - 2y + 2\pi = 0$.

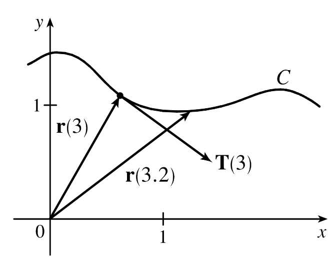
16. (a) The average velocity over $[3, 3.2]$ is given by

$\dfrac{\mathbf{r}(3.2) - \mathbf{r}(3)}{3.2 - 3} = 5[\mathbf{r}(3.2) - \mathbf{r}(3)]$, so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2) - \mathbf{r}(3)]$.



(b) $\mathbf{v}(3) = \mathbf{r}'(3) = \lim\limits_{h\to 0}\dfrac{\mathbf{r}(3 + h) - \mathbf{r}(3)}{h}$

(c) $\mathbf{T}(3) = \dfrac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}$, a unit vector in the same direction as $\mathbf{r}'(3)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing t, and with length 1.



17. $\mathbf{r}(t) = t\ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $\quad \mathbf{v}(t) = \mathbf{r}'(t) = (1+\ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k}$,

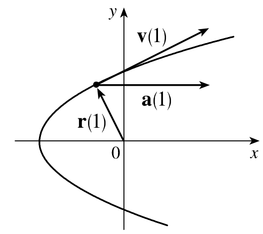
$|\mathbf{v}\,(t)| = \sqrt{(1+\ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\quad \mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t}\,\mathbf{i} + e^{-t}\,\mathbf{k}$

18. $\mathbf{r}(t) = (2t^2 - 3)\,\mathbf{i} + 2t\,\mathbf{j} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 4t\,\mathbf{i} + 2\,\mathbf{j}$,

speed $= |\mathbf{v}(t)| = \sqrt{16t^2 + 4} = 2\sqrt{4t^2 + 1}$, and $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = 4\,\mathbf{i}$.

At $t = 1$ we have $\mathbf{r}(1) = -\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{v}(1) = 4\,\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{a}(1) = 4\,\mathbf{i}$.

Notice that $y/2 = t \quad \Rightarrow \quad x = 2\,(y/2)^2 - 3 = \frac{1}{2}y^2 - 3$, so the path is a parabola.



19. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k})\,dt = 3t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - 3t^2\,\mathbf{k} + \mathbf{C}$, but $\mathbf{i} - \mathbf{j} + 3\,\mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{C}$,

so $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v}(t) = (3t^2 + 1)\,\mathbf{i} + (4t^3 - 1)\,\mathbf{j} + (3 - 3t^2)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k} + \mathbf{D}$.

But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$.

20. We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a 45° angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \quad \Rightarrow \quad \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 13.4.5 we have $\mathbf{a} = -g\,\mathbf{j}$ where here $g \approx 32$ ft/s^2. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \quad \Rightarrow \quad \mathbf{v}\,(t) = \frac{43}{\sqrt{2}}\,\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so

$\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\,\mathbf{j} \quad \Rightarrow \quad \mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\,\mathbf{i} + 3.8\,\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \quad \Rightarrow$

$t = \dfrac{43}{\sqrt{2}\,g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\,\mathbf{i} + 21.4\,\mathbf{j}$, so the maximum height is approximately 21.4 ft.

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \quad \Rightarrow$

$-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \quad \Rightarrow \quad t \approx 2.11$ s. $\quad \mathbf{r}(2.11) \approx 64.2\,\mathbf{i} - 0.08\,\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

21. Example 13.4.5 showed that the maximum horizontal range is achieved with an angle of elevation of $45\,^\circ$. In this case, however, the projectile would hit the top of the tunnel using that angle. The horizontal range will be maximized with the largest angle of elevation that keeps the projectile within a height of 30 m. From Example 13.4.5 we know that the position function of the projectile is $\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$ and the velocity is

$\mathbf{v}(t) = \mathbf{r}'(t) = (v_0 \cos\alpha)\,\mathbf{i} + [(v_0 \sin\alpha) - gt]\,\mathbf{j}$. The projectile achieves its maximum height when the vertical component of velocity is zero, so $(v_0 \sin\alpha) - gt = 0 \;\Rightarrow\; t = \dfrac{v_0 \sin\alpha}{g}$. We want the vertical height of the projectile at that time to be 30 m: $\quad (v_0 \sin\alpha)\left(\dfrac{v_0 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\dfrac{v_0 \sin\alpha}{g}\right)^2 = 30 \;\Rightarrow$

$$\left(\frac{v_0^2 \sin^2\alpha}{g}\right) - \frac{1}{2}\left(\frac{v_0^2 \sin^2\alpha}{g}\right) = 30 \;\Rightarrow\; \frac{v_0^2 \sin^2\alpha}{2g} = 30 \;\Rightarrow\; \sin^2\alpha = \frac{30(2g)}{v_0^2} = \frac{60(9.8)}{40^2} = 0.3675 \;\Rightarrow$$

$\sin\alpha = \sqrt{0.3675}$. Thus the desired angle of elevation is $\alpha = \sin^{-1}\sqrt{0.3675} \approx 37.3\,^\circ$.

From the same example, the horizontal distance traveled is $d = \dfrac{v_0^2 \sin 2\alpha}{g} \approx \dfrac{40^2 \sin(74.6\,^\circ)}{9.8} \approx 157.4$ m.

22. $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$, $\quad \mathbf{r}'(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}$, $\quad \mathbf{r}''(t) = 2\,\mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{1+4+4t^2} = \sqrt{4t^2+5}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2+5}}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{|4\,\mathbf{i} - 2\,\mathbf{j}|}{\sqrt{4t^2+5}} = \dfrac{2\sqrt{5}}{\sqrt{4t^2+5}}$.

23. (a) Instead of proceeding directly, we use Formula 3 of Theorem 13.2.3: $\mathbf{r}(t) = t\,\mathbf{R}(t) \;\Rightarrow$

$\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have

$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] \;\Rightarrow$$

$$\begin{aligned}\mathbf{a} = \mathbf{v}' &= e^{-t}[\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t}[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}\end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

24. (a) $F(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ \sqrt{2} - x & \text{if } x \ge \frac{1}{\sqrt{2}} \end{cases} \quad\Rightarrow\quad F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \quad\Rightarrow$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

since $\dfrac{d}{dx}[-x(1-x^2)^{-1/2}] = -(1-x^2)^{-1/2} - x^2(1-x^2)^{-3/2} = -(1-x^2)^{-3/2}$.

Now $\lim\limits_{x\to 0^+}\sqrt{1-x^2} = 1 = F(0)$ and $\lim\limits_{x\to(1/\sqrt{2})^-}\sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so F is continuous. Also, since

$\lim_{x\to 0^+} F'(x) = 0 = \lim_{x\to 0^-} F'(x)$ and $\lim_{x\to (1/\sqrt{2})^-} F'(x) = -1 = \lim_{x\to (1/\sqrt{2})^+} F'(x)$, F' is continuous. But

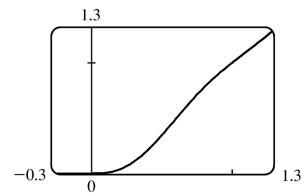
$\lim_{x\to 0^+} F''(x) = -1 \neq 0 = \lim_{x\to 0^-} F''(x)$, so F'' is not continuous at $x = 0$. (The same is true at $x = \frac{1}{\sqrt{2}}$.)

So F does not have continuous curvature.

(b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on P are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere

$\kappa(x) = \dfrac{|P''(x)|}{(1 + [P'(x)]^2)^{3/2}}$, so we need $P''(0) = 0$ and $P''(1) = 0$.

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$. The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$. Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.



☐ PROBLEMS PLUS

1. (a) $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{r}'(t) = -\omega R\sin\omega t\,\mathbf{i} + \omega R\cos\omega t\,\mathbf{j}$, so $\mathbf{r} = R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$ and $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$. $\mathbf{v}\cdot\mathbf{r} = \omega R^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$, so $\mathbf{v}\perp\mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v}\perp\mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.

(b) In (a), we wrote $\mathbf{v}$ in the form $\omega R\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j}$. Clearly $|\mathbf{v}| = \omega R\,|\mathbf{u}| = \omega R$. At speed ωR, the particle completes one revolution, a distance $2\pi R$, in time $T = \dfrac{2\pi R}{\omega R} = \dfrac{2\pi}{\omega}$.

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 R\cos\omega t\,\mathbf{i} - \omega^2 R\sin\omega t\,\mathbf{j} = -\omega^2 R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$, so $\mathbf{a} = -\omega^2\mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2\,|\mathbf{r}| = \omega^2 R$.

(d) By Newton's Second Law (see Section 13.4), $\mathbf{F} = m\mathbf{a}$, so $|\mathbf{F}| = m\,|\mathbf{a}| = mR\omega^2 = \dfrac{m\,(\omega R)^2}{R} = \dfrac{m\,|\mathbf{v}|^2}{R}$.

2. (a) Dividing the equation $|\mathbf{F}|\sin\theta = \dfrac{mv_R^2}{R}$ by the equation $|\mathbf{F}|\cos\theta = mg$, we obtain $\tan\theta = \dfrac{v_R^2}{Rg}$, so $v_R^2 = Rg\tan\theta$.

(b) $R = 400$ ft and $\theta = 12^\circ$, so $v_R = \sqrt{Rg\tan\theta} \approx \sqrt{400\cdot 32\cdot\tan 12^\circ} \approx 52.16$ ft/s ≈ 36 mi/h.

(c) We want to choose a new radius R_1 for which the new rated speed is $\frac{3}{2}$ of the old one: $\sqrt{R_1 g\tan 12^\circ} = \frac{3}{2}\sqrt{Rg\tan 12^\circ}$. Squaring, we get $R_1 g\tan 12^\circ = \frac{9}{4}Rg\tan 12^\circ$, so $R_1 = \frac{9}{4}R = \frac{9}{4}(400) = 900$ ft.

3. (a) The projectile reaches maximum height when $0 = \dfrac{dy}{dt} = \dfrac{d}{dt}\left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right] = v_0\sin\alpha - gt$; that is, when $t = \dfrac{v_0\sin\alpha}{g}$ and $y = (v_0\sin\alpha)\left(\dfrac{v_0\sin\alpha}{g}\right) - \dfrac{1}{2}g\left(\dfrac{v_0\sin\alpha}{g}\right)^2 = \dfrac{v_0^2\sin^2\alpha}{2g}$. This is the maximum height attained when the projectile is fired with an angle of elevation α. This maximum height is largest when $\alpha = 90^\circ$. In that case, $\sin\alpha = 1$ and the maximum height is $\dfrac{v_0^2}{2g}$.

(b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. The points on or inside this parabola are those for which $-R \le x \le R$ and $0 \le y \le \dfrac{-1}{2R}x^2 + \dfrac{R}{2}$. When the projectile is fired at angle of elevation α, the points (x, y) along its path satisfy the relations $x = (v_0\cos\alpha)\,t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, where $0 \le t \le (2v_0\sin\alpha)/g$ (as in Example 13.4.5). Thus

$|x| \le \left|v_0\cos\alpha\left(\dfrac{2v_0\sin\alpha}{g}\right)\right| = \left|\dfrac{v_0^2}{g}\sin 2\alpha\right| \le \left|\dfrac{v_0^2}{g}\right| = |R|$. This shows that $-R \le x \le R$.

For t in the specified range, we also have $y = t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = \frac{1}{2}gt\left(\dfrac{2v_0\sin\alpha}{g} - t\right) \ge 0$ and

$y = (v_0 \sin\alpha)\,\dfrac{x}{v_0\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \dfrac{g}{2v_0^2\cos^2\alpha}x^2 = -\dfrac{1}{2R\cos^2\alpha}\,x^2 + (\tan\alpha)\,x$. Thus

$$\begin{aligned} y - \left(\frac{-1}{2R}x^2 + \frac{R}{2}\right) &= \frac{-1}{2R\cos^2\alpha}\,x^2 + \frac{1}{2R}x^2 + (\tan\alpha)\,x - \frac{R}{2} \\ &= \frac{x^2}{2R}\left(1 - \frac{1}{\cos^2\alpha}\right) + (\tan\alpha)\,x - \frac{R}{2} = \frac{x^2(1-\sec^2\alpha) + 2R\,(\tan\alpha)\,x - R^2}{2R} \\ &= \frac{-(\tan^2\alpha)\,x^2 + 2R\,(\tan\alpha)\,x - R^2}{2R} = \frac{-\left[(\tan\alpha)\,x - R\right]^2}{2R} \le 0 \end{aligned}$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola $y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$.

Now let (a,b) be any point on or inside the parabola $y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$. Then $-R \le a \le R$ and $0 \le b \le -\dfrac{1}{2R}\,a^2 + \dfrac{R}{2}$.

We seek an angle α such that (a,b) lies in the path of the projectile; that is, we wish to find an angle α such that

$b = -\dfrac{1}{2R\cos^2\alpha}\,a^2 + (\tan\alpha)\,a$ or equivalently $b = \dfrac{-1}{2R}\,(\tan^2\alpha + 1)a^2 + (\tan\alpha)\,a$. Rearranging this equation we get

$\dfrac{a^2}{2R}\tan^2\alpha - a\tan\alpha + \left(\dfrac{a^2}{2R} + b\right) = 0$ or $a^2(\tan\alpha)^2 - 2aR(\tan\alpha) + (a^2 + 2bR) = 0$ $(\star)$. This quadratic equation

for $\tan\alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2 - 4AC \ge 0 \quad\Leftrightarrow$

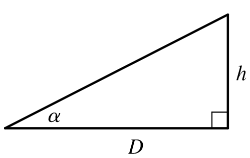
$(-2aR)^2 - 4a^2(a^2 + 2bR) \ge 0 \quad\Leftrightarrow\quad 4a^2(R^2 - a^2 - 2bR) \ge 0 \quad\Leftrightarrow\quad -a^2 - 2bR + R^2 \ge 0 \quad\Leftrightarrow$

$b \le \dfrac{1}{2R}\,(R^2 - a^2) \quad\Leftrightarrow\quad b \le \dfrac{-1}{2R}\,a^2 + \dfrac{R}{2}$. This condition is satisfied since (a,b) is on or inside the parabola

$y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$. It follows that (a,b) lies in the path of the projectile when $\tan\alpha$ satisfies $(\star)$, that is, when

$\tan\alpha = \dfrac{2aR \pm \sqrt{4a^2(R^2 - a^2 - 2bR)}}{2a^2} = \dfrac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}$.

(c)



If the gun is pointed at a target with height h at a distance D downrange, then $\tan\alpha = h/D$. When the projectile reaches a distance D downrange (remember we are assuming that it doesn't hit the ground first), we have $D = x = (v_0\cos\alpha)t$, so $t = \dfrac{D}{v_0\cos\alpha}$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2\cos^2\alpha}$.

Meanwhile, the target, whose x-coordinate is also D, has fallen from height h to height

$h - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2\cos^2\alpha}$. Thus the projectile hits the target.

4. (a) As in Problem 3, $\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$, so $x = (v_0\cos\alpha)t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$. The difference here is that the projectile travels until it reaches a point where $x > 0$ and $y = -(\tan\theta)x$. (Here $0 \le \theta \le \frac{\pi}{2}$.)

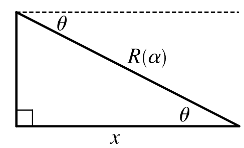
From the parametric equations, we obtain $t = \dfrac{x}{v_0\cos\alpha}$ and $y = \dfrac{(v_0\sin\alpha)x}{v_0\cos\alpha} - \dfrac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha)x - \dfrac{gx^2}{2v_0^2\cos^2\alpha}$.

Thus the projectile hits the inclined plane at the point where $(\tan\alpha)x - \dfrac{gx^2}{2v_0^2\cos^2\alpha} = -(\tan\theta)x$. Since

$\dfrac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha + \tan\theta)x$ and $x > 0$, we must have $\dfrac{gx}{2v_0^2\cos^2\alpha} = \tan\alpha + \tan\theta$. It follows that

$x = \dfrac{2v_0^2 \cos^2 \alpha}{g}(\tan\alpha + \tan\theta)$ and $t = \dfrac{x}{v_0 \cos\alpha} = \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)$. This means that the parametric equations are defined for t in the interval $\left[0, \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)\right]$.

(b) The downhill range (that is, the distance to the projectile's landing point as measured along the inclined plane) is $R(\alpha) = x\sec\theta$, where x is the coordinate of the landing point calculated in part (a). Thus



$$R(\alpha) = \frac{2v_0^2 \cos^2\alpha}{g}(\tan\alpha + \tan\theta)\sec\theta = \frac{2v_0^2}{g}\left(\frac{\sin\alpha\,\cos\alpha}{\cos\theta} + \frac{\cos^2\alpha\,\sin\theta}{\cos^2\theta}\right)$$
$$= \frac{2v_0^2\cos\alpha}{g\cos^2\theta}(\sin\alpha\,\cos\theta + \cos\alpha\,\sin\theta) = \frac{2v_0^2\cos\alpha\,\sin(\alpha+\theta)}{g\cos^2\theta}$$

$R(\alpha)$ is maximized when

$$0 = R'(\alpha) = \frac{2v_0^2}{g\cos^2\theta}\left[-\sin\alpha\,\sin(\alpha+\theta) + \cos\alpha\,\cos(\alpha+\theta)\right]$$
$$= \frac{2v_0^2}{g\cos^2\theta}\cos[(\alpha+\theta)+\alpha] = \frac{2v_0^2\cos(2\alpha+\theta)}{g\cos^2\theta}$$

This condition implies that $\cos(2\alpha+\theta) = 0 \quad\Rightarrow\quad 2\alpha+\theta = \frac{\pi}{2} \quad\Rightarrow\quad \alpha = \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$.

(c) The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where $x > 0$ and $y = (\tan\theta)x$. Since $\tan\theta = -\tan(-\theta)$, we obtain the solution from the previous one by replacing θ with $-\theta$. The desired angle is $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$.

(d) As observed in part (c), firing the projectile up an inclined plane with angle of inclination θ involves the same equations as in parts (a) and (b) but with θ replaced by $-\theta$. So if R is the distance up an inclined plane, we know from part (b) that $R = \dfrac{2v_0^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2(-\theta)} \quad\Rightarrow\quad v_0^2 = \dfrac{Rg\cos^2\theta}{2\cos\alpha\sin(\alpha-\theta)}$. v_0^2 is minimized (and hence v_0 is minimized) with respect to α when

$$0 = \frac{d}{d\alpha}\left(v_0^2\right) = \frac{Rg\cos^2\theta}{2}\cdot\frac{-(\cos\alpha\cos(\alpha-\theta) - \sin\alpha\sin(\alpha-\theta))}{[\cos\alpha\sin(\alpha-\theta)]^2}$$
$$= \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos[\alpha+(\alpha-\theta)]}{[\cos\alpha\sin(\alpha-\theta)]^2} = \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos(2\alpha-\theta)}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

Since $\theta < \alpha < \frac{\pi}{2}$, this implies $\cos(2\alpha-\theta) = 0 \quad\Leftrightarrow\quad 2\alpha-\theta = \frac{\pi}{2} \quad\Rightarrow\quad \alpha = \frac{1}{2}\left(\frac{\pi}{2}+\theta\right)$. Thus the initial speed, and hence the energy required, is minimized for $\alpha = \frac{1}{2}\left(\frac{\pi}{2}+\theta\right)$.

5. (a) $\mathbf{a} = -g\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{v}_0 - gt\,\mathbf{j} = 2\,\mathbf{i} - gt\,\mathbf{j} \quad\Rightarrow\quad \mathbf{s} = \mathbf{s}_0 + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} = 3.5\,\mathbf{j} + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} \quad\Rightarrow$ $\mathbf{s} = 2t\,\mathbf{i} + \left(3.5 - \frac{1}{2}gt^2\right)\mathbf{j}$. Therefore $y = 0$ when $t = \sqrt{7/g}$ seconds. At that instant, the ball is $2\sqrt{7/g} \approx 0.94$ ft to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are $(0.94, 0)$. At impact, the velocity is $\mathbf{v} = 2\,\mathbf{i} - \sqrt{7g}\,\mathbf{j}$, so the speed is $|\mathbf{v}| = \sqrt{4+7g} \approx 15$ ft/s.

(b) The slope of the curve when $t = \sqrt{\frac{7}{g}}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt}{2} = \frac{-g\sqrt{7/g}}{2} = \frac{-\sqrt{7g}}{2}$. Thus $\cot\theta = \frac{\sqrt{7g}}{2}$ and $\theta \approx 7.6°$.

(c) From (a), $|\mathbf{v}| = \sqrt{4+7g}$. So the ball rebounds with speed $0.8\sqrt{4+7g} \approx 12.08$ ft/s at angle of inclination $90° - \theta \approx 82.3886°$. By Example 13.4.5, the horizontal distance traveled between bounces is $d = \frac{v_0^2 \sin 2\alpha}{g}$, where $v_0 \approx 12.08$ ft/s and $\alpha \approx 82.3886°$. Therefore, $d \approx 1.197$ ft. So the ball strikes the floor at about $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table's edge.

6. By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \left\langle \sin\left(\frac{1}{2}\pi t^2\right), \cos\left(\frac{1}{2}\pi t^2\right)\right\rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$. Thus $\mathbf{T}'(t) = \pi t\left\langle \cos\left(\frac{1}{2}\pi t^2\right), -\sin\left(\frac{1}{2}\pi t^2\right)\right\rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t)^2(1)} = \pi\,|t|$.

7. The trajectory of the projectile is given by $\mathbf{r}(t) = (v\cos\alpha)t\,\mathbf{i} + \left[(v\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$, so $\mathbf{v}(t) = \mathbf{r}'(t) = v\cos\alpha\,\mathbf{i} + (v\sin\alpha - gt)\,\mathbf{j}$ and

$$|\mathbf{v}(t)| = \sqrt{(v\cos\alpha)^2 + (v\sin\alpha - gt)^2} = \sqrt{v^2 - (2vg\sin\alpha)\,t + g^2t^2} = \sqrt{g^2\left(t^2 - \frac{2v}{g}(\sin\alpha)\,t + \frac{v^2}{g^2}\right)}$$

$$= g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2} - \frac{v^2}{g^2}\sin^2\alpha} = g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha}$$

The projectile hits the ground when $(v\sin\alpha)t - \frac{1}{2}gt^2 = 0 \quad\Rightarrow\quad t = \frac{2v}{g}\sin\alpha$, so the distance traveled by the projectile is

$$L(\alpha) = \int_0^{(2v/g)\sin\alpha} |\mathbf{v}(t)|\,dt = \int_0^{(2v/g)\sin\alpha} g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha}\,dt$$

$$= g\left[\frac{t-(v/g)\sin\alpha}{2}\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} + \frac{[(v/g)\cos\alpha]^2}{2}\ln\left(t - \frac{v}{g}\sin\alpha + \sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]_0^{(2v/g)\sin\alpha}$$

[using Formula 21 in the Table of Integrals]

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} + \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right.$$
$$\left. + \frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} - \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(-\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]$$

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\cdot\frac{v}{g} + \frac{v^2}{g^2}\cos^2\alpha\ln\left(\frac{v}{g}\sin\alpha + \frac{v}{g}\right) + \frac{v}{g}\sin\alpha\cdot\frac{v}{g} - \frac{v^2}{g^2}\cos^2\alpha\ln\left(-\frac{v}{g}\sin\alpha + \frac{v}{g}\right)\right]$$

$$= \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{(v/g)\sin\alpha + v/g}{-(v/g)\sin\alpha + v/g}\right) = \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

We want to maximize $L(\alpha)$ for $0 \le \alpha \le \pi/2$.

$$L'(\alpha) = \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha \cdot \frac{1-\sin\alpha}{1+\sin\alpha} \cdot \frac{2\cos\alpha}{(1-\sin\alpha)^2} - 2\cos\alpha\,\sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

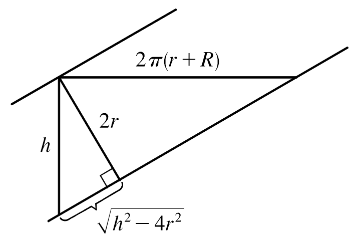
$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha \cdot \frac{2}{\cos\alpha} - 2\cos\alpha\,\sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{g}\cos\alpha\left[1 - \sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right] = \frac{v^2}{g}\cos\alpha\left[2 - \sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$L(\alpha)$ has critical points for $0 < \alpha < \pi/2$ when $L'(\alpha) = 0 \;\Rightarrow\; 2 - \sin\alpha\ln\left(\dfrac{1+\sin\alpha}{1-\sin\alpha}\right) = 0$ [since $\cos\alpha \neq 0$].

Solving by graphing (or using a CAS) gives $\alpha \approx 0.9855$. Compare values at the critical point and the endpoints: $L(0) = 0$, $L(\pi/2) = v^2/g$, and $L(0.9855) \approx 1.20v^2/g$. Thus the distance traveled by the projectile is maximized for $\alpha \approx 0.9855$ or $\approx 56°$.

8. As the cable is wrapped around the spool, think of the top or bottom of the cable forming a helix of radius $R + r$. Let h be the vertical distance between coils. Then, from similar triangles,



$$\frac{2r}{\sqrt{h^2-4r^2}} = \frac{2\pi(r+R)}{h} \quad\Rightarrow\quad h^2r^2 = \pi^2(r+R)^2(h^2-4r^2) \quad\Rightarrow$$

$$h = \frac{2\pi r(r+R)}{\sqrt{\pi^2(r+R)^2 - r^2}}.$$

If we parametrize the helix by $x(t) = (R+r)\cos t$, $y(t) = (R+r)\sin t$, then we must have $z(t) = [h/(2\pi)]t$. The length of one complete cycle is

$$\ell = \int_0^{2\pi}\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_0^{2\pi}\sqrt{(R+r)^2 + \left(\frac{h}{2\pi}\right)^2}\,dt = 2\pi\sqrt{(R+r)^2 + \left(\frac{h}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{(R+r)^2 + \frac{r^2(R+r)^2}{\pi^2(R+r)^2 - r^2}} = 2\pi(R+r)\sqrt{1 + \frac{r^2}{\pi^2(R+r)^2 - r^2}} = \frac{2\pi^2(R+r)^2}{\sqrt{\pi^2(R+r)^2 - r^2}}$$

The number of complete cycles is $[\![L/\ell]\!]$, and so the shortest length along the spool is

$$h\left[\!\!\left[\frac{L}{\ell}\right]\!\!\right] = \frac{2\pi r(R+r)}{\sqrt{\pi^2(R+r)^2 - r^2}}\left[\!\!\left[\frac{L\sqrt{\pi^2(R+r)^2 - r^2}}{2\pi^2(R+r)^2}\right]\!\!\right]$$

9. We can write the vector equation as $\mathbf{r}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$. Then $\mathbf{r}'(t) = 2t\,\mathbf{a} + \mathbf{b}$ which says that each tangent vector is the sum of a scalar multiple of $\mathbf{a}$ and the vector $\mathbf{b}$. Thus the tangent vectors are all parallel to the plane determined by $\mathbf{a}$ and $\mathbf{b}$ so the curve must be parallel to this plane. [Here we assume that $\mathbf{a}$ and $\mathbf{b}$ are nonparallel. Otherwise the tangent vectors are all parallel and the curve lies along a single line.] A normal vector for the plane is $\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$. The point (c_1, c_2, c_3) lies on the plane (when $t = 0$), so an equation of the plane is

$$(a_2b_3 - a_3b_2)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2) + (a_1b_2 - a_2b_1)(z - c_3) = 0$$

or

$$(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z = a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$$

14 □ PARTIAL DERIVATIVES

14.1 Functions of Several Variables

1. (a) $f(x,y) = \dfrac{x^2 y}{2x - y^2} \quad\Rightarrow\quad f(1,3) = \dfrac{1^2(3)}{2(1) - 3^2} = -\dfrac{3}{7}$

(b) $f(-2,-1) = \dfrac{(-2)^2(-1)}{2(-2) - (-1)^2} = \dfrac{4}{5}$

(c) $f(x+h,y) = \dfrac{(x+h)^2 y}{2(x+h) - y^2}$

(d) $f(x,x) = \dfrac{x^2 x}{2x - x^2} = \dfrac{x^3}{x(2-x)} = \dfrac{x^2}{2-x}$

2. (a) $g(x,y) = x\sin y + y\sin x \quad\Rightarrow\quad g(\pi,0) = \pi\sin 0 + 0\sin\pi = \pi\cdot 0 + 0\cdot 0 = 0$

(b) $g\left(\frac{\pi}{2},\frac{\pi}{4}\right) = \frac{\pi}{2}\sin\frac{\pi}{4} + \frac{\pi}{4}\sin\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{4}(1) = \dfrac{\pi\left(\sqrt{2}+1\right)}{4}$

(c) $g(0,y) = 0\sin y + y\sin 0 = 0 + y\cdot 0 = 0$

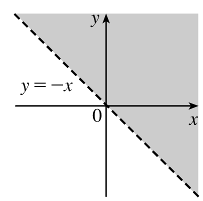
(d) $g(x,y+h) = x\sin(y+h) + (y+h)\sin x$

3. (a) $g(x,y) = x^2\ln(x+y) \quad\Rightarrow\quad g(3,1) = 3^2\ln(3+1) = 9\ln 4$

(b) $\ln(x+y)$ is defined only when $x+y>0 \quad\Rightarrow\quad y>-x$.

Thus, the domain of g is $\{(x,y) \mid y>-x\}$.

(c) The range of $\ln(x+y)$ is $\mathbb{R}$, so the range of g is $\mathbb{R}$.

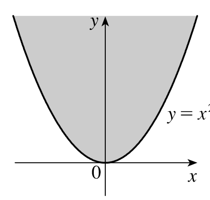


4. (a) $h(x,y) = e^{\sqrt{y-x^2}} \quad\Rightarrow\quad h(-2,5) = e^{\sqrt{5-(-2)^2}} = e^{\sqrt{1}} = e$

(b) $\sqrt{y-x^2}$ is defined only when $y-x^2\geq 0 \quad\Rightarrow\quad y\geq x^2$.

Thus, the domain of h is $\{(x,y) \mid y\geq x^2\}$.

(c) We know $\sqrt{y-x^2}\geq 0 \quad\Rightarrow\quad e^{\sqrt{y-x^2}}\geq 1$. Thus, the range of h is $[1,\infty]$.



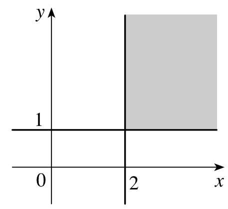
5. (a) $F(x,y,z) = \sqrt{y} - \sqrt{x-2z} \quad\Rightarrow\quad F(3,4,1) = \sqrt{4} - \sqrt{3-2(1)} = 2-1 = 1$

(b) $\sqrt{y}$ is defined only when $y\geq 0$. $\sqrt{x-2z}$ is defined only when $x-2z\geq 0 \quad\Rightarrow\quad z\leq\frac{1}{2}x$. Thus, the domain is $\{(x,y,z) \mid x\geq 2z, y\geq 0\}$, which is the set of points on or below the plane $z=\frac{1}{2}x$ and on or to the right of the xz-plane.

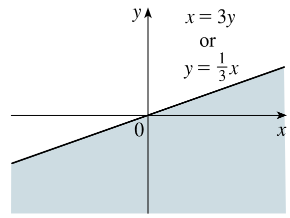
6. (a) $f(x,y,z) = \ln\left(z - \sqrt{x^2+y^2}\right) \quad\Rightarrow\quad f(4,-3,6) = \ln\left(6 - \sqrt{4^2+(-3)^2}\right) = \ln\left(6-\sqrt{25}\right) = \ln 1 = 0$

(b) $\ln\left(z-\sqrt{x^2+y^2}\right)$ is defined only when $z-\sqrt{x^2+y^2}>0 \quad\Leftrightarrow\quad z>\sqrt{x^2+y^2} \quad\Rightarrow\quad z^2>x^2+y^2$. Thus, the domain is $\left\{(x,y,z) \mid z>\sqrt{x^2+y^2}\right\}$, which is the set of points inside the top half of the cone $z^2=x^2+y^2$.

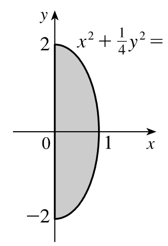
7. $f(x, y) = \sqrt{x-2} + \sqrt{y-1}$. $\sqrt{x-2}$ is defined only when $x - 2 \ge 0$, or $x \ge 2$, and $\sqrt{y-1}$ is defined only when $y - 1 \ge 0$, or $y \ge 1$. So the domain of f is $\{(x, y) \mid x \ge 2,\ y \ge 1\}$.



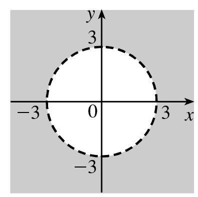
8. $f(x, y) = \sqrt[4]{x - 3y}$. $\sqrt[4]{x-3y}$ is defined only when $x - 3y \ge 0$, or $x \ge 3y$. So the domain of f is $\{(x, y) \mid x \ge 3y\}$ or equivalently $\left\{(x, y) \mid y \le \frac{1}{3}x\right\}$.



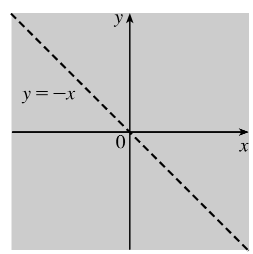
9. $q(x, y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$. $\sqrt{x}$ is defined only when $x \ge 0$. $\sqrt{4 - 4x^2 - y^2}$ is defined only when $4 - 4x^2 - y^2 \ge 0 \quad \Leftrightarrow \quad 1 \ge x^2 + \frac{1}{4}y^2$. So the domain of q is $\left\{(x, y) \mid x^2 + \frac{1}{4}y^2 \le 1, x \ge 0\right\}$.



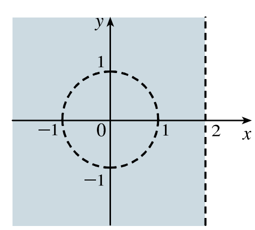
10. $g(x, y) = \ln(x^2 + y^2 - 9)$. $\ln(x^2 + y^2 - 9)$ is defined only when $x^2 + y^2 - 9 > 0 \quad \Leftrightarrow \quad x^2 + y^2 > 9$. So the domain of g is $\{(x, y) \mid x^2 + y^2 > 9\}$.



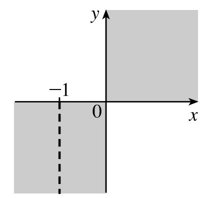
11. $g(x, y) = \dfrac{x - y}{x + y}$. g is not defined if $x + y = 0 \quad \Leftrightarrow \quad y = -x$ (and is defined otherwise). Thus, the domain of g is $\{(x, y) \mid y \ne -x\}$, the set of all points in $\mathbb{R}^2$ that are not on the line $y = -x$.



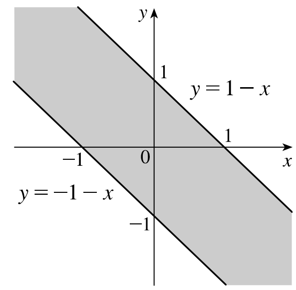
12. $g(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$. $\ln(2 - x)$ is defined only when $2 - x > 0 \quad \Leftrightarrow \quad x < 2$. In addition, g is not defined if $1 - x^2 - y^2 = 0 \quad \Leftrightarrow \quad x^2 + y^2 = 1$. Thus, the domain of g is $\left\{(x, y) \mid x < 2,\, x^2 + y^2 \ne 1\right\}$, the set of all points to the left of the line $x = 2$ and not on the unit circle.



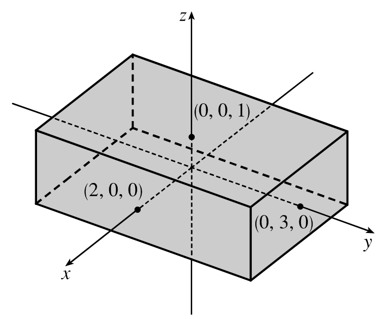
13. $p(x,y) = \dfrac{\sqrt{xy}}{x+1}$. $\sqrt{xy}$ is defined only when $xy \geq 0$. Further, p is defined only when $x + 1 \neq 0 \quad \Leftrightarrow \quad x \neq -1$. So the domain of p is $\{(x,y) \mid xy \geq 0, x \neq -1\}$.



14. $f(x,y) = \sin^{-1}(x+y)$. $\sin^{-1}(x+y)$ is defined only when $-1 \leq x + y \leq 1 \quad \Leftrightarrow \quad -1 - x \leq y \leq 1 - x$. Thus, the domain of f is $\{(x,y) \mid -1 - x \leq y \leq 1 - x\}$, which consists of those points on or between the parallel lines $y = -1 - x$ and $y = 1 - x$.

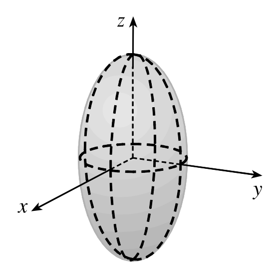


15. $f(x,y,z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$. f is defined only when $4 - x^2 \geq 0 \quad \Leftrightarrow \quad -2 \leq x \leq 2$, and $9 - y^2 \geq 0 \quad \Leftrightarrow$ $-3 \leq y \leq 3$, and $1 - z^2 \geq 0 \quad \Leftrightarrow \quad -1 \leq z \leq 1$. Thus, the domain of f is $\{(x,y,z) \mid -2 \leq x \leq 2,\ -3 \leq y \leq 3,\ -1 \leq z \leq 1\}$, which is a solid rectangular box with vertices $(\pm 2, \pm 3, \pm 1)$ (all 8 combinations).



16. $f(x,y,z) = \ln(16 - 4x^2 - 4y^2 - z^2)$. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0 \quad \Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1$. Thus,

$D = \left\{ (x,y,z) \,\middle|\, \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1 \right\}$, that is, the points inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.



17. (a) $f(160, 70) = 0.1091(160)^{0.425}(70)^{0.725} \approx 20.5$, which means that the surface area of a person 70 inches (5 feet 10 inches) tall who weighs 160 pounds is approximately 20.5 square feet.

(b) Answers will vary depending on the height and weight of the reader.

18. $P(120, 20) = 1.47(120)^{0.65}(20)^{0.35} \approx 94.2$, so when the manufacturer invests \$20 million in capital and 120,000 hours of labor are completed yearly, the monetary value of the production is about \$94.2 million.

19. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°\text{C}$ and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°\text{C}$ without wind.

(b) The question is asking: when the temperature is $-20°\text{C}$, what wind speed gives a wind-chill index of $-30°\text{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of -49°C? From Table 1, the temperature is -35°C.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is -5°C. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

20. (a) From Table 3, $f(95, 70) = 124$, which means that when the actual temperature is 95°F and the relative humidity is 70%, the perceived air temperature is approximately 124°F.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is 80°F. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is 100°F. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately 1°F per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of 5°F per 10% relative humidity) and at an increasing rate (approximately 12°F per 10% relative humidity for larger values of h).

21. (a) According to Table 4, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

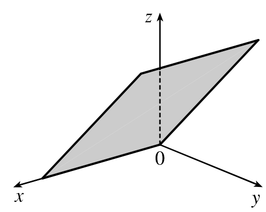
(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

22. (a) The cost of making x small boxes, y medium boxes, and z large boxes is $C = f(x, y, z) = 8000 + 2.5x + 4y + 4.5z$ dollars.

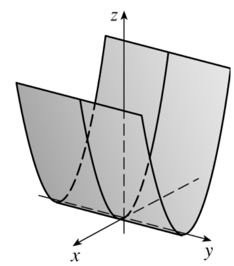
(b) $f(3000, 5000, 4000) = 8000 + 2.5(3000) + 4(5000) + 4.5(4000) = 53{,}500$ which means that it costs \$53,500 to make 3000 small boxes, 5000 medium boxes, and 4000 large boxes.

(c) Because no partial boxes will be produced, each of x, y, and z must be a positive integer or zero.

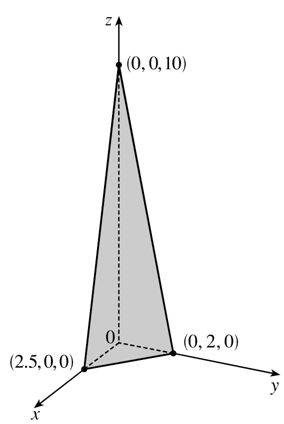
23. The graph of f has equation $z = y$, a plane which intersects the yz-plane in the line $z = y$, $x = 0$. The portion of this plane in the first octant is shown.



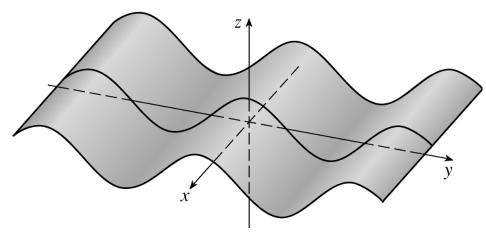
24. The graph of f has equation $z = x^2$, a parabolic cylinder.



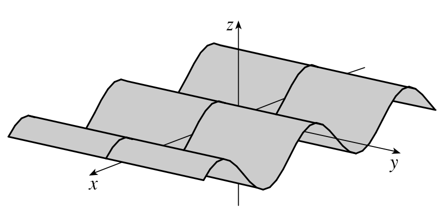
25. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



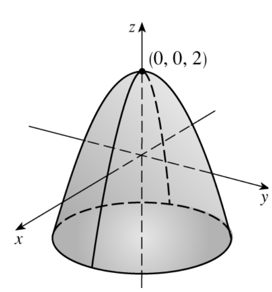
26. $z = \cos y$, a cylinder.



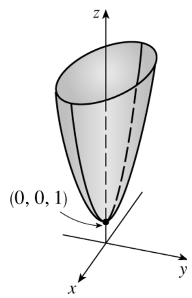
27. $z = \sin x$, a cylinder.



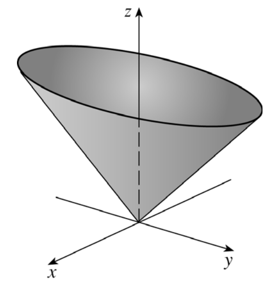
28. $z = 2 - x^2 - y^2$, a circular paraboloid opening downward with vertex at $(0, 0, 2)$.



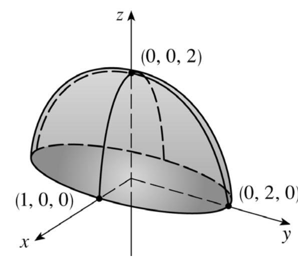
29. $z = x^2 + 4y^2 + 1$, an elliptic paraboloid opening upward with vertex at $(0, 0, 1)$.



30. $z = \sqrt{4x^2 + y^2}$ so $4x^2 + y^2 = z^2$ and $z \geq 0$, the top half of an elliptic cone.



31. $z = \sqrt{4 - 4x^2 - y^2}$ so $4x^2 + y^2 + z^2 = 4$ or $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$ and $z \geq 0$, the top half of an ellipsoid.



32. (a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and the trace in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. The only possibility is graph III. Notice also that the level curves of f are $\dfrac{1}{1 + x^2 + y^2} = k \quad \Leftrightarrow \quad x^2 + y^2 = \dfrac{1}{k} - 1$, a family of circles for $k < 1$.

(b) $f(x, y) = \dfrac{1}{1 + x^2 y^2}$. The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x, y) = \ln(x^2 + y^2)$. The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of f are $\ln(x^2 + y^2) = k \quad \Leftrightarrow \quad x^2 + y^2 = e^k$, a family of circles. In addition, f is large negative when $x^2 + y^2$ is small, so this is graph IV.

(d) $f(x, y) = \cos\sqrt{x^2 + y^2}$. The trace in $x = 0$ is $z = \cos\sqrt{y^2} = \cos|y| = \cos y$, and the trace in $y = 0$ is $z = \cos\sqrt{x^2} = \cos|x| = \cos x$. Notice also that the level curve $f(x, y) = 0$ is $\cos\sqrt{x^2 + y^2} = 0 \quad \Leftrightarrow$ $x^2 + y^2 = \left(\frac{\pi}{2} + n\pi\right)^2$, a family of circles, so this is graph V.

(e) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and the trace in $y = 0$ is $z = 0$, so it must be graph VI.

(f) $f(x, y) = \cos(xy)$. The trace in $x = 0$ is $z = \cos 0 = 1$, and the trace in $y = 0$ is $z = 1$. As mentioned in part (b), these traces match both graphs I and II. Here z can be negative, so the graph is II. (Also notice that the trace in $x = 1$ is $z = \cos y$, and the trace in $y = 1$ is $z = \cos x$.)

33. The point $(-3,3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z=60$, we estimate that $f(-3,3)\approx 56$. The point $(3,-2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3,-2)\approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

34. (a) C (Chicago) lies between level curves with pressures 1012 and 1016 mb, and since C appears to be located about one-fourth the distance from the 1012 mb isobar to the 1016 mb isobar, we estimate the pressure at Chicago to be about 1013 mb. N lies very close to a level curve with pressure 1012 mb so we estimate the pressure at Nashville to be approximately 1012 mb. S appears to be just about halfway between level curves with pressures 1008 and 1012 mb, so we estimate the pressure at San Francisco to be about 1010 mb. V lies close to a level curve with pressure 1016 mb but we can't see a level curve to its left so it is more difficult to make an accurate estimate. There are lower pressures to the right of V and V is a short distance to the left of the level curve with pressure 1016 mb, so we might estimate that the pressure at Vancouver is about 1017 mb.

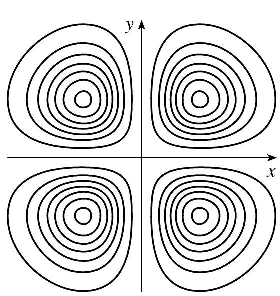
(b) Winds are stronger where the isobars are closer together (see Figure 13), and the level curves are closer near S than at the other locations, so the winds were strongest at San Francisco.

35. The point $(160,10)$, corresponding to day 160 and a depth of 10 m, lies between the isothermals with temperature values of 8 and 12°C. Since the point appears to be located about three-fourths the distance from the 8°C isothermal to the 12°C isothermal, we estimate the temperature at that point to be approximately 11°C. The point $(180,5)$ lies between the 16 and 20°C isothermals, very close to the 20°C level curve, so we estimate the temperature there to be about 19.5°C.

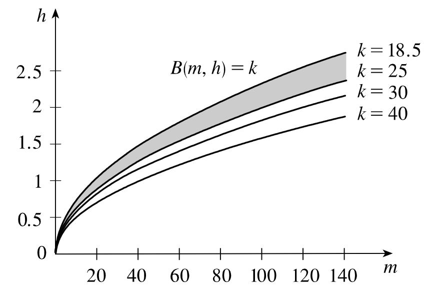
36. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

37. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

38.



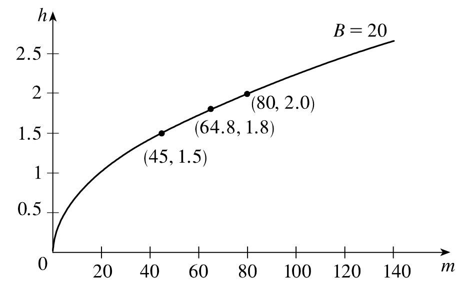
39. The level curves of $B(m,h)=\dfrac{m}{h^2}$ are $\dfrac{m}{h^2}=k \quad\Leftrightarrow\quad m=kh^2$ or equivalently $h=\sqrt{m/k}=\dfrac{1}{\sqrt{k}}\sqrt{m}$ since $m>0$, $h>0$. We draw the level curves for $k=18.5$, 25, 30, and 40.



The shaded region corresponds to BMI values between 18.5 and 25, those considered optimal. For a mass of 62 kg and a height of 152 cm (1.52 m), the BMI is $B(62,1.52)=\dfrac{62}{1.52^2}\approx 26.8$, which is outside the optimal range.

40. From Exercise 39, the body mass index function is $B(m,h)=m/h^2$. The BMI for a person 200 cm (2.0 m, about 6 ft 7 in) tall and with mass 80 kg (about 176 lb) is $B(80,2.0)=80/(2.0)^2=20$. The level curve $B(m,h)=20 \quad\Leftrightarrow\quad m=20h^2$ is shown in the graph.

A person 1.5 m tall (about 4 ft 11 in) has a BMI on the same level curve if their mass is $m=20(1.5)^2=45$ kg (about 99 lb), and a person 1.8 m (about 5 ft 11 in) tall would have mass $m=20(1.8)^2=64.8$ kg (about 143 lb). (See the graph.)



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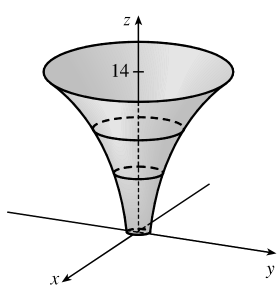

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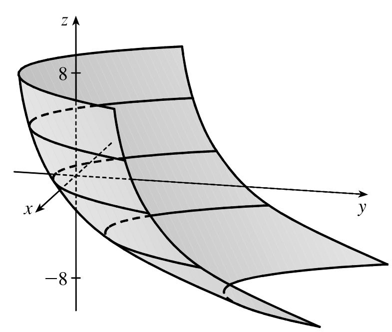

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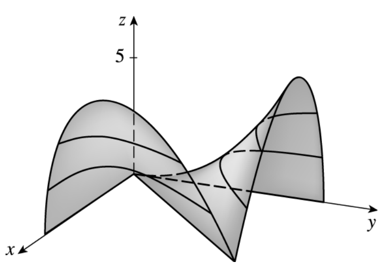

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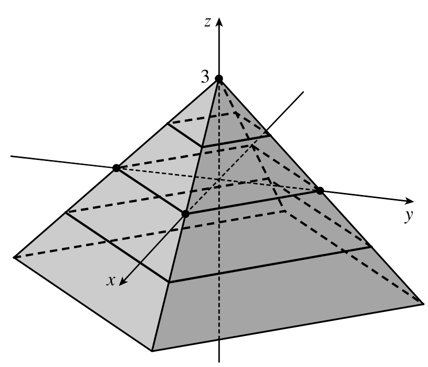
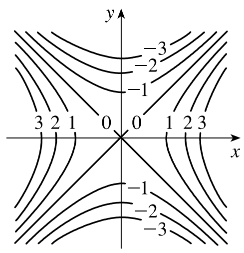
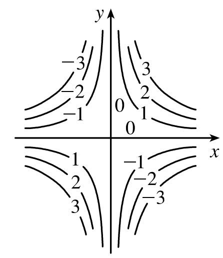


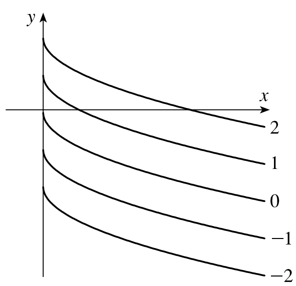
45. The level curves are $x^2 - y^2 = k$. When $k = 0$ the level curve is the pair of lines $y = \pm x$, and when $k \neq 0$ the level curves are a family of hyperbolas (oriented differently for $k > 0$ than for $k < 0$).



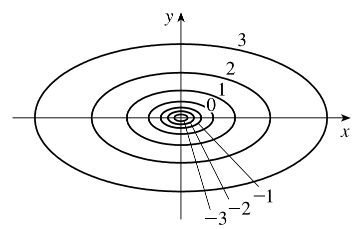
46. The level curves are $xy = k$ or $y = k/x$. When $k \neq 0$ the level curves are a family of hyperbolas. When $k = 0$ the level curve is the pair of lines $x = 0$, $y = 0$.



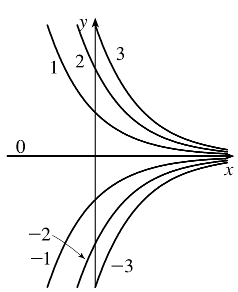
47. The level curves are $\sqrt{x} + y = k$ or $y = -\sqrt{x} + k$, a family of vertical translations of the graph of the root function $y = -\sqrt{x}$.



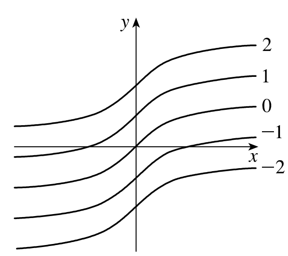
48. The level curves are $\ln(x^2 + 4y^2) = k$ or $x^2 + 4y^2 = e^k$, a family of ellipses.



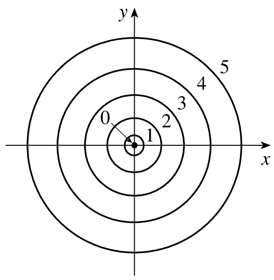
49. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.



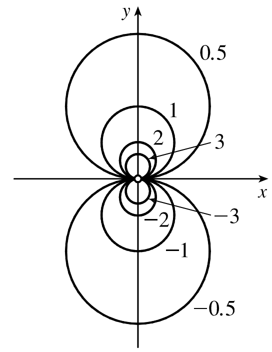
50. The level curves are $y - \arctan x = k$ or $y = (\arctan x) + k$, a family of vertical translations of the graph of the inverse tangent function $y = \arctan x$.



51. The level curves are $\sqrt[3]{x^2+y^2}=k$ or $x^2+y^2=k^3$ $(k \geq 0)$, a family of circles centered at the origin with radius $k^{3/2}$.

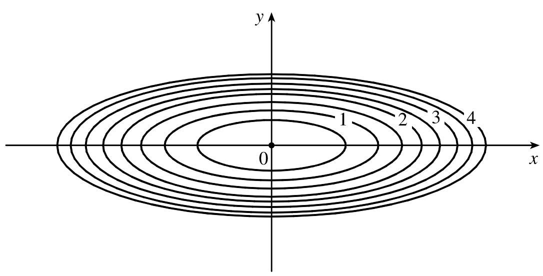


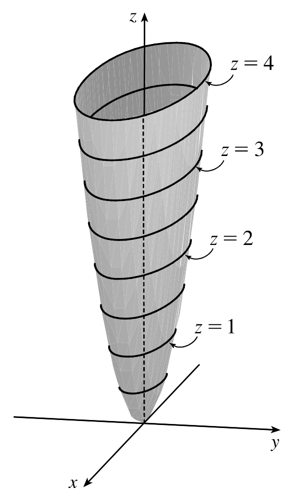
52. For $k \neq 0$ and $(x,y) \neq (0,0)$, $k = \dfrac{y}{x^2+y^2}$ $\Leftrightarrow$ $x^2+y^2-\dfrac{y}{k}=0$ $\Leftrightarrow$ $x^2+\left(y-\frac{1}{2k}\right)^2=\dfrac{1}{4k^2}$, a family of circles with center $\left(0,\frac{1}{2k}\right)$ and radius $\frac{1}{2k}$ (without the origin). If $k=0$, the level curve is the x-axis.



53. The contour map consists of the level curves $k=x^2+9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k=0$, the origin.)

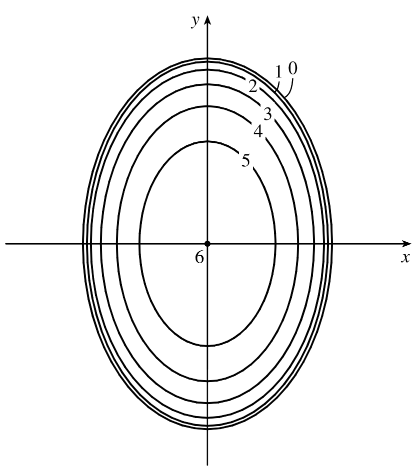
The graph of $f(x,y)$ is the surface $z=x^2+9y^2$, an elliptic paraboloid.



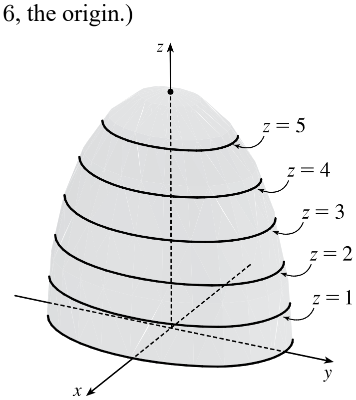


If we visualize lifting each ellipse $k=x^2+9y^2$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f.

54.



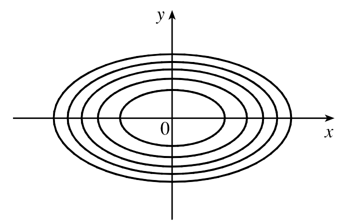
The contour map consists of the level curves $k=\sqrt{36-9x^2-4y^2}$ $\Rightarrow$ $9x^2+4y^2=36-k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k=6$, the origin.)



[continued]

The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

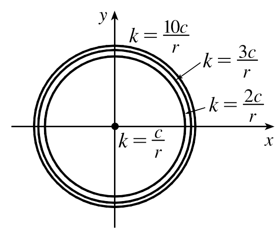
55. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.



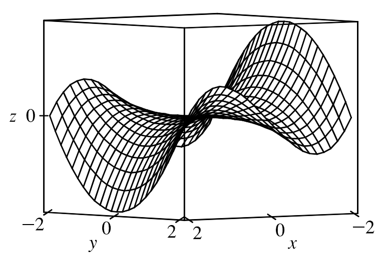
56. The equipotential curves are $k = \dfrac{c}{\sqrt{r^2 - x^2 - y^2}}$ or

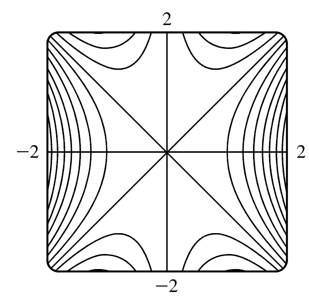
$x^2 + y^2 = r^2 - \left(\dfrac{c}{k}\right)^2$, a family of circles ($k \geq c/r$).

Note: As $k \to \infty$, the radius of the circle approaches r.



57. $f(x, y) = xy^2 - x^3$

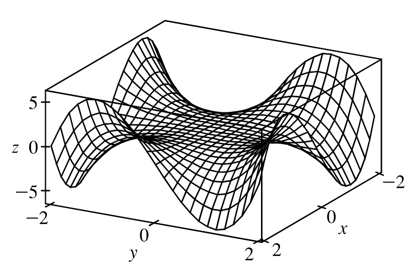


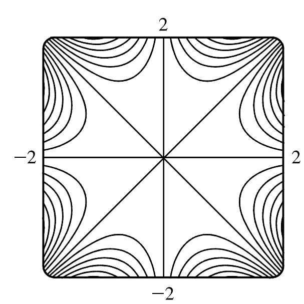


The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

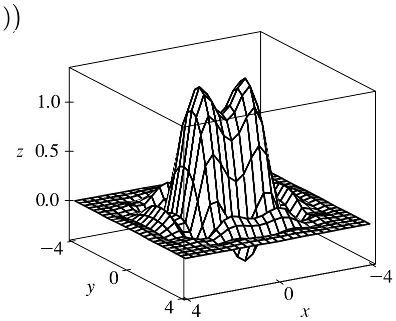
58. $f(x, y) = xy^3 - yx^3$

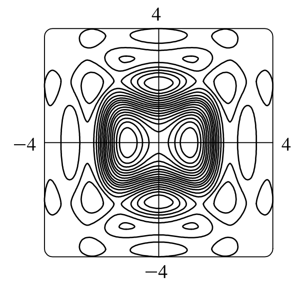
The traces parallel to either the yz-plane or the xz-plane are cubic curves.





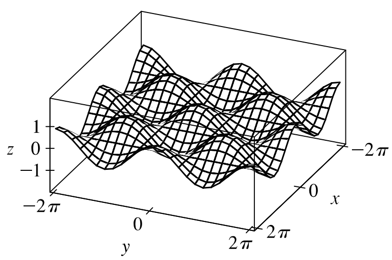
59. $f(x,y) = e^{-(x^2+y^2)/3}\left(\sin(x^2) + \cos(y^2)\right)$

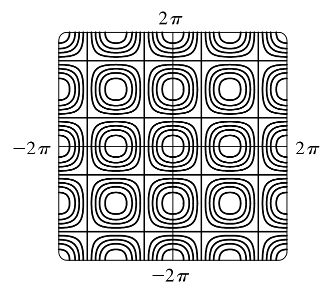




60. $f(x,y) = \cos x \, \cos y$

The traces parallel to either the yz- or xz-plane are cosine curves with amplitudes that vary from 0 to 1.





61. $z = \sin(xy)$ (a) C (b) II

Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

62. $z = e^x \cos y$ (a) A (b) IV

Reasons: This function is periodic in y but not x, a condition satisfied only by A and IV. Also, note that traces in $x = k$ are cosine curves with amplitude that increases as x increases.

63. $z = \sin(x - y)$ (a) F (b) I

Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

64. $z = \sin x - \sin y$ (a) E (b) III

Reasons: This function is periodic in both x and y, but unlike the function in Exercise 63, it is not constant along lines such as $y = x + \pi$, so the contour map is III. Also notice that traces in $y = k$ are vertically shifted copies of the sine wave $z = \sin x$, so the graph must be E.

65. $z = (1 - x^2)(1 - y^2)$ (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$ (a) D (b) V

Reasons: This function is not periodic, ruling out the graphs in A, C, E, and F. Also, the values of z approach 0 as we use points farther from the origin. The only graph that shows this behavior is D, which corresponds to V.

67. $k = 2y - z + 1$ is a family of parallel planes with normal vector $\langle 0, 2, -1 \rangle$.

68. $k = x + y^2 - z^2$ is a family of hyperbolic paraboloids with saddle point $(k, 0, 0)$.

69. Equations for the level surfaces are $k = x^2 + y^2 - z^2$. For $k = 0$, the equation becomes $z^2 = x^2 + y^2$ and the surface is a right circular cone with center the origin and axis the z-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the z-axis. For $k < 0$ we have a family of hyperboloids of two sheets with axis the z-axis.

70. $k = x^2 + 2y^2 + 3z^2$ is a family of ellipsoids with major axis the x-axis for $k > 0$ and the origin for $k = 0$.

71. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

(c) The graph of g is the graph of f reflected about the xy-plane.

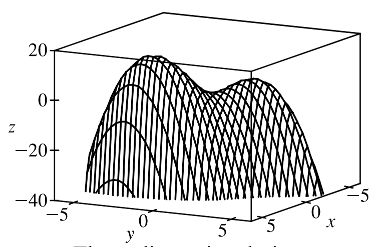
(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

72. (a) The graph of g is the graph of f shifted 2 units in the positive x-direction.

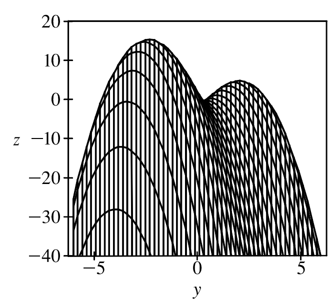
(b) The graph of g is the graph of f shifted 2 units in the negative y-direction.

(c) The graph of g is the graph of f shifted 3 units in the negative x-direction and 4 units in the positive y-direction.

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



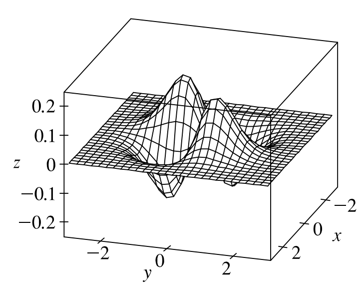
Three-dimensional view



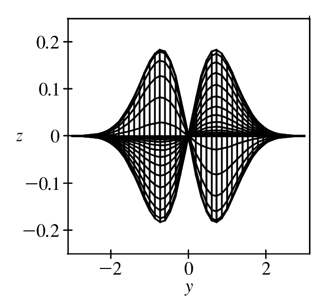
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

74. $f(x, y) = xye^{-x^2 - y^2}$



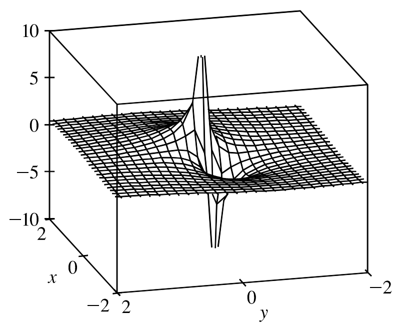
Three-dimensional view



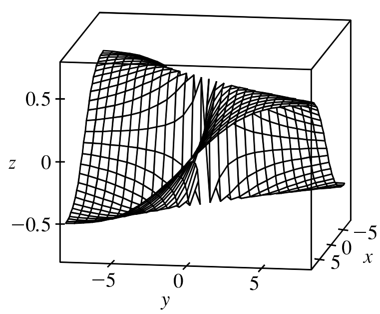
Front view

The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the

front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f.

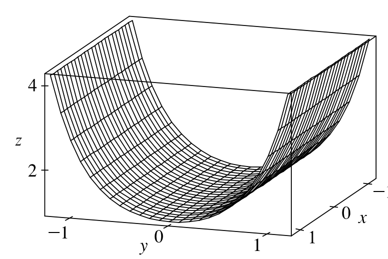
75. 

$f(x,y) = \dfrac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x,y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y) \to \infty$, while in others $f(x,y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y = -x$.

76. 

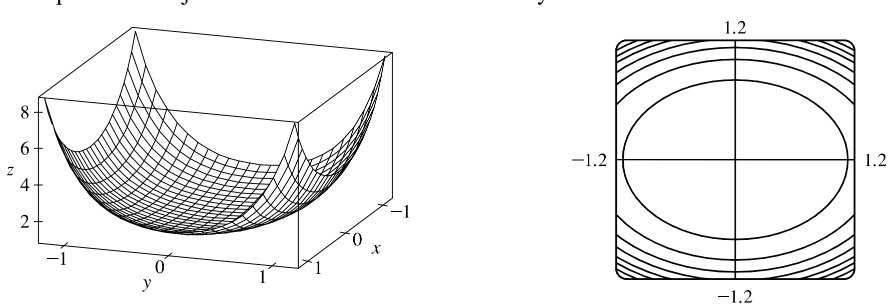
$f(x,y) = \dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x,y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x,y) = 0$ for all values of (x,y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

77. $f(x,y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



$c = 0$

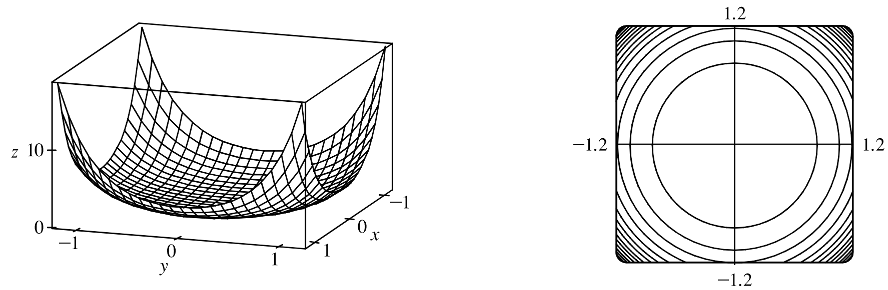
For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.



$c = 0.5$ (level curves in increments of 1)

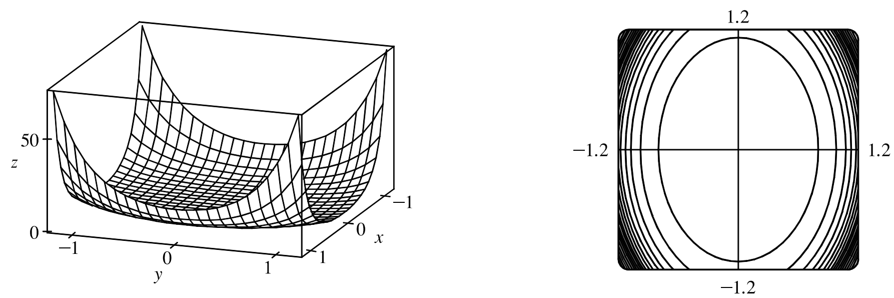
[continued]

For $c = 1$ the level curves are circles centered at the origin.



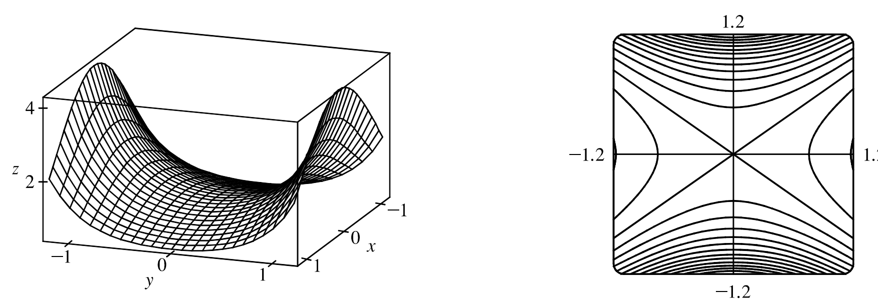
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

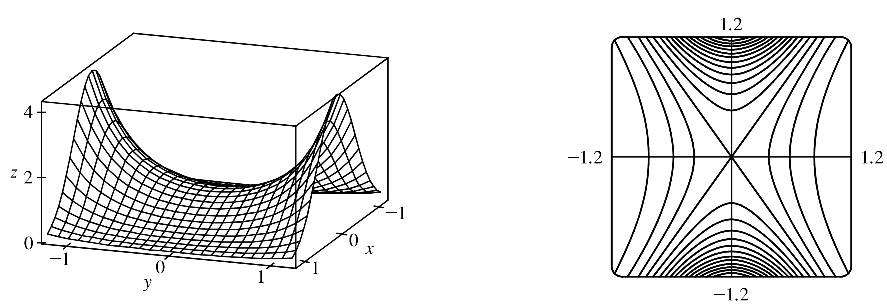


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.

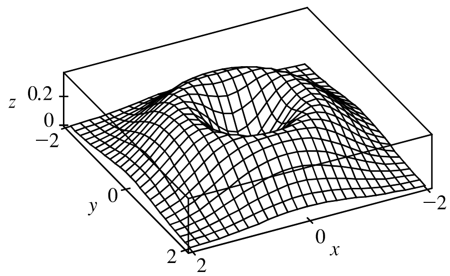


$c = -0.5$ (level curves in increments of 0.25)

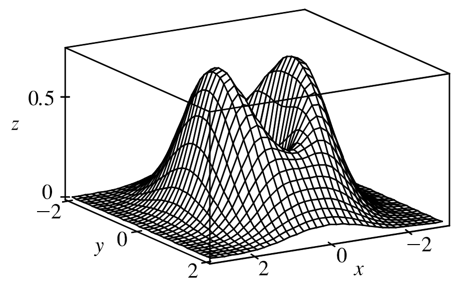


$c = -2$ (level curves in increments of 0.25)

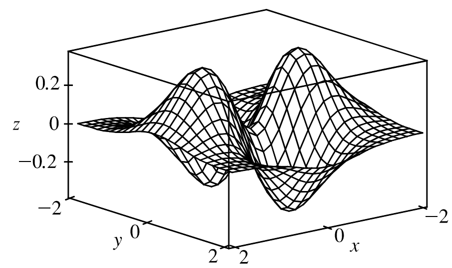
78. $z = (ax^2 + by^2)e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy-plane). Interchanging a and b rotates the graph by $90°$ about the z-axis.



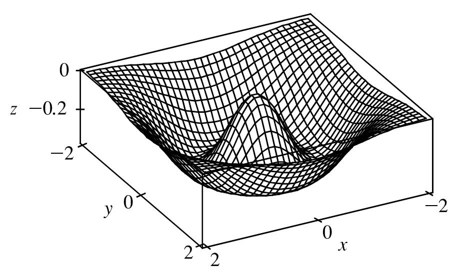
$a = 1, b = 1$



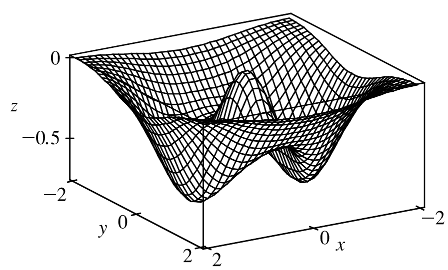
$a = 2, b = 1$



$a = 1, b = -1$



$a = -1, b = -1$



$a = -2, b = -1$

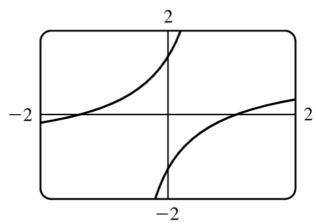
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

79. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See the following graph.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

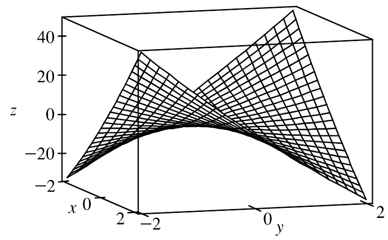
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the

[continued]

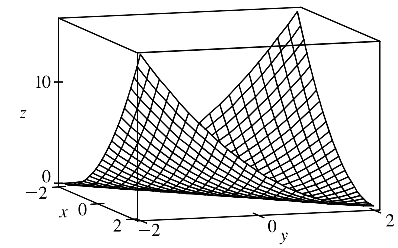
plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minimums of 0 on the line $y = x$.



$c = -5,\ z = 2$



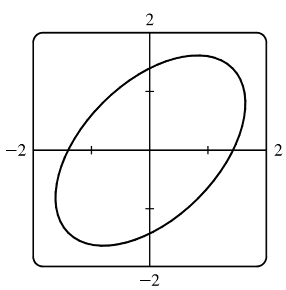
$c = -10$



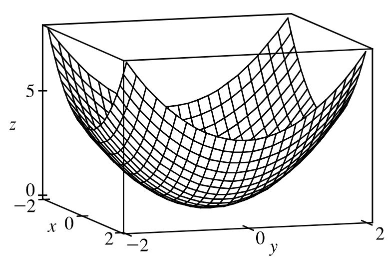
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

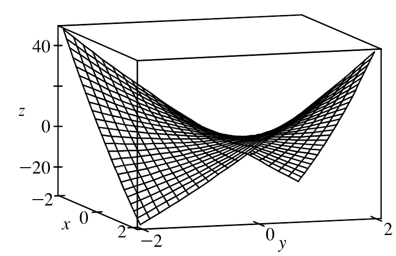
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1,\ z = 2$



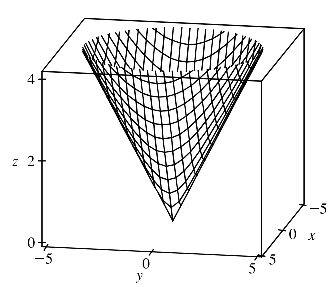
$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

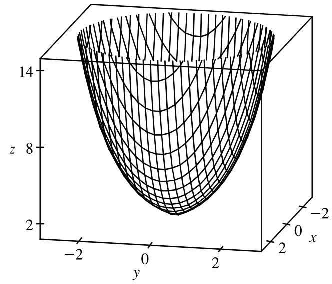
80. First, we graph $f(x, y) = \sqrt{x^2 + y^2}$.



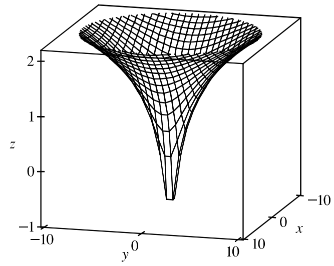
$f(x, y) = \sqrt{x^2 + y^2}$

[continued]

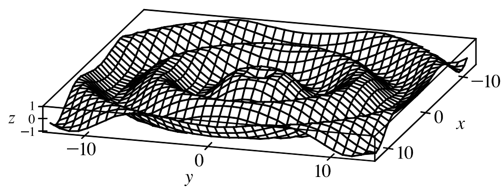
Graphs of the other four functions follow.



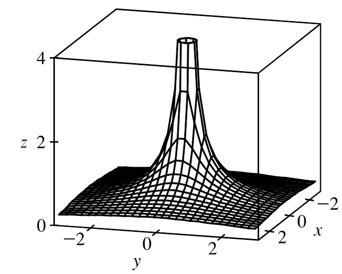
$f(x, y) = e^{\sqrt{x^2+y^2}}$



$f(x, y) = \ln \sqrt{x^2 + y^2}$



$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$



$f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$

Notice that each graph $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ exhibits radial symmetry about the z-axis and the trace in the xz-plane for $x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ is obtained from the graph of g by graphing $z = g(x)$ in the xz-plane and rotating the curve about the z-axis.

81. (a) $P = bL^{\alpha}K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^{\alpha}K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^{\alpha} \quad \Rightarrow \quad \ln\dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^{\alpha}\right) \quad \Rightarrow$

$$\ln\frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	−0.02	−0.06
1901	−0.04	−0.02
1902	−0.04	0
1903	−0.07	−0.05
1904	−0.13	−0.12
1905	−0.18	−0.04
1906	−0.20	−0.07
1907	−0.23	−0.15
1908	−0.41	−0.38
1909	−0.33	−0.24
1910	−0.35	−0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	−0.38	−0.34
1912	−0.38	−0.24
1913	−0.41	−0.25
1914	−0.47	−0.37
1915	−0.53	−0.34
1916	−0.49	−0.28
1917	−0.53	−0.39
1918	−0.60	−0.50
1919	−0.68	−0.57
1920	−0.74	−0.57
1921	−1.05	−0.85
1922	−0.98	−0.59

[continued]

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \quad\Rightarrow\quad b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^{\alpha}K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

1. In general, we can't say anything about $f(3,1)$! $\lim\limits_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as (x, y) approaches, but is not equal to, $(3,1)$. If f is continuous, we know that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim\limits_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x, y) points near the origin.

x \ y	-0.2	-0.1	-0.05	0	0.05	0.1	0.2
-0.2	-2.551	-2.525	-2.513	-2.500	-2.488	-2.475	-2.451
-0.1	-2.525	-2.513	-2.506	-2.500	-2.494	-2.488	-2.475
-0.05	-2.513	-2.506	-2.503	-2.500	-2.497	-2.494	-2.488
0	-2.500	-2.500	-2.500		-2.500	-2.500	-2.500
0.05	-2.488	-2.494	-2.497	-2.500	-2.503	-2.506	-2.513
0.1	-2.475	-2.488	-2.494	-2.500	-2.506	-2.513	-2.525
0.2	-2.451	-2.475	-2.488	-2.500	-2.513	-2.525	-2.551

As the table shows, the values of $f(x,y)$ seem to approach -2.5 as (x, y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -2.5$. Since f is a rational function, it is continuous on its domain. f is defined at $(0,0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^20^3 + 0^30^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y)=\dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x)=\dfrac{2x^2}{x^2+2x^2}=\dfrac{2}{3}$ $(x\neq 0)$, so $f(x,y)\to\frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y)=x^2y^3-4y^2$ is a polynomial, and hence continuous, so we can find the limit by direct substitution:
$\lim\limits_{(x,y)\to(3,2)} f(x,y)=f(3,2)=(3)^2(2)^3-4(2)^2=56.$

6. $f(x,y)=x^2y+3xy^2+4$ is a polynomial, and hence continuous, so we can find the limit by direct substitution:
$\lim\limits_{(x,y)\to(5,-2)} f(x,y)=f(5,-2)=5^2(-2)+3(5)(-2)^2+4=14.$

7. $f(x,y)=\dfrac{x^2y-xy^3}{x-y+2}$ is a rational function, and hence, continuous on its domain. $(-3,1)$ is in the domain of f, so we can find the limit by direct substitution: $\lim\limits_{(x,y)\to(-3,1)} f(x,y)=f(-3,1)=\dfrac{(-3)^2(1)-(-3)(1)^3}{-3-1+2}=\dfrac{12}{-2}=-6.$

8. $f(x,y)=\dfrac{x^2y+xy^2}{x^2-y^2}$ is a rational function, and hence, continuous on its domain. $(2,-1)$ is in the domain of f, so we can find the limit by direct substitution: $\lim\limits_{(x,y)\to(2,-1)} f(x,y)=f(2,-1)=\dfrac{(2)^2(-1)+(2)(-1)^2}{(2)^2-(-1)^2}=-\dfrac{2}{3}.$

9. $x-y$ is a polynomial and therefore continuous. Since $\sin t$ is a continuous function, the composition $\sin(x-y)$ is also continuous. The function y is a polynomial, and hence continuous, and the product of continuous functions is continuous, so $f(x,y)=y\sin(x-y)$ is a continuous function. Then $\lim\limits_{(x,y)\to(\pi,\pi/2)} f(x,y)=f\left(\pi,\frac{\pi}{2}\right)=\frac{\pi}{2}\sin\left(\pi-\frac{\pi}{2}\right)=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}.$

10. $2x-y$ is a polynomial and therefore continuous. Since $\sqrt{t}$ is continuous for $t\geq 0$, the composition $\sqrt{2x-y}$ is continuous where $2x-y\geq 0$. The function e^u is continuous everywhere, so the composition $f(x,y)=e^{\sqrt{2x-y}}$ is a continuous function for $2x-y\geq 0$. If $x=3$ and $y=2$ then $2x-y\geq 0$, so $\lim\limits_{(x,y)\to(3,2)} f(x,y)=f(3,2)=e^{\sqrt{2(3)-2}}=e^2.$

11. $f(x,y) = \dfrac{x^2y^3 - x^3y^2}{x^2 - y^2} = \dfrac{x^2y^2(y-x)}{(x-y)(x+y)} = -\dfrac{x^2y^2}{x+y}$ for $x - y \neq 0$; in particular, $(x,y) \neq (1,1)$. Thus,

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} \left(-\frac{x^2y^2}{x+y}\right) = -\frac{(1)^2(1)^2}{1+1} = -\frac{1}{2}.$$

12. $f(x,y) = \dfrac{\cos y - \sin 2y}{\cos x \cos y} = \dfrac{\cos y - 2\sin y \cos y}{\cos x \cos y} = \dfrac{1 - 2\sin y}{\cos x}$ for $\cos y \neq 0$; in particular, $(x,y) \neq (\pi, \pi/2)$. Thus,

$$\lim_{(x,y)\to(\pi,\pi/2)} f(x,y) = \lim_{(x,y)\to(\pi,\pi/2)} \frac{1 - 2\sin y}{\cos x} = \frac{1 - 2\sin(\pi/2)}{\cos \pi} = \frac{-1}{-1} = 1.$$

13. $f(x,y) = \dfrac{y^2}{x^2 + y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the y-axis. Then $f(0,y) = y^2/y^2 = 1$ for $y \neq 0$, so $f(x,y) \to 1$. Since f has two different limits along two different lines, the limit does not exist.

14. $f(x,y) = \dfrac{2xy}{x^2 + 3y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = 2x^2/4x^2 = 1/2$ for $x \neq 0$. Since f has two different limits along two different lines, the limit does not exist.

15. $f(x,y) = \dfrac{(x+y)^2}{x^2 + y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^2/x^2 = 1$ for $x \neq 0$, so $f(x,y) \to 1$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = 4x^2/(2x^2) = 2$ for $x \neq 0$, so $f(x,y) \to 2$. Since f has two different limits along two different lines, the limit does not exist.

16. $f(x,y) = \dfrac{x^2 + xy^2}{x^4 + y^2}$. First approach $(0,0)$ along the y-axis. Then $f(0,y) = 0/y^2 = 0$ for $y \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = \dfrac{x^2 + x^3}{x^4 + x^2} = \dfrac{x^2(1+x)}{x^2(x^2+1)} = \dfrac{1+x}{1+x^2}$ for $x \neq 0$, so $f(x,y) \to 1$. Since f has two different limits along two different lines, the limit does not exist.

17. $f(x,y) = \dfrac{y^2 \sin^2 x}{x^4 + y^4}$. First approach $(0,0)$ along the y-axis. Then $f(0,y) = 0/y^4 = 0$ for $y \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = \dfrac{x^2 \sin^2 x}{2x^4} = \dfrac{1}{2}\left(\dfrac{\sin x}{x}\right)^2$ for $x \neq 0$, so by Equation 3.3.5, $f(x,y) \to \frac{1}{2}(1)^2 = \frac{1}{2}$. Since f has two different limits along two different lines, the limit does not exist.

18. $f(x,y) = \dfrac{y - x}{1 - y + \ln x}$. First approach $(1,1)$ along the line $x = 1$. Then $f(1,y) = \dfrac{y-1}{1 - y + 0} = -1$ for $y \neq 1$, so $f(x,y) \to -1$. Now approach $(1,1)$ along the line $y = x$. Then $f(x,x) = \dfrac{x - x}{1 - x + \ln x} = 0$ for $1 - x + \ln x \neq 0$. So $f(x,y) \to 0$. Since f has two different limits along two different lines, the limit does not exist.

19. $x^2y - xy^2 + 3$ is a polynomial and therefore continuous. t^3 is also a polynomial and therefore continuous. so the composition $(x^2y - xy^2 + 3)^3$ is continuous. Thus, $\lim\limits_{(x,y)\to(-1,-2)} (x^2y - xy^2 + 3)^3 = [(-1)^2(-2) - (-1)(-2)^2 + 3]^3 = 5^3 = 125.$

20. e^{xy} is a composition of continuous functions and therefore continuous. $\sin xy$ is also a composition of continuous functions and therefore continuous. Thus, the product $e^{xy} \sin xy$ is continuous, and $\lim\limits_{(x,y)\to(\pi,1/2)} e^{xy} \sin xy = e^{\pi/2} \sin \frac{\pi}{2} = e^{\pi/2}$.

21. $f(x,y) = \dfrac{3x - 2y}{4x^2 - y^2}$ is a rational function and continuous on its domain. $(2,3)$ is in the domain of f, so

$$\lim_{(x,y)\to(2,3)} f(x,y) = \frac{3(2) - 2(3)}{4(2)^2 - 3^2} = \frac{0}{7} = 0.$$

22. $f(x,y) = \dfrac{2x - y}{4x^2 - y^2} = \dfrac{2x - y}{(2x - y)(2x + y)} = \dfrac{1}{2x + y}$ for $2x - y \neq 0$. Thus, $\lim\limits_{(x,y)\to(1,2)} f(x,y) = \dfrac{1}{2(1) + 2} = \dfrac{1}{4}$.

23. Let $f(x,y) = \dfrac{xy^2 \cos y}{x^2 + y^4}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y = x$ also gives a limit of 0. But $f\left(y^2, y\right) = \dfrac{y^2y^2 \cos y}{(y^2)^2 + y^4} = \dfrac{y^4 \cos y}{2y^4} = \dfrac{\cos y}{2}$ for $y \neq 0$, so $f(x,y) \to \frac{1}{2} \cos 0 = \frac{1}{2}$ as $(x,y) \to (0,0)$ along the parabola $x = y^2$. Thus the limit doesn't exist.

24. $f(x,y) = \dfrac{x^3 - y^3}{x^2 + xy + y^2} = \dfrac{(x - y)(x^2 + xy + y^2)}{x^2 + xy + y^2} = x - y$ for $(x,y) \neq (0,0)$. [Note that $x^2 + xy + y^2 = 0$ only when $(x,y) = (0,0)$.] Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} (x - y) = 0 - 0 = 0$.

25.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} &= \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} \\ &= \lim_{(x,y)\to(0,0)} \frac{\left(x^2 + y^2\right)\left(\sqrt{x^2 + y^2 + 1} + 1\right)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2 + y^2 + 1} + 1\right) = 2\end{aligned}$$

26. $f(x,y) = \dfrac{xy^4}{x^2 + y^8}$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the curve $x = y^4$ gives $f(y^4, y) = y^8/2y^8 = \frac{1}{2}$ for $y \neq 0$, so along this path $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$. Thus the limit does not exist.

27. $x + z$ is a polynomial and therefore, continuous. $\sqrt{t}$ is continuous on its domain. Thus, the composition is continuous at $(6, 1, -2)$. $\cos \pi y$ is continuous, so the product of $\sqrt{x + z}$ and $\cos \pi y$ is also continuous. Then

$$\lim_{(x,y,z)\to(6,1,-2)} \sqrt{x + z} \cos \pi y = \sqrt{6 + (-2)} \cos \pi = -2.$$

28. $f(x,y,z) = \dfrac{xy+yz}{x^2+y^2+z^2}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

29. $f(x,y,z) = \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

30. $f(x,y,z) = \dfrac{x^4+y^2+z^3}{x^4+2y^2+z}$. Then $f(x,0,0) = x^4/x^4 = 1$ for $x \neq 0$, so $f(x,y,z) \to 1$ as $(x,y,z) \to (0,0,0)$ along the x-axis. But $f(0,y,0) = y^2/(2y^2) = \frac{1}{2}$ for $y \neq 0$, so $f(x,y,z) \to \frac{1}{2}$ as $(x,y,z) \to (0,0,0)$ along the y-axis. Since f has two different limits along two different lines, the limit does not exist.

31. $-1 \leq \sin\left(\dfrac{1}{x^2+y^2}\right) \leq 1 \quad \Rightarrow \quad -xy \leq xy\sin\left(\dfrac{1}{x^2+y^2}\right) \leq xy$ for $xy > 0$. If $xy < 0$, we have $-xy \geq xy\sin\left(\dfrac{1}{x^2+y^2}\right) \geq xy$. In either case, $\displaystyle\lim_{(x,y)\to(0,0)} xy = 0$ and $\displaystyle\lim_{(x,y)\to(0,0)} (-xy) = 0$. Thus, $\displaystyle\lim_{(x,y)\to(0,0)} xy\sin\left(\dfrac{1}{x^2+y^2}\right) = 0$ by the Squeeze Theorem.

32. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. Since $|y| \leq \sqrt{x^2+y^2}$, we have $\dfrac{|y|}{\sqrt{x^2+y^2}} \leq 1$ and so $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$. Now $|x| \to 0$ as $(x,y) \to (0,0)$, so $\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \to 0$ and hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

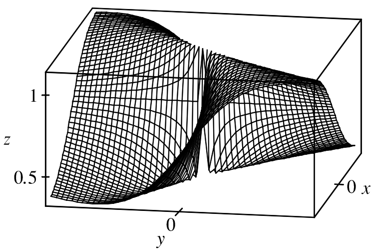
33. We use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = 0$:

$0 \leq \dfrac{|x|\,y^4}{x^4+y^4} \leq |x|$ since $0 \leq \dfrac{y^4}{x^4+y^4} \leq 1$, and $|x| \to 0$ as $(x,y) \to (0,0)$, so $\dfrac{|x|\,y^4}{x^4+y^4} \to 0 \quad \Rightarrow \quad \dfrac{xy^4}{x^4+y^4} \to 0$ as $(x,y) \to (0,0)$.

34. We use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2} = 0$:

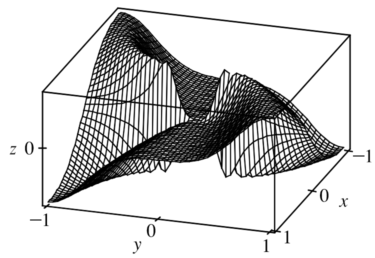
$0 \leq \dfrac{x^2y^2z^2}{x^2+y^2+z^2} \leq x^2y^2$ since $0 \leq \dfrac{z^2}{x^2+y^2+z^2} \leq 1$, and $x^2y^2 \to 0$ as $(x,y,z) \to (0,0,0)$, so $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} \to 0$ as $(x,y,z) \to (0,0,0)$.

35.



From the ridges on the graph, we see that as $(x, y) \to (0, 0)$ along the lines under the two ridges, $f(x, y) = \dfrac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ approaches different values. Since the function approaches different values depending on the path of approach, the limit does not exist.

36.

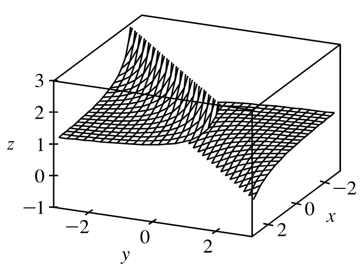


From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$ is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0, 0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

37. $h(x, y) = g(f(x, y)) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus, h is continuous on its domain $\{(x, y) \mid 2x + 3y - 6 \geq 0\} = \left\{(x, y) \mid y \geq -\frac{2}{3}x + 2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

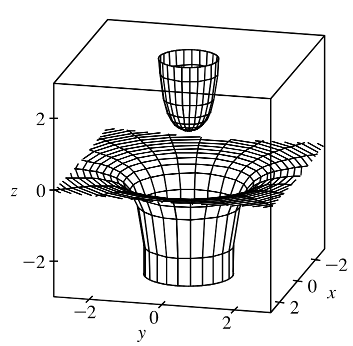
38. $h(x, y) = g(f(x, y)) = \dfrac{1 - xy}{1 + x^2y^2} + \ln\left(\dfrac{1 - xy}{1 + x^2y^2}\right)$. f is a rational function, so it is continuous on its domain. Because $1 + x^2y^2 > 0$, the domain of f is $\mathbb{R}^2$, so f is continuous everywhere. g is continuous on its domain $\{t \mid t > 0\}$. Thus, h is continuous on its domain $\left\{(x, y) \;\middle|\; \dfrac{1 - xy}{1 + x^2y^2} > 0\right\} = \{(x, y) \mid xy < 1\}$, which consists of all points between (but not on) the two branches of the hyperbola $y = 1/x$.

39.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x, y) = e^{1/(x-y)}$ as a composition of functions, $g(x, y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \quad \Leftrightarrow \quad y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x, y)) = e^{1/(x-y)} = f(x, y)$ is continuous except along the line $y = x$, as we suspected.

40.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. Since $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \quad \Leftrightarrow$ $x^2 + y^2 = 1$, confirming our observation that f is discontinuous on the circle $x^2 + y^2 = 1$.

41. The functions xy and $1+e^{x-y}$ are continuous everywhere, and $1+e^{x-y}$ is never zero, so $F(x,y)=\dfrac{xy}{1+e^{x-y}}$ is continuous on its domain $\mathbb{R}^2$.

42. $F(x,y)=\cos\sqrt{1+x-y}=g(f(x,y))$ where $f(x,y)=\sqrt{1+x-y}$, continuous on its domain $\{(x,y)\mid 1+x-y\geq 0\}=\{(x,y)\mid y\leq x+1\}$, and $g(t)=\cos t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y)\mid y\leq x+1\}$.

43. $F(x,y)=\dfrac{1+x^2+y^2}{1-x^2-y^2}$ is a rational function and thus is continuous on its domain $\left\{(x,y)\mid 1-x^2-y^2\neq 0\right\}=\left\{(x,y)\mid x^2+y^2\neq 1\right\}$.

44. The functions e^x+e^y and $e^{xy}-1$ are continuous everywhere, so $H(x,y)=\dfrac{e^x+e^y}{e^{xy}-1}$ is continuous except where $e^{xy}-1=0 \quad\Rightarrow\quad xy=0 \quad\Rightarrow\quad x=0$ or $y=0$. Thus H is continuous on its domain $\{(x,y)\mid x\neq 0,\ y\neq 0\}$.

45. $\sqrt{x}$ is continuous on its domain $\{(x,y)\mid x\geq 0\}$ and $\sqrt{1-x^2-y^2}$ is continuous on its domain $\left\{(x,y)\mid 1-x^2-y^2\geq 0\right\}=\left\{(x,y)\mid x^2+y^2\leq 1\right\}$, so the sum $G(x,y)=\sqrt{x}+\sqrt{1-x^2-y^2}$ is continuous for $x\geq 0$ and $x^2+y^2\leq 1$, that is, $\left\{(x,y)\mid x^2+y^2\leq 1,\ x\geq 0\right\}$. This is the right half of the unit disk.

46. $G(x,y)=\ln(1+x-y)=g(f(x,y))$ where $f(x,y)=1+x-y$, a polynomial and hence continuous on $\mathbb{R}^2$, and $g(t)=\ln t$, continuous on its domain $\{t\mid t>0\}$. Thus G is continuous on its domain $\{(x,y)\mid 1+x-y>0\}=\{(x,y)\mid y<x+1\}$, the region in $\mathbb{R}^2$ below the line $y=x+1$.

47. $f(x,y,z)=h(g(x,y,z))$ where $g(x,y,z)=x^2+y^2+z^2$, a polynomial that is continuous everywhere, and $h(t)=\arcsin t$, continuous on $[-1,1]$. Thus f is continuous on its domain $\left\{(x,y,z)\mid -1\leq x^2+y^2+z^2\leq 1\right\}=\left\{(x,y,z)\mid x^2+y^2+z^2\leq 1\right\}$, so f is continuous on the unit ball.

48. $\sqrt{y-x^2}$ is continuous on its domain $\left\{(x,y)\mid y-x^2\geq 0\right\}=\left\{(x,y)\mid y\geq x^2\right\}$ and $\ln z$ is continuous on its domain $\{z\mid z>0\}$, so the product $f(x,y,z)=\sqrt{y-x^2}\,\ln z$ is continuous for $y\geq x^2$ and $z>0$, that is, $\left\{(x,y,z)\mid y\geq x^2,\, z>0\right\}$.

49. $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2\leq 2x^2+y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right|\leq\left|y^3\right|$. We know that $\left|y^3\right|\to 0$ as $(x,y)\to(0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^3}{2x^2+y^2}=0$. But $f(0,0)=1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y)\mid (x,y)\neq(0,0)\}$.

50. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0)=0/x^2=0$ for $x\neq 0$, so $f(x,y)\to 0$ as

$(x, y) \to (0, 0)$ along the x-axis. But $f(x, x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x, y) \to \frac{1}{3}$ as $(x, y) \to (0, 0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ doesn't exist, so f is not continuous at $(0, 0)$ and the largest set on which f is continuous is $\{(x, y) \mid (x, y) \neq (0, 0)\}$.

51. $$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta) = 0$$

52. $$\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2) = \lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2} = \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \quad \text{[using l'Hospital's Rule]}$$
$$= \lim_{r\to 0^+} (-r^2) = 0$$

53. $$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} = \lim_{r\to 0^+} \frac{e^{-r^2} - 1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}(-2r)}{2r} \quad \text{[using l'Hospital's Rule]}$$
$$= \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$$

54. **1.** $\lim\limits_{(x,y)\to(a,b)} x = a$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|x - a| < \varepsilon$.

But $|x - a| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta$, so choose $\varepsilon = \delta$. Then $|x - a| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$.

Thus, $\lim\limits_{(x,y)\to(a,b)} x = a$.

2. $\lim\limits_{(x,y)\to(a,b)} y = b$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|y - b| < \varepsilon$.

But $|y - b| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta$, so choose $\varepsilon = \delta$. Then $|y - b| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$.

Thus, $\lim\limits_{(x,y)\to(a,b)} y = b$.

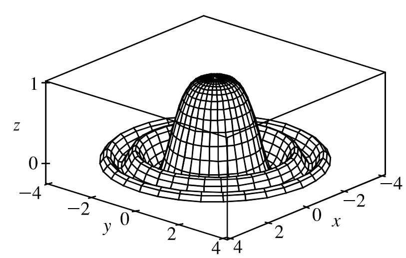
3. $\lim\limits_{(x,y)\to(a,b)} c = c$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|c - c| < \varepsilon$.

But $|c - c| = 0$, so this will be true no matter what δ we choose.

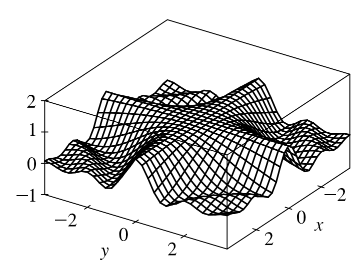
55. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim\limits_{r\to 0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$$

Or: Use the fact that $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.



56. $$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$



From the graph, it appears that f is continuous everywhere. We know xy is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin xy$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin xy}{xy}$ is continuous on $\mathbb{R}^2$ except

possibly where $xy = 0$. To show that f is continuous at those points, consider any point (a, b) in $\mathbb{R}^2$ where $ab = 0$. Because xy is continuous, $xy \to ab = 0$ as $(x, y) \to (a, b)$. If we let $t = xy$, then $t \to 0$ as $(x, y) \to (a, b)$ and $\displaystyle\lim_{(x,y)\to(a,b)} \frac{\sin xy}{xy} = \lim_{t\to 0} \frac{\sin t}{t} = 1$ by Equation 3.3.5. Thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ and f is continuous on $\mathbb{R}^2$.

57. (a) $f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$ Consider the path $y = mx^a$, $0 < a < 4$. [The path does not pass through $(0, 0)$ if $a \le 0$ except for the trivial case where $m = 0$.] If $mx^a \le 0$ then $f(x, mx^a) = 0$. If $mx^a > 0$ then $mx^a = |mx^a| = |m|\,|x^a|$ and $mx^a \ge x^4 \quad\Leftrightarrow\quad |m|\,|x^a| \ge x^4 \quad\Leftrightarrow\quad \dfrac{x^4}{|x^a|} \le |m| \quad\Leftrightarrow\quad |x|^{4-a} \le |m|$ whenever x^a is defined. Then $mx^a \ge x^4 \quad\Leftrightarrow\quad |x| \le |m|^{1/(4-a)}$ so $f(x, mx^a) = 0$ for $|x| \le |m|^{1/(4-a)}$ and $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along this path.

(b) If we approach $(0, 0)$ along the path $y = x^5$, $x > 0$ then we have $f(x, x^5) = 1$ for $0 < x < 1$ because $0 < x^5 < x^4$ there. Thus $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along this path, but in part (a) we found a limit of 0 along other paths, so $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist and f is discontinuous at $(0, 0)$.

(c) First we show that f is discontinuous at any point $(a, 0)$ on the x-axis. If we approach $(a, 0)$ along the path $x = a$, $y > 0$ then $f(a, y) = 1$ for $0 < y < a^4$, so $f(x, y) \to 1$ as $(x, y) \to (a, 0)$ along this path. If we approach $(a, 0)$ along the path $x = a$, $y < 0$ then $f(a, y) = 0$ since $y < 0$ and $f(x, y) \to 0$ as $(x, y) \to (a, 0)$. Thus the limit does not exist and f is discontinuous on the line $y = 0$. f is also discontinuous on the curve $y = x^4$: For any point (a, a^4) on this curve, approaching the point along the path $x = a$, $y > a^4$ gives $f(a, y) = 0$ since $y > a^4$, so $f(x, y) \to 0$ as $(x, y) \to (a, a^4)$. But approaching the point along the path $x = a$, $y < a^4$ gives $f(a, y) = 1$ for $y > 0$, so $f(x, y) \to 1$ as $(x, y) \to (a, a^4)$ and the limit does not exist there.

58. Since $|\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}|\cos\theta \ge |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}|$. Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}| < \delta = \varepsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

59. $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, then $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \varepsilon$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 12.3.61 (the Cauchy-Schwartz Inequality). Set $\delta = \varepsilon / |\mathbf{c}|$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$,
$|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\varepsilon / |\mathbf{c}|) = \varepsilon$. So f is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

1. By Definition 4, $f_T(92,60) = \lim\limits_{h\to 0} \dfrac{f(92+h,60) - f(92,60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92,60) \approx \dfrac{f(94,60) - f(92,60)}{2} = \dfrac{111-105}{2} = 3$, $f_T(92,60) \approx \dfrac{f(90,60) - f(92,60)}{-2} = \dfrac{100-105}{-2} = 2.5$. Averaging these values, we estimate $f_T\,(92,60)$ to be approximately 2.75. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 2.75°F for every degree that the actual temperature rises.

Similarly, $f_H(92,60) = \lim\limits_{h\to 0} \dfrac{f(92,60+h) - f(92,60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92,60) \approx \dfrac{f(92,65) - f(92,60)}{5} = \dfrac{108-105}{5} = 0.6$, $f_H(92,60) \approx \dfrac{f(92,55) - f(92,60)}{-5} = \dfrac{103-105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92,60)$ to be approximately 0.5. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 0.5°F for every percent that the relative humidity increases.

2. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim\limits_{h\to 0} \dfrac{f(40+h,15) - f(40,15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table: $f_v(40,15) \approx \dfrac{f(50,15) - f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$, $f_v(40,15) \approx \dfrac{f(30,15) - f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly, $f_t(40,15) = \lim\limits_{h\to 0} \dfrac{f(40,15+h) - f(40,15)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_t(40,15) \approx \dfrac{f(40,20) - f(40,15)}{5} = \dfrac{28-25}{5} = 0.6$, $f_t(40,15) \approx \dfrac{f(40,10) - f(40,15)}{-5} = \dfrac{21-25}{-5} = 0.8$. Averaging these values, we have $f_t(40,15) \approx 0.7$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim\limits_{t\to\infty} (\partial h/\partial t) = 0$.

3. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude $158°$W, latitude $21°$N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

4. (a) If we start at $(1, 2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1, 2)$ is positive.

(b) If we start at $(1, 2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1, 2)$ is negative.

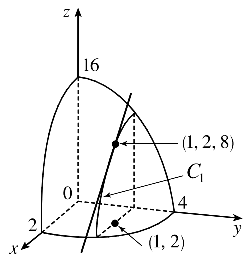
5. (a) The graph of f decreases if we start at $(-1, 2)$ and move in the positive x-direction, so $f_x(-1, 2)$ is negative.

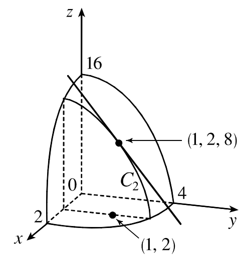
(b) The graph of f decreases if we start at $(-1, 2)$ and move in the positive y-direction, so $f_y(-1, 2)$ is negative.

6. $f_x(2, 1)$ is the rate of change of f at $(2, 1)$ in the x-direction. If we start at $(2, 1)$, where $f(2, 1) = 10$, and move in the positive x-direction, we reach the next contour line [where $f(x, y) = 12$] after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2, 1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2, 1)$ would be the average of these two, so $f_x(2, 1) \approx 2.8$. Similarly, $f_y(2, 1)$ is the rate of change of f at $(2, 1)$ in the y-direction. If we approach $(2, 1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2, 1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2, 1) \approx -2.1$.

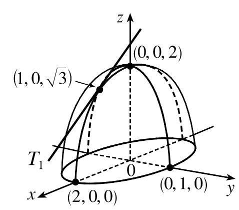
7. $f(x, y) = 16 - 4x^2 - y^2 \quad \Rightarrow \quad f_x(x, y) = -8x$ and $f_y(x, y) = -2y \quad \Rightarrow \quad f_x(1, 2) = -8$ and $f_y(1, 2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$

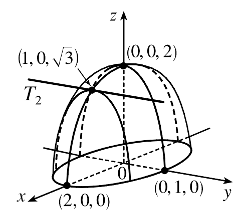
(the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.





8. $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \;\Rightarrow\; f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2} \;\Rightarrow$ $f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1, 0) = 0$.





9. $f(x, y) = x^4 + 5xy^3 \;\Rightarrow\; f_x(x, y) = 4x^3 + 5y^3,\; f_y(x, y) = 0 + 5x \cdot 3y^2 = 15xy^2$

10. $f(x, y) = x^2 y - 3y^4 \;\Rightarrow\; f_x(x, y) = 2x \cdot y - 0 = 2xy,\; f_y(x, y) = x^2 \cdot 1 - 3 \cdot 4y^3 = x^2 - 12y^3$

11. $g(x, y) = x^3 \sin y \;\Rightarrow\; g_x(x, y) = 3x^2 \sin y,\; g_y(x, y) = x^3 \cos y$

12. $g(x, t) = e^{xt} \;\Rightarrow\; g_x(x, t) = e^{xt} \cdot t = te^{xt},\; g_t(x, t) = e^{xt} \cdot x = xe^{xt}$

13. $z = \ln(x + t^2) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2} \cdot 1 = \dfrac{1}{x + t^2},\; \dfrac{\partial z}{\partial t} = \dfrac{1}{x + t^2} \cdot 2t = \dfrac{2t}{x + t^2}$

14. $w = \dfrac{u}{v^2} \;\Rightarrow\; w_u = \dfrac{1}{v^2} \cdot 1 = \dfrac{1}{v^2},\; w_v = u \cdot (-2v^{-3}) = -\dfrac{2u}{v^3}$

15. $f(x, y) = ye^{xy} \;\Rightarrow\; f_x(x, y) = y \cdot e^{xy} \cdot y = y^2 e^{xy},\; f_y(x, y) = y \cdot e^{xy} \cdot x + e^{xy} \cdot 1 = e^{xy} + xye^{xy}$

16. $g(x, y) = (x^2 + xy)^3 \;\Rightarrow\; g_x(x, y) = 3(x^2 + xy)^2(2x + y),\; g_y(x, y) = 3(x^2 + xy)^2(0 + x) = 3x(x^2 + xy)^2$

17. $g(x, y) = y(x + x^2 y)^5 \;\Rightarrow\; g_x(x, y) = 5y(x + x^2 y)^4(1 + 2xy)$,

$g_y(x, y) = y \cdot 5(x + x^2 y)^4 \cdot x^2 + (x + x^2 y)^5 \cdot 1 = 5x^2 y(x + x^2 y)^4 + (x + x^2 y)^5$

18. $f(x, y) = \dfrac{x}{(x + y)^2} \;\Rightarrow\; f_x(x, y) = \dfrac{(x + y)^2(1) - (x)(2)(x + y)}{[(x + y)^2]^2} = \dfrac{x + y - 2x}{(x + y)^3} = \dfrac{y - x}{(x + y)^3}$,

$f_y(x, y) = \dfrac{(x + y)^2(0) - (x)(2)(x + y)}{[(x + y)^2]^2} = -\dfrac{2x}{(x + y)^3}$

19. $f(x,y)=\dfrac{ax+by}{cx+dy} \quad\Rightarrow\quad f_x(x,y)=\dfrac{(cx+dy)(a)-(ax+by)(c)}{(cx+dy)^2}=\dfrac{(ad-bc)y}{(cx+dy)^2}$,

$f_y(x,y)=\dfrac{(cx+dy)(b)-(ax+by)(d)}{(cx+dy)^2}=\dfrac{(bc-ad)x}{(cx+dy)^2}$

20. $w=\dfrac{e^v}{u+v^2} \quad\Rightarrow\quad \dfrac{\partial w}{\partial u}=\dfrac{0(u+v^2)-e^v(1)}{(u+v^2)^2}=-\dfrac{e^v}{(u+v^2)^2},\quad \dfrac{\partial w}{\partial v}=\dfrac{e^v(u+v^2)-e^v(2v)}{(u+v^2)^2}=\dfrac{e^v(u+v^2-2v)}{(u+v^2)^2}$

21. $g(u,v)=(u^2v-v^3)^5 \quad\Rightarrow\quad g_u(u,v)=5(u^2v-v^3)^4\cdot 2uv=10uv(u^2v-v^3)^4$,

$g_v(u,v)=5(u^2v-v^3)^4(u^2-3v^2)=5(u^2-3v^2)(u^2v-v^3)^4$

22. $u(r,\theta)=\sin(r\cos\theta) \quad\Rightarrow\quad u_r(r,\theta)=\cos(r\cos\theta)\cdot\cos\theta=\cos\theta\cos(r\cos\theta)$,

$u_\theta(r,\theta)=\cos(r\cos\theta)(-r\sin\theta)=-r\sin\theta\cos(r\cos\theta)$

23. $R(p,q)=\tan^{-1}(pq^2) \quad\Rightarrow\quad R_p(p,q)=\dfrac{1}{1+(pq^2)^2}\cdot q^2=\dfrac{q^2}{1+p^2q^4},\quad R_q(p,q)=\dfrac{1}{1+(pq^2)^2}\cdot 2pq=\dfrac{2pq}{1+p^2q^4}$

24. $f(x,y)=x^y \quad\Rightarrow\quad f_x(x,y)=yx^{y-1},\ f_y(x,y)=x^y\ln x$

25. $F(x,y)=\displaystyle\int_y^x \cos(e^t)\,dt \quad\Rightarrow\quad F_x(x,y)=\dfrac{\partial}{\partial x}\int_y^x \cos(e^t)\,dt=\cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;

$F_y(x,y)=\dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(e^t)\,dt=\dfrac{\partial}{\partial y}\left[-\int_x^y \cos(e^t)\,dt\right]=-\dfrac{\partial}{\partial y}\int_x^y \cos(e^t)\,dt=-\cos(e^y)$.

26. $F(\alpha,\beta)=\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt \quad\Rightarrow$

$F_\alpha(\alpha,\beta)=\dfrac{\partial}{\partial\alpha}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt=\dfrac{\partial}{\partial\alpha}\left[-\int_\beta^\alpha \sqrt{t^3+1}\,dt\right]=-\dfrac{\partial}{\partial\alpha}\int_\beta^\alpha \sqrt{t^3+1}\,dt=-\sqrt{\alpha^3+1}$ by the Fundamental Theorem of Calculus, Part 1; $\quad F_\beta(\alpha,\beta)=\dfrac{\partial}{\partial\beta}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt=\sqrt{\beta^3+1}$.

27. $f(x,y,z)=x^3yz^2+2yz \quad\Rightarrow\quad f_x(x,y,z)=3x^2yz^2,\ f_y(x,y,z)=x^3z^2+2z,\ f_z(x,y,z)=2x^3yz+2y$

28. $f(x,y,z)=xy^2e^{-xz} \quad\Rightarrow\quad f_x(x,y,z)=y^2\left[x\cdot e^{-xz}(-z)+e^{-xz}\cdot 1\right]=(1-xz)y^2e^{-xz},\quad f_y(x,y,z)=2xye^{-xz}$,

$f_z(x,y,z)=xy^2e^{-xz}(-x)=-x^2y^2e^{-xz}$

29. $w=\ln(x+2y+3z) \quad\Rightarrow\quad \dfrac{\partial w}{\partial x}=\dfrac{1}{x+2y+3z},\quad \dfrac{\partial w}{\partial y}=\dfrac{2}{x+2y+3z},\quad \dfrac{\partial w}{\partial z}=\dfrac{3}{x+2y+3z}$

30. $w=y\tan(x+2z) \quad\Rightarrow\quad \dfrac{\partial w}{\partial x}=y\left[\sec^2(x+2z)\right](1)=y\sec^2(x+2z),\quad \dfrac{\partial w}{\partial y}=\tan(x+2z)$,

$\dfrac{\partial w}{\partial z}=y\left[\sec^2(x+2z)\right](2)=2y\sec^2(x+2z)$

31. $p=\sqrt{t^4+u^2\cos v} \quad\Rightarrow\quad \dfrac{\partial p}{\partial t}=\frac{1}{2}(t^4+u^2\cos v)^{-1/2}(4t^3)=\dfrac{2t^3}{\sqrt{t^4+u^2\cos v}}$,

$\dfrac{\partial p}{\partial u}=\frac{1}{2}(t^4+u^2\cos v)^{-1/2}(2u\cos v)=\dfrac{u\cos v}{\sqrt{t^4+u^2\cos v}},\quad \dfrac{\partial p}{\partial v}=\frac{1}{2}(t^4+u^2\cos v)^{-1/2}[u^2(-\sin v)]=-\dfrac{u^2\sin v}{2\sqrt{t^4+u^2\cos v}}$

32. $u = x^{y/z} \quad \Rightarrow \quad u_x = \dfrac{y}{z}\, x^{(y/z)-1},\ u_y = x^{y/z} \ln x \cdot \dfrac{1}{z} = \dfrac{x^{y/z}}{z} \ln x,\ \ u_z = x^{y/z} \ln x \cdot \dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2} \ln x$

33. $h(x, y, z, t) = x^2 y \cos(z/t) \quad \Rightarrow \quad h_x(x, y, z, t) = 2xy \cos(z/t),\ h_y(x, y, z, t) = x^2 \cos(z/t),$

$h_z(x, y, z, t) = -x^2 y \sin(z/t)(1/t) = (-x^2 y/t) \sin(z/t),\ h_t(x, y, z, t) = -x^2 y \sin(z/t)(-zt^{-2}) = (x^2 yz/t^2) \sin(z/t)$

34. $\phi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2} \quad \Rightarrow \quad \phi_x(x, y, z, t) = \dfrac{1}{\gamma z + \delta t^2}(\alpha) = \dfrac{\alpha}{\gamma z + \delta t^2},$

$\phi_y(x, y, z, t) = \dfrac{1}{\gamma z + \delta t^2}(2\beta y) = \dfrac{2\beta y}{\gamma z + \delta t^2},\ \phi_z(x, y, z, t) = \dfrac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(\gamma)}{(\gamma z + \delta t^2)^2} = \dfrac{-\gamma(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2},$

$\phi_t(x, y, z, t) = \dfrac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(2\delta t)}{(\gamma z + \delta t^2)^2} = -\dfrac{2\delta t(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}$

35. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

36. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i \cos(x_1 + 2x_2 + \cdots + nx_n)$.

37. $R(s, t) = te^{s/t} \quad \Rightarrow \quad R_t(s, t) = t \cdot e^{s/t}(-s/t^2) + e^{s/t} \cdot 1 = \left(1 - \dfrac{s}{t}\right) e^{s/t}$, so $R_t(0, 1) = \left(1 - \dfrac{0}{1}\right) e^{0/1} = 1$.

38. $f(x, y) = y \sin^{-1}(xy) \quad \Rightarrow \quad f_y(x, y) = y \cdot \dfrac{1}{\sqrt{1 - (xy)^2}}(x) + \sin^{-1}(xy) \cdot 1 = \dfrac{xy}{\sqrt{1 - x^2 y^2}} + \sin^{-1}(xy),$

so $f_y\left(1, \frac{1}{2}\right) = \dfrac{1 \cdot \frac{1}{2}}{\sqrt{1 - 1^2\left(\frac{1}{2}\right)^2}} + \sin^{-1}\left(1 \cdot \frac{1}{2}\right) = \dfrac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1}\frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}$.

39. $f(x, y, z) = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}} \quad \Rightarrow$

$$f_y(x, y, z) = \frac{1}{\dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}} \cdot \frac{\left(1 + \sqrt{x^2 + y^2 + z^2}\right)\left(-\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y\right) - \left(1 - \sqrt{x^2 + y^2 + z^2}\right)\left(\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y\right)}{\left(1 + \sqrt{x^2 + y^2 + z^2}\right)^2}$$

$$= \frac{1 + \sqrt{x^2 + y^2 + z^2}}{1 - \sqrt{x^2 + y^2 + z^2}} \cdot \frac{-y(x^2 + y^2 + z^2)^{-1/2}\left(1 + \sqrt{x^2 + y^2 + z^2} + 1 - \sqrt{x^2 + y^2 + z^2}\right)}{\left(1 + \sqrt{x^2 + y^2 + z^2}\right)^2}$$

$$= \frac{-y(x^2 + y^2 + z^2)^{-1/2}\,(2)}{\left(1 - \sqrt{x^2 + y^2 + z^2}\right)\left(1 + \sqrt{x^2 + y^2 + z^2}\right)} = \frac{-2y}{\sqrt{x^2 + y^2 + z^2}\,[1 - (x^2 + y^2 + z^2)]}$$

so $f_y(1, 2, 2) = \dfrac{-2(2)}{\sqrt{1^2 + 2^2 + 2^2}\,[1 - (1^2 + 2^2 + 2^2)]} = \dfrac{-4}{\sqrt{9}\,(1 - 9)} = \dfrac{1}{6}$.

40. $f(x,y,z)=x^{yz} \;\Rightarrow\; f_z(x,y,z)=(x^{yz}\ln x)(y)=yx^{yz}\ln x$, so $f_z(e,1,0)=1e^{(1)(0)}\ln e=1$.

41. $x^2+2y^2+3z^2=1 \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(x^2+2y^2+3z^2\right)=\dfrac{\partial}{\partial x}(1) \;\Rightarrow\; 2x+0+6z\,\dfrac{\partial z}{\partial x}=0 \;\Rightarrow\; 6z\,\dfrac{\partial z}{\partial x}=-2x \;\Rightarrow$

$\dfrac{\partial z}{\partial x}=\dfrac{-2x}{6z}=-\dfrac{x}{3z}$, and $\dfrac{\partial}{\partial y}\left(x^2+2y^2+3z^2\right)=\dfrac{\partial}{\partial y}(1) \;\Rightarrow\; 0+4y+6z\,\dfrac{\partial z}{\partial y}=0 \;\Rightarrow\; 6z\,\dfrac{\partial z}{\partial y}=-4y \;\Rightarrow$

$\dfrac{\partial z}{\partial y}=\dfrac{-4y}{6z}=-\dfrac{2y}{3z}$.

42. $x^2-y^2+z^2-2z=4 \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(x^2-y^2+z^2-2z\right)=\dfrac{\partial}{\partial x}(4) \;\Rightarrow\; 2x-0+2z\,\dfrac{\partial z}{\partial x}-2\,\dfrac{\partial z}{\partial x}=0 \;\Rightarrow$

$(2z-2)\,\dfrac{\partial z}{\partial x}=-2x \;\Rightarrow\; \dfrac{\partial z}{\partial x}=\dfrac{-2x}{2z-2}=\dfrac{x}{1-z}$, and $\dfrac{\partial}{\partial y}\left(x^2-y^2+z^2-2z\right)=\dfrac{\partial}{\partial y}(4) \;\Rightarrow$

$0-2y+2z\,\dfrac{\partial z}{\partial y}-2\,\dfrac{\partial z}{\partial y}=0 \;\Rightarrow\; (2z-2)\,\dfrac{\partial z}{\partial y}=2y \;\Rightarrow\; \dfrac{\partial z}{\partial y}=\dfrac{2y}{2z-2}=\dfrac{y}{z-1}$.

43. $e^z=xyz \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(e^z\right)=\dfrac{\partial}{\partial x}(xyz) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial x}=y\left(x\,\dfrac{\partial z}{\partial x}+z\cdot 1\right) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial x}-xy\,\dfrac{\partial z}{\partial x}=yz \;\Rightarrow$

$\left(e^z-xy\right)\dfrac{\partial z}{\partial x}=yz$, so $\dfrac{\partial z}{\partial x}=\dfrac{yz}{e^z-xy}$.

$\dfrac{\partial}{\partial y}\left(e^z\right)=\dfrac{\partial}{\partial y}(xyz) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial y}=x\left(y\,\dfrac{\partial z}{\partial x}+z\cdot 1\right) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial y}-xy\,\dfrac{\partial z}{\partial y}=xz \;\Rightarrow\; \left(e^z-xy\right)\dfrac{\partial z}{\partial y}=xz$, so

$\dfrac{\partial z}{\partial y}=\dfrac{xz}{e^z-xy}$.

44. $yz+x\ln y=z^2 \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(yz+x\ln y\right)=\dfrac{\partial}{\partial x}\left(z^2\right) \;\Rightarrow\; y\,\dfrac{\partial z}{\partial x}+\ln y=2z\,\dfrac{\partial z}{\partial x} \;\Rightarrow\; \ln y=2z\,\dfrac{\partial z}{\partial x}-y\,\dfrac{\partial z}{\partial x} \;\Rightarrow$

$\ln y=(2z-y)\,\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x}=\dfrac{\ln y}{2z-y}$.

$\dfrac{\partial}{\partial y}\left(yz+x\ln y\right)=\dfrac{\partial}{\partial y}\left(z^2\right) \;\Rightarrow\; y\,\dfrac{\partial z}{\partial y}+z\cdot 1+x\cdot\dfrac{1}{y}=2z\,\dfrac{\partial z}{\partial y} \;\Rightarrow\; z+\dfrac{x}{y}=2z\,\dfrac{\partial z}{\partial y}-y\,\dfrac{\partial z}{\partial y} \;\Rightarrow$

$z+\dfrac{x}{y}=(2z-y)\,\dfrac{\partial z}{\partial y}$, so $\dfrac{\partial z}{\partial y}=\dfrac{z+(x/y)}{2z-y}=\dfrac{x+yz}{y(2z-y)}$.

45. (a) $z=f(x)+g(y) \;\Rightarrow\; \dfrac{\partial z}{\partial x}=f'(x),\quad \dfrac{\partial z}{\partial y}=g'(y)$

(b) $z=f(x+y)$. Let $u=x+y$. Then $\dfrac{\partial z}{\partial x}=\dfrac{df}{du}\dfrac{\partial u}{\partial x}=\dfrac{df}{du}(1)=f'(u)=f'(x+y)$,

$\dfrac{\partial z}{\partial y}=\dfrac{df}{du}\dfrac{\partial u}{\partial y}=\dfrac{df}{du}(1)=f'(u)=f'(x+y)$.

46. (a) $z = f(x)g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x)g(y), \quad \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du} \cdot y = yf'(u) = yf'(xy)$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

47. $f(x,y) = x^4y - 2x^3y^2 \quad \Rightarrow \quad f_x(x,y) = 4x^3y - 6x^2y^2,\ \ f_y(x,y) = x^4 - 4x^3y$. Then $f_{xx}(x,y) = 12x^2y - 12xy^2$,

$f_{xy}(x,y) = 4x^3 - 12x^2y,\ \ f_{yx}(x,y) = 4x^3 - 12x^2y$, and $f_{yy}(x,y) = -4x^3$.

48. $f(x,y) = \ln(ax+by) \quad \Rightarrow \quad f_x(x,y) = \dfrac{a}{ax+by} = a(ax+by)^{-1},\ \ f_y(x,y) = \dfrac{b}{ax+by} = b(ax+by)^{-1}$. Then

$f_{xx}(x,y) = -a(ax+by)^{-2}(a) = -\dfrac{a^2}{(ax+by)^2},\ \ f_{xy}(x,y) = -a(ax+by)^{-2}(b) = -\dfrac{ab}{(ax+by)^2}$,

$f_{yx}(x,y) = -b(ax+by)^{-2}(a) = -\dfrac{ab}{(ax+by)^2}$, and $f_{yy}(x,y) = -b(ax+by)^{-2}(b) = -\dfrac{b^2}{(ax+by)^2}$.

49. $z = \dfrac{y}{2x+3y} = y(2x+3y)^{-1} \quad \Rightarrow \quad z_x = y(-1)(2x+3y)^{-2}(2) = -\dfrac{2y}{(2x+3y)^2}$,

$z_y = \dfrac{(2x+3y)\cdot 1 - y\cdot 3}{(2x+3y)^2} = \dfrac{2x}{(2x+3y)^2}$. Then $z_{xx} = -2y(-2)(2x+3y)^{-3}(2) = \dfrac{8y}{(2x+3y)^3}$,

$z_{xy} = -\dfrac{(2x+3y)^2\cdot 2 - 2y\cdot 2(2x+3y)(3)}{[(2x+3y)^2]^2} = -\dfrac{(2x+3y)\,(4x+6y-12y)}{(2x+3y)^4} = \dfrac{6y-4x}{(2x+3y)^3}$,

$z_{yx} = \dfrac{(2x+3y)^2\cdot 2 - 2x\cdot 2(2x+3y)(2)}{[(2x+3y)^2]^2} = \dfrac{6y-4x}{(2x+3y)^3},\ \ z_{yy} = 2x(-2)(2x+3y)^{-3}(3) = -\dfrac{12x}{(2x+3y)^3}$.

50. $T = e^{-2r}\cos\theta \quad \Rightarrow \quad T_r = -2e^{-2r}\cos\theta,\ \ T_\theta = -e^{-2r}\sin\theta$. Then $T_{rr} = -2e^{-2r}(-2)\cos\theta = 4e^{-2r}\cos\theta$,

$T_{r\theta} = 2e^{-2r}\sin\theta,\ \ T_{\theta r} = -e^{-2r}(-2)\sin\theta = 2e^{-2r}\sin\theta,\ \ T_{\theta\theta} = -e^{-2r}\cos\theta$.

51. $v = \sin(s^2-t^2) \quad \Rightarrow \quad v_s = \cos(s^2-t^2)\cdot 2s = 2s\cos(s^2-t^2),\ v_t = \cos(s^2-t^2)\cdot(-2t) = -2t\cos(s^2-t^2)$. Then

$v_{ss} = 2s\left[-\sin(s^2-t^2)\cdot 2s\right] + \cos(s^2-t^2)\cdot 2 = 2\cos(s^2-t^2) - 4s^2\sin(s^2-t^2)$,

$v_{st} = 2s\left[-\sin(s^2-t^2)\cdot(-2t)\right] + \cos(s^2-t^2)\cdot 0 = 4st\sin(s^2-t^2)$,

$v_{ts} = -2t\left[-\sin(s^2-t^2)\cdot 2s\right] + \cos(s^2-t^2)\cdot 0 = 4st\sin(s^2-t^2)$,

$v_{tt} = -2t\left[-\sin(s^2-t^2)\cdot(-2t)\right] + \cos(s^2-t^2)\cdot(-2) = -2\cos(s^2-t^2) - 4t^2\sin(s^2-t^2)$.

52. $z = \arctan \dfrac{x+y}{1-xy} \quad \Rightarrow$

$$z_x = \frac{1}{1 + \left(\dfrac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2} = \frac{1+y^2}{1+x^2+y^2+x^2y^2}$$

$$= \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2},$$

$$z_y = \frac{1}{1 + \left(\dfrac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2+(x+y)^2} = \frac{1+x^2}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2}.$$

Then $z_{xx} = -(1+x^2)^{-2} \cdot 2x = -\dfrac{2x}{(1+x^2)^2}$, $z_{xy} = 0, z_{yx} = 0$, $z_{yy} = -(1+y^2)^{-2} \cdot 2y = -\dfrac{2y}{(1+y^2)^2}$.

53. $u = x^4y^3 - y^4 \quad \Rightarrow \quad u_x = 4x^3y^3$, $u_{xy} = 12x^3y^2$ and $u_y = 3x^4y^2 - 4y^3$, $u_{yx} = 12x^3y^2$.

Thus $u_{xy} = u_{yx}$.

54. $u = e^{xy}\sin y \quad \Rightarrow \quad u_x = ye^{xy}\sin y$, $u_{xy} = ye^{xy}\cos y + (\sin y)(y \cdot xe^{xy} + e^{xy} \cdot 1) = e^{xy}(y\cos y + xy\sin y + \sin y)$,

$u_y = e^{xy}\cos y + (\sin y)(xe^{xy}) = e^{xy}(\cos y + x\sin y)$,

$u_{yx} = e^{xy} \cdot \sin y + (\cos y + x\sin y) \cdot ye^{xy} = e^{xy}(\sin y + y\cos y + xy\sin y)$. Thus $u_{xy} = u_{yx}$.

55. $u = \cos(x^2y) \quad \Rightarrow \quad u_x = -\sin(x^2y) \cdot 2xy = -2xy\sin(x^2y)$,

$u_{xy} = -2xy \cdot \cos(x^2y) \cdot x^2 + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$ and

$u_y = -\sin(x^2y) \cdot x^2 = -x^2\sin(x^2y)$, $u_{yx} = -x^2 \cdot \cos(x^2y) \cdot 2xy + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$.

Thus $u_{xy} = u_{yx}$.

56. $u = \ln(x+2y) \quad \Rightarrow \quad u_x = \dfrac{1}{x+2y} = (x+2y)^{-1}$, $u_{xy} = (-1)(x+2y)^{-2}(2) = -\dfrac{2}{(x+2y)^2}$ and

$u_y = \dfrac{1}{x+2y} \cdot 2 = 2(x+2y)^{-1}$, $u_{yx} = (-2)(x+2y)^{-2} = -\dfrac{2}{(x+2y)^2}$. Thus $u_{xy} = u_{yx}$.

57. $f(x,y) = x^4y^2 - x^3y \quad \Rightarrow \quad f_x = 4x^3y^2 - 3x^2y$, $f_{xx} = 12x^2y^2 - 6xy$, $f_{xxx} = 24xy^2 - 6y$ and

$f_{xy} = 8x^3y - 3x^2$, $f_{xyx} = 24x^2y - 6x$.

58. $f(x,y) = \sin(2x+5y) \quad \Rightarrow \quad f_y = \cos(2x+5y) \cdot 5 = 5\cos(2x+5y)$, $f_{yx} = -5\sin(2x+5y) \cdot 2 = -10\sin(2x+5y)$,

$f_{yxy} = -10\cos(2x+5y) \cdot 5 = -50\cos(2x+5y)$

59. $f(x,y,z) = e^{xyz^2} \quad \Rightarrow \quad f_x = e^{xyz^2} \cdot yz^2 = yz^2e^{xyz^2}$, $f_{xy} = yz^2 \cdot e^{xyz^2}(xz^2) + e^{xyz^2} \cdot z^2 = (xyz^4 + z^2)e^{xyz^2}$,

$f_{xyz} = (xyz^4 + z^2) \cdot e^{xyz^2}(2xyz) + e^{xyz^2} \cdot (4xyz^3 + 2z) = (2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$.

60. $g(r,s,t) = e^r\sin(st) \quad \Rightarrow \quad g_r = e^r\sin(st)$, $g_{rs} = e^r\cos(st) \cdot t = te^r\cos(st)$,

$g_{rst} = te^r(-\sin(st) \cdot s) + \cos(st) \cdot e^r = e^r[\cos(st) - st\sin(st)]$.

61. $W = \sqrt{u+v^2} \quad \Rightarrow \quad \dfrac{\partial W}{\partial v} = \frac{1}{2}(u+v^2)^{-1/2}(2v) = v(u+v^2)^{-1/2}$,

$$\frac{\partial^2 W}{\partial u\,\partial v} = v\left(-\tfrac{1}{2}\right)(u+v^2)^{-3/2}(1) = -\tfrac{1}{2}v(u+v^2)^{-3/2}, \quad \frac{\partial^3 W}{\partial u^2\,\partial v} = -\tfrac{1}{2}v\left(-\tfrac{3}{2}\right)(u+v^2)^{-5/2}(1) = \tfrac{3}{4}v(u+v^2)^{-5/2}.$$

62. $V = \ln(r+s^2+t^3) \quad \Rightarrow \quad \dfrac{\partial V}{\partial t} = \dfrac{3t^2}{r+s^2+t^3} = 3t^2(r+s^2+t^3)^{-1}$,

$$\frac{\partial^2 V}{\partial s\,\partial t} = 3t^2(-1)(r+s^2+t^3)^{-2}(2s) = -6st^2(r+s^2+t^3)^{-2},$$

$$\frac{\partial^3 V}{\partial r\,\partial s\,\partial t} = -6st^2(-2)(r+s^2+t^3)^{-3}(1) = 12st^2(r+s^2+t^3)^{-3} = \frac{12st^2}{(r+s^2+t^3)^3}.$$

63. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = (y+2z)^{-1}, \quad \dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}$,

$$\frac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2}, \quad \frac{\partial^3 w}{\partial x^2\,\partial y} = 0.$$

64. $u = x^a y^b z^c$. If $a=0$, or if $b=0$ or 1, or if $c=0$, 1, or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise $\dfrac{\partial u}{\partial z} = cx^a y^b z^{c-1}$,

$$\frac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}, \quad \frac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}, \quad \frac{\partial^4 u}{\partial y\,\partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3},$$

$\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

65. $f(x,y) = xy^2 - x^3y \quad \Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)y^2-(x+h)^3y-(xy^2-x^3y)}{h} \\ &= \lim_{h\to 0}\frac{h(y^2-3x^2y-3xyh-yh^2)}{h} = \lim_{h\to 0}(y^2-3x^2y-3xyh-yh^2) = y^2-3x^2y \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x(y+h)^2-x^3(y+h)-(xy^2-x^3y)}{h} = \lim_{h\to 0}\frac{h(2xy+xh-x^3)}{h} \\ &= \lim_{h\to 0}(2xy+xh-x^3) = 2xy-x^3 \end{aligned}$$

66. $f(x,y) = \dfrac{x}{x+y^2} \quad \Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{\frac{x+h}{x+h+y^2}-\frac{x}{x+y^2}}{h}\cdot\frac{(x+h+y^2)(x+y^2)}{(x+h+y^2)(x+y^2)} \\ &= \lim_{h\to 0}\frac{(x+h)(x+y^2)-x(x+h+y^2)}{h(x+h+y^2)(x+y^2)} = \lim_{h\to 0}\frac{y^2h}{h(x+h+y^2)(x+y^2)} \\ &= \lim_{h\to 0}\frac{y^2}{(x+h+y^2)(x+y^2)} = \frac{y^2}{(x+y^2)^2} \end{aligned}$$

[continued]

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{\frac{x}{x+(y+h)^2}-\frac{x}{x+y^2}}{h}\cdot\frac{\left[x+(y+h)^2\right](x+y^2)}{\left[x+(y+h)^2\right](x+y^2)} \\ &= \lim_{h\to 0}\frac{x(x+y^2)-x\left[x+(y+h)^2\right]}{h[x+(y+h)^2](x+y^2)} = \lim_{h\to 0}\frac{h(-2xy-xh)}{h[x+(y+h)^2](x+y^2)} \\ &= \lim_{h\to 0}\frac{-2xy-xh}{\left[x+(y+h)^2\right](x+y^2)} = \frac{-2xy}{(x+y^2)^2} \end{aligned}$$

67. Assuming that the third partial derivatives of f are continuous (easily verified), we can write $f_{xzy} = f_{yxz}$. Then

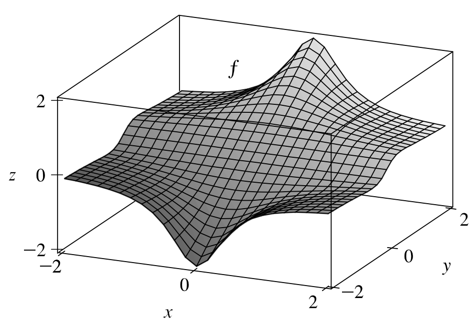
$f(x,y,z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right) \quad\Rightarrow\quad f_y = 2xyz^3 + 0$, $f_{yx} = 2yz^3$, and $f_{yxz} = 6yz^2 = f_{xzy}$.

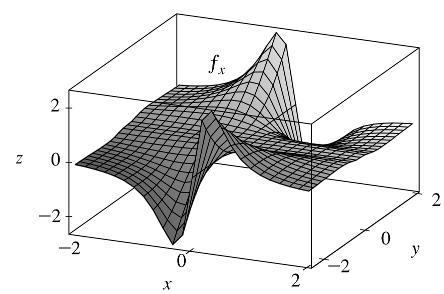
68. Let $f(x,y,z) = \sqrt{1+xz}$ and $h(x,y,z) = \sqrt{1-xy}$ so that $g = f + h$. Then $f_y = 0 = f_{yx} = f_{yxz}$ and $h_z = 0 = h_{zx} = h_{zxy}$. But (since the partial derivatives are continous on their domains) $f_{xyz} = f_{yxz}$ and $h_{xyz} = h_{zxy}$, so $g_{xyz} = f_{xyz} + h_{xyz} = 0 + 0 = 0$.

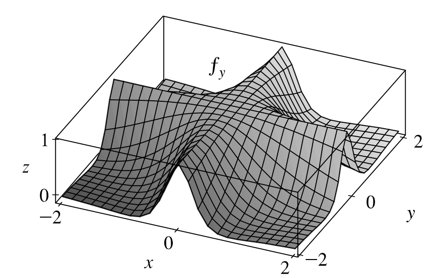
69. First of all, if we start at the point $(3,-3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

70. $f(x,y) = \dfrac{y}{1+x^2y^2} \quad\Rightarrow\quad f_x = \dfrac{(1+x^2y^2)(0) - y(2xy^2)}{(1+x^2y^2)^2} = -\dfrac{2xy^3}{(1+x^2y^2)^2}$,

$f_y = \dfrac{(1+x^2y^2)(1) - y(2x^2y)}{(1+x^2y^2)^2} = \dfrac{1-x^2y^2}{(1+x^2y^2)^2}$

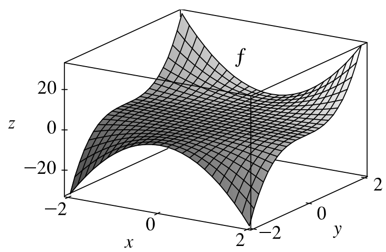


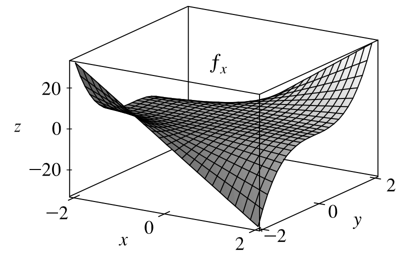


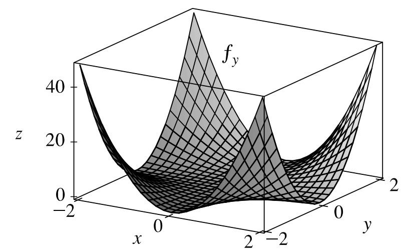


Note that traces of f in planes parallel to the xz-plane have only one extreme value (a minimum for $y < 0$, a maximum for $y > 0$), and the traces of f_x in these planes have only one zero (going from negative to positive if $y < 0$ and from positive to negative if $y > 0$). The traces of f in planes parallel to the yz-plane have two extreme values, and the traces of f_y in these planes have two zeros.

71. $f(x,y) = x^2y^3 \quad \Rightarrow \quad f_x = 2xy^3, \quad f_y = 3x^2y^2$







Note that traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < 0$ and upward for $y > 0$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < 0$ and positive slopes for $y > 0$. The traces of f in planes parallel to the yz-plane are cubic curves, and the traces of f_y in these planes are parabolas.

72. (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1,2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.

(b) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1,2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

(c) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so f_{xy} is the rate of change of f_x in the y-direction. f_x is positive at $(1,2)$ and if we move in the positive y-direction, the surface becomes steeper, looking in the positive x-direction. Thus the values of f_x are increasing and $f_{xy}(1,2)$ is positive.

(d) f_x is negative at $(-1,2)$ and if we move in the positive y-direction, the surface gets steeper (with negative slope), looking in the positive x-direction. This means that the values of f_x are decreasing as y increases, so $f_{xy}(-1,2)$ is negative.

73. By Definition 4, $f_x(3,2) = \lim\limits_{h \to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2.

Similarly, $f_x(3,2.2) = \lim\limits_{h \to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$, $f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$.

Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

[continued]

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$$f_x(3,1.8) \approx \frac{f(3.5,1.8)-f(3,1.8)}{0.5} = \frac{20.0-18.1}{0.5} = 3.8,\ f_x(3,1.8) \approx \frac{f(2.5,1.8)-f(3,1.8)}{-0.5} = \frac{12.5-18.1}{-0.5} = 11.2.$$

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{f_x(x,y+h)-f_x(x,y)}{h} \quad\Rightarrow$

$f_{xy}(3,2) = \lim\limits_{h\to 0} \dfrac{f_x(3,2+h)-f_x(3,2)}{h}$. We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2)-f_x(3,2)}{0.2} = \frac{16.8-12.2}{0.2} = 23,\ f_{xy}(3,2) \approx \frac{f_x(3,1.8)-f_x(3,2)}{-0.2} = \frac{7.5-12.2}{-0.2} = 23.5.$$

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

74. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

75. (a) $f(x,y) = 4 - x^2 - 2y^2$. In the plane $y = 1$, $f(x,1) = 4 - x^2 - 2(1^2) = 2 - x^2$, so a vector equation for C_1 is given by $\mathbf{r}(t) = \langle t, 1, 2-t^2\rangle$ where the point $(1,1,1)$ corresponds to $t = 1$. Then $\mathbf{r}'(t) = \langle 1, 0, -2t\rangle \quad\Rightarrow\quad \mathbf{r}'(1) = \langle 1, 0, -2\rangle$ and parametric equations of the tangent line are $x = t+1$, $y = 1$, $z = -2t+1$. Thus, $x = t+1 \quad\Rightarrow\quad x-1 = t \quad\Rightarrow$

$z = -2(x-1)+1 = -2x+3$. So the equation of the tangent line can be given by $z = -2x+3$, $y = 1$ which has a slope $m = -2$. Therefore, $f_x(1,1) = -2$.

(b) In the plane $x = 1$, $f(1,y) = 4 - 1^2 - 2y^2 = 3 - 2y^2$, so a vector equation for C_2 is given by $\mathbf{r}(t) = \langle 1, t, 3-2t^2 \rangle$ where the point $(1,1,1)$ corresponds to $t = 1$. Then $\mathbf{r}'(t) = \langle 0, 1, -4t \rangle \;\Rightarrow\; \mathbf{r}'(1) = \langle 0, 1, -4 \rangle$ and parametric equations of the tangent line are $x = 1, y = t+1, z = -4t+1$. Thus, $y = t+1 \;\Rightarrow\; y - 1 = t \;\Rightarrow$ $z = -4(y-1)+1 = -4y+5$. So the equation of the tangent line can be given by $z = -4y+5$, $x = 1$ which has a slope $m = -4$. Therefore, $f_y(1,1) = -4$.

76. For each i, $i = 1, \ldots, n$, $\partial u/\partial x_i = a_i e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ and $\partial^2 u/\partial x_i^2 = a_i^2 e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$.

Then $\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = u$

since $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

77. (a) $u = \sin(kx)\,\sin(akt) \;\Rightarrow\; u_t = ak\sin(kx)\,\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\,\sin(akt)$, $u_x = k\cos(kx)\,\sin(akt)$, $u_{xx} = -k^2\sin(kx)\,\sin(akt)$. Thus $u_{tt} = a^2 u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2 - x^2} \;\Rightarrow\; u_t = \dfrac{(a^2t^2 - x^2) - t(2a^2t)}{(a^2t^2 - x^2)^2} = -\dfrac{a^2t^2 + x^2}{(a^2t^2 - x^2)^2}$,

$$u_{tt} = \frac{-2a^2t(a^2t^2 - x^2)^2 + (a^2t^2 + x^2)(2)(a^2t^2 - x^2)(2a^2t)}{(a^2t^2 - x^2)^4} = \frac{2a^4t^3 + 6a^2tx^2}{(a^2t^2 - x^2)^3},$$

$$u_x = t(-1)(a^2t^2 - x^2)^{-2}(-2x) = \frac{2tx}{(a^2t^2 - x^2)^2},$$

$$u_{xx} = \frac{2t(a^2t^2 - x^2)^2 - 2tx\,(2)(a^2t^2 - x^2)(-2x)}{(a^2t^2 - x^2)^4} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{(a^2t^2 - x^2)^3} = \frac{2a^2t^3 + 6tx^2}{(a^2t^2 - x^2)^3}.$$

Thus $u_{tt} = a^2 u_{xx}$.

(c) $u = (x - at)^6 + (x + at)^6 \;\Rightarrow\; u_t = -6a(x - at)^5 + 6a(x + at)^5$, $u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$. Thus $u_{tt} = a^2 u_{xx}$.

(d) $u = \sin(x - at) + \ln(x + at) \;\Rightarrow\; u_t = -a\cos(x - at) + \dfrac{a}{x + at}$, $u_{tt} = -a^2\sin(x - at) - \dfrac{a^2}{(x + at)^2}$,

$u_x = \cos(x - at) + \dfrac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \dfrac{1}{(x + at)^2}$. Thus $u_{tt} = a^2 u_{xx}$.

78. (a) $u = x^2 + y^2 \;\Rightarrow\; u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln\sqrt{x^2+y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}}\left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, so $u_{xx}+u_{yy}=0$.

(e) $u = \sin x\,\cosh y + \cos x\,\sinh y$ is a solution: $u_x = \cos x\,\cosh y - \sin x\,\sinh y$, $u_{xx} = -\sin x\,\cosh y - \cos x\,\sinh y$,

and $u_y = \sin x\,\sinh y + \cos x\,\cosh y$, $u_{yy} = \sin x\,\cosh y + \cos x\,\sinh y$.

(f) $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution: $u_x = -e^{-x}\cos y + e^{-y}\sin x$, $\quad u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, and

$u_y = -e^{-x}\sin y + e^{-y}\cos x$, $\quad u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

79. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus, $u_{xx}+u_{yy}+u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

80. $u = e^{-\alpha^2k^2t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2k^2t}\cos kx$, $\quad u_{xx} = -k^2e^{-\alpha^2k^2t}\sin kx$, $\quad$ and $\quad u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$.

Thus, $\alpha^2 u_{xx} = u_t$.

81. $c(x,t) = \dfrac{1}{\sqrt{4\pi Dt}}\,e^{-x^2/(4Dt)} \quad\Rightarrow$

$$\begin{aligned}\frac{\partial c}{\partial t} &= \frac{1}{\sqrt{4\pi Dt}}\cdot e^{-x^2/(4Dt)}\left[-x^2(-1)(4Dt)^{-2}(4D)\right] + e^{-x^2/(4Dt)}\cdot\left(-\tfrac{1}{2}\right)(4\pi Dt)^{-3/2}(4\pi D)\\ &= (4\pi Dt)^{-3/2}\left(4\pi Dt\cdot\frac{x^2}{4Dt^2} - 2\pi D\right)e^{-x^2/(4Dt)} = \frac{2\pi D}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)},\end{aligned}$$

$$\frac{\partial c}{\partial x} = \frac{1}{\sqrt{4\pi Dt}}\,e^{-x^2/(4Dt)}\cdot\frac{-2x}{4Dt} = \frac{-2\pi x}{(4\pi Dt)^{3/2}}\,e^{-x^2/(4Dt)}, \text{ and}$$

$$\begin{aligned}\frac{\partial^2 c}{\partial x^2} &= \frac{-2\pi}{(4\pi Dt)^{3/2}}\left[x\cdot e^{-x^2/(4Dt)}\cdot\frac{-2x}{4Dt} + e^{-x^2/(4Dt)}\cdot 1\right]\\ &= \frac{-2\pi}{(4\pi Dt)^{3/2}}\left(-\frac{x^2}{2Dt}+1\right)e^{-x^2/(4Dt)} = \frac{2\pi}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)}.\end{aligned}$$

Thus, $\dfrac{\partial c}{\partial t} = \dfrac{2\pi D}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)} = D\left[\dfrac{2\pi}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)}\right] = D\,\dfrac{\partial^2 c}{\partial x^2}$.

82. (a) $T(x,y) = \dfrac{60}{1+x^2+y^2} \quad\Rightarrow\quad \dfrac{\partial T}{\partial x} = -\dfrac{60(2x)}{(1+x^2+y^2)^2}$, so at $(2,1)$, $T_x = -\dfrac{240}{(1+4+1)^2} = -\dfrac{20}{3}$.

(b) $\dfrac{\partial T}{\partial y} = -\dfrac{60(2y)}{(1+x^2+y^2)^2}$, so at $(2,1)$, $T_y = -\dfrac{120}{36} = -\dfrac{10}{3}$. Thus, from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}\,^\circ\mathrm{C/m}$ in the x-direction and is decreasing at a rate of $\frac{10}{3}\,^\circ\mathrm{C/m}$ in the y-direction.

83. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus, $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

84. (a) $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

(b) By part (a), $PV = mRT \quad\Rightarrow\quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad\Rightarrow\quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.

Since $T = \dfrac{PV}{mR}$, we have $T\,\dfrac{\partial P}{\partial T}\,\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P} = mR$.

85. $\left(P + \dfrac{n^2a}{V^2}\right)(V-nb) = nRT \quad\Rightarrow\quad T = \dfrac{1}{nR}\left(P + \dfrac{n^2a}{V^2}\right)(V-nb)$, so $\dfrac{\partial T}{\partial P} = \dfrac{1}{nR}(1)(V-nb) = \dfrac{V-nb}{nR}$.

We can also write $P + \dfrac{n^2a}{V^2} = \dfrac{nRT}{V-nb} \quad\Rightarrow\quad P = \dfrac{nRT}{V-nb} - \dfrac{n^2a}{V^2} = nRT(V-nb)^{-1} - n^2aV^{-2}$, so

$\dfrac{\partial P}{\partial V} = -nRT(V-nb)^{-2}(1) + 2n^2aV^{-3} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V-nb)^2}$.

86. $W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15^\circ\mathrm{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately $1.3^\circ\mathrm{C}$ if the actual temperature decreases by $1^\circ\mathrm{C}$. $\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15^\circ\mathrm{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately $0.16^\circ\mathrm{C}$ if the wind speed increases by 1 km/h.

87. (a) $S = f(w,h) = 0.1091w^{0.425}h^{0.725} \quad\Rightarrow\quad \dfrac{\partial S}{\partial w} = 0.1091(0.425)w^{0.425-1}h^{0.725} = 0.0463675w^{-0.575}h^{0.725}$, so

$\dfrac{\partial S}{\partial w}(160,70) = 0.0463675(160)^{-0.575}(70)^{0.725} \approx 0.0545$. This means that for a person 70 inches tall who weighs 160 pounds, an increase in weight (while height remains constant) causes the surface area to increase at a rate of about 0.0545 square feet (about 7.85 square inches) per pound.

(b) $\dfrac{\partial S}{\partial h} = 0.1091(0.725)w^{0.425}h^{0.725-1} = 0.0790975w^{0.425}h^{-0.275}$, so

$\dfrac{\partial S}{\partial h}(160,70) = 0.0790975(160)^{0.425}(70)^{-0.275} \approx 0.213$. This means that for a person 70 inches tall who weighs

160 pounds, an increase in height (while weight remains unchanged at 160 pounds) causes the surface area to increase at a rate of about 0.213 square feet (about 30.7 square inches) per inch of height.

88. $R = C\dfrac{L}{r^4} \quad\Rightarrow\quad \dfrac{\partial R}{\partial L} = \dfrac{C}{r^4}$ and $\dfrac{\partial R}{\partial r} = CL\left(-4r^{-5}\right) = -4C\dfrac{L}{r^5}$.

$\partial R/\partial L$ is the rate at which the resistance of the flowing blood increases with respect to the length of the artery when the radius stays constant. $\partial R/\partial r$ is the rate of change of the resistance with respect to the radius of the artery when the length remains unchanged. Because $\partial R/\partial r$ is negative, the resistance decreases if the radius increases.

89. $P(v,x,m) = Av^3 + \dfrac{B(mg/x)^2}{v} = Av^3 + Bm^2g^2x^{-2}v^{-1}$.

$\partial P/\partial v = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant. $\partial P/\partial x = -2Bm^2g^2x^{-3}v^{-1} = -\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes with respect to the fraction of time spent in flapping mode when the mass and velocity are held constant. $\partial P/\partial m = 2Bmg^2x^{-2}v^{-1} = \dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power with respect to mass when the velocity and fraction of flapping time remain constant.

90. $T(x,t) = T_0 + T_1e^{-\lambda x}\sin(\omega t - \lambda x)$

(a) $\partial T/\partial x = T_1e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda)\right] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x) = -\lambda T_1 e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]$.

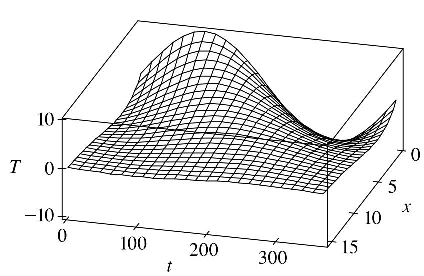
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1e^{-\lambda x}\left[\cos(\omega t - \lambda x)(\omega)\right] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c)
$$\begin{aligned} T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \\ &= -\lambda T_1\left(e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)\right] + e^{-\lambda x}(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\right) \\ &= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x) \end{aligned}$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2}T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, when $\lambda = 0.2$, the highest temperature at the surface is reached when $t \approx 91$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 149$, and at a depth of 10 feet, at $t \approx 207$.

91. $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m} \cdot \dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

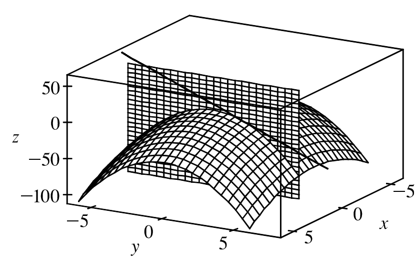
92. $E(m,v) = 2.65m^{0.66} + \dfrac{3.5m^{0.75}}{v} \quad \Rightarrow$

$E_m(m,v) = 2.65(0.66)m^{0.66-1} + \dfrac{3.5(0.75)m^{0.75-1}}{v} = 1.749m^{-0.34} + \dfrac{2.625m^{-0.25}}{v}$,

$E_v(m,v) = 3.5m^{0.75}\left(-v^{-2}\right) = -\dfrac{3.5m^{0.75}}{v^2}$. Then $E_m(400,8) = 1.749(400)^{-0.34} + \dfrac{2.625(400)^{-0.25}}{8} \approx 0.301$ which means that the average energy needed for a lizard to walk or run 1 km increases at a rate of about 0.301 kcal per gram of body mass increase from 400 g if the speed is 8 km/h. $E_v(400,8) = -\dfrac{3.5(400)^{0.75}}{8^2} \approx -4.89$, which means that the average energy needed by a lizard with body mass 400 g decreases at a rate of about 4.89 kcal per km/h when the speed increases from 8 km/h.

93. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \quad \Rightarrow \quad \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

94. $z = 6 - x - x^2 - 2y^2$. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



95. $f_x(x,y) = x + 4y \quad \Rightarrow \quad f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \quad \Rightarrow \quad f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

96. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a} = \dfrac{\partial(b^2 + c^2 - 2ab\cos A)}{\partial a}$ or $2a = -2bc\,(-\sin A)\,\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0 = 2b - 2c(\cos A) - 2bc\,(-\sin A)\,\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry, $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

97. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

98. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth-order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth-order partial derivatives with p partials with respect to x and $n - p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth-order partial derivative can range from 0 to n, a function of two variables has $n + 1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

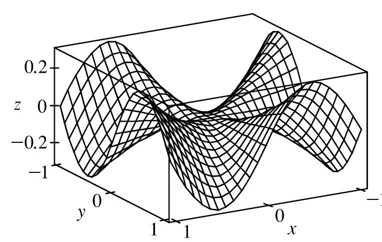
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth-order partial derivatives of a function of three variables.

99. $f(x, y) = x(x^2 + y^2)^{-3/2} e^{\sin(x^2 y)}$. Let $g(x) = f(x, 0) = x(x^2)^{-3/2} e^0 = x\,|x|^{-3}$. To find $f_x(1, 0)$, we are using the point $(1, 0)$, so near $(1, 0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using Equation 1, we have $f_x(1, 0) = g'(1) = -2$.

100. $f(x, y) = \sqrt[3]{x^3 + y^3}$. $f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(0 + h, 0) - f(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(h^3 + 0)^{1/3} - 0}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$.

Or: Let $g(x) = f(x, 0) = \sqrt[3]{x^3 + 0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0, 0) = g'(0) = 1$.

101. (a) $f(x, y) = \begin{cases} \dfrac{x^3 y - x y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$



(b) For $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - (x^3 y - x y^3)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

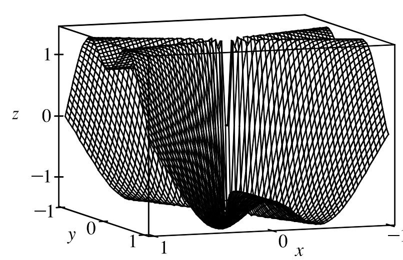
and, by symmetry, $f_y(x, y) = \dfrac{x^5 - 4x^3 y^2 - x y^4}{(x^2 + y^2)^2}$.

(c) $f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0, 0) = \lim\limits_{h \to 0} \dfrac{f(0, h) - f(0, 0)}{h} = 0$.

(d) By (3), $f_{xy}(0, 0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h \to 0} \dfrac{f_x(0, h) - f_x(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0, 0) = \frac{\partial f_y}{\partial x} = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x, y) \neq (0, 0)$, we use a CAS to compute



$$f_{xy}(x, y) = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}$$

Now as $(x, y) \to (0, 0)$ along the x-axis, $f_{xy}(x, y) \to 1$ while as $(x, y) \to (0, 0)$ along the y-axis, $f_{xy}(x, y) \to -1$. Thus f_{xy} isn't

continuous at $(0, 0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.

DISCOVERY PROJECT Deriving the Cobb-Douglas Production Function

1. For $K = K_0$, we have $\dfrac{dP}{dL} = \alpha \dfrac{P}{L} \;\Rightarrow\; \dfrac{dP}{P} = \alpha \dfrac{dL}{L} \;\Rightarrow\; \displaystyle\int \frac{dP}{P} = \int \alpha \frac{dL}{L} \;\Rightarrow\; \ln|P| = \alpha \ln|L| + A_1(K_0)$.

Then $e^{\ln|P|} = e^{\ln|L|^{\alpha} + A_1(K_0)} \;\Rightarrow\; P = e^{A_1(K_0)} L^{\alpha} \;\Rightarrow\; P(L, K_0) = C_1(K_0) L^{\alpha}$, where $e^{A_1(K_0)} = C_1(K_0)$.

2. For $L = L_0$, we have $\dfrac{dP}{dK} = \beta \dfrac{P}{K} \;\Rightarrow\; \dfrac{dP}{P} = \beta \dfrac{dK}{K} \;\Rightarrow\; \displaystyle\int \frac{dP}{P} = \int \beta \frac{dK}{K} \;\Rightarrow\; \ln|P| = \beta \ln|K| + A_2(L_0)$.

Then $e^{\ln|P|} = e^{\ln|K|^{\beta} + A_2(L_0)} \;\Rightarrow\; P = e^{A_2(L_0)} K^{\beta} \;\Rightarrow\; P(L_0, K) = C_2(L_0) K^{\beta}$, where $e^{A_2(L_0)} = C_2(L_0)$.

3. Suppose both labor and capital are increased by a factor of m. Then

$$\begin{aligned} P(mL, mk) &= b(mL)^{\alpha}(mK)^{1-\alpha} = bm^{\alpha}L^{\alpha}m^{1-\alpha}K^{1-\alpha} = m^{\alpha}m^{1-\alpha}bL^{\alpha}K^{1-\alpha} \\ &= m(bL^{\alpha}K^{1-\alpha}) = mP(L, K) \end{aligned}$$

Therefore, production is also increased by a factor of m.

4. For $P(L, K) = bL^{\alpha}K^{1-\alpha}$,

$$\begin{aligned} L\,\frac{\partial P}{\partial L} + K\,\frac{\partial P}{\partial K} &= L(b\alpha L^{\alpha-1}K^{1-\alpha}) + K(bL^{\alpha}(1-\alpha)K^{1-\alpha-1}) \\ &= b\alpha L^{\alpha}K^{1-\alpha} + b(1-\alpha)L^{\alpha}K^{1-\alpha} = bL^{\alpha}K^{1-\alpha}[\alpha + (1-\alpha)] \\ &= bL^{\alpha}K^{1-\alpha} = P(L, K) \end{aligned}$$

5. $P(L, K) = 1.01L^{0.75}K^{0.25}$. Marginal productivity of labor is given by $\partial P/\partial L = 1.01(0.75)L^{-0.25}K^{0.25}$. With $L = 194$ and $K = 407$, we have $\partial P/\partial L \approx 0.911656$. When capital is held constant at $K = 407$, as labor increases from $L = 194$, production will increase 0.911656 per unit of labor.

Marginal productivity of capital is given by $\partial P/\partial K = 1.01(0.25)L^{0.75}K^{-0.75}$. With $L = 194$ and $K = 407$, we have $\partial P/\partial K \approx 0.144849$. When labor is held constant at $L = 194$, as capital increases from $K = 407$, production will increase 0.144849 per unit of capital.

As the marginal productivity of labor is higher, it would be more beneficial to increase labor.

14.4 Tangent Planes and Linear Approximations

1. $z = f(x, y) = 16 - x^2 - y^2 \;\Rightarrow\; f_x(x, y) = -2x, f_y(x, y) = -2y$, so $f_x(2, 2) = -4$ and $f_y(2, 2) = -4$. By Equation 2, an equation of the tangent plane is $z - 8 = f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) \;\Rightarrow\; z - 8 = -4(x - 2) - 4(y - 2)$, or $z = -4x - 4y + 24$.

2. $z = f(x, y) = y^2 \sin x \quad \Rightarrow \quad f_x(x, y) = y^2 \cos x$, $f_y(x, y) = 2y \sin x$, so $f_x(\pi/2, -2) = 0$ and $f_y(\pi/2, -2) = -4$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(\pi/2, -2)(x - \pi/2) + f_y(\pi/2, -2)(y - (-2)) \quad \Rightarrow$ $z - 4 = -4(y + 2)$, or $z = -4y - 4$.

3. $z = f(x, y) = 2x^2 + y^2 - 5y \quad \Rightarrow \quad f_x(x, y) = 4x,\ f_y(x, y) = 2y - 5,$ so $f_x(1, 2) = 4$ and $f_y(1, 2) = -1$. By Equation 2, an equation of the tangent plane is $z - (-4) = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \quad \Rightarrow$ $z + 4 = 4(x - 1) + (-1)(y - 2)$, or $z = 4x - y - 6$.

4. $z = f(x, y) = (x + 2)^2 - 2(y - 1)^2 - 5 \quad \Rightarrow \quad f_x(x, y) = 2(x + 2),\ \ f_y(x, y) = -4(y - 1),$ so $f_x(2, 3) = 8$ and $f_y(2, 3) = -8$. By Equation 2, an equation of the tangent plane is $z - 3 = f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3) \quad \Rightarrow$ $z - 3 = 8(x - 2) + (-8)(y - 3)$, or $z = 8x - 8y + 11$.

5. $z = f(x, y) = e^{x-y} \quad \Rightarrow \quad f_x(x, y) = e^{x-y}(1) = e^{x-y},\ \ f_y(x, y) = e^{x-y}(-1) = -e^{x-y},$ so $f_x(2, 2) = 1$ and $f_y(2, 2) = -1$. Thus, an equation of the tangent plane is $z - 1 = f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) \quad \Rightarrow$ $z - 1 = 1(x - 2) + (-1)(y - 2)$, or $z = x - y + 1$.

6. $z = f(x, y) = y^2 e^x \quad \Rightarrow \quad f_x(x, y) = y^2 e^x$, $f_y(x, y) = 2ye^x$, so $f_x(0, 3) = 9$ and $f_y(0, 3) = 6$. Thus, an equation of the tangent plane is $z - 9 = f_x(0, 3)(x - 0) + f_y(0, 3)(y - 3) \quad \Rightarrow \quad z - 9 = 9x + 6(y - 3)$, or $z = 9x + 6y - 9$.

7. $z = f(x, y) = \dfrac{2\sqrt{y}}{x} \quad \Rightarrow \quad f_x(x, y) = -\dfrac{2\sqrt{y}}{x^2}, f_y(x, y) = \dfrac{1}{x\sqrt{y}}$, so $f_x(-1, 1) = -2$ and $f_y(-1, 1) = -1$. Thus, an equation of the tangent plane is $z - (-2) = f_x(-1, 1)(x - (-1)) + f_y(-1, 1)(y - 1) \quad \Rightarrow$ $z + 2 = -2(x + 1) - 1(y - 1)$, or $z = -2x - y - 3$.

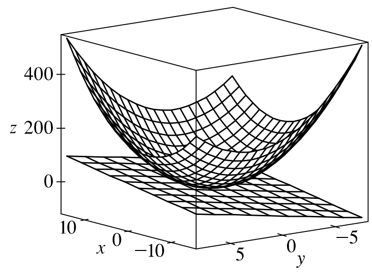
8. $z = f(x, y) = x/y^2 = xy^{-2} \quad \Rightarrow \quad f_x(x, y) = 1/y^2,\ \ f_y(x, y) = -2xy^{-3} = -2x/y^3,$ so $f_x(-4, 2) = \frac{1}{4}$ and $f_y(-4, 2) = 1$. Thus, an equation of the tangent plane is $z - (-1) = f_x(-4, 2)\,[x - (-4)] + f_y(-4, 2)(y - 2) \quad \Rightarrow$ $z + 1 = \frac{1}{4}(x + 4) + 1(y - 2)$, or $z = \frac{1}{4}x + y - 2$.

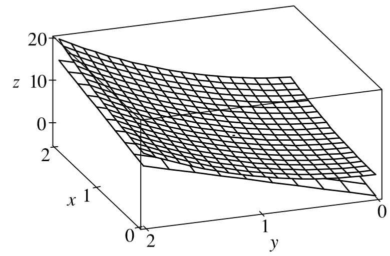
9. $z = f(x, y) = x \sin(x + y) \quad \Rightarrow \quad f_x(x, y) = x \cdot \cos(x + y) \cdot 1 + \sin(x + y) \cdot 1 = x \cos(x + y) + \sin(x + y)$ and $f_y(x, y) = x \cos(x + y) \cdot 1$, so $f_x(-1, 1) = (-1) \cos 0 + \sin 0 = -1,\ \ f_y(-1, 1) = (-1) \cos 0 = -1$. Thus, an equation of the tangent plane is $z - 0 = f_x(-1, 1)(x - (-1)) + f_y(-1, 1)(y - 1) \quad \Rightarrow \quad z = (-1)(x + 1) + (-1)(y - 1)$, or $x + y + z = 0$.

10. $z = f(x, y) = \ln(x - 2y) \quad \Rightarrow \quad f_x(x, y) = 1/(x - 2y),\ f_y(x, y) = -2/(x - 2y),$ so $f_x(3, 1) = 1$ and $f_y(3, 1) = -2$. Thus, an equation of the tangent plane is $z - 0 = f_x(3, 1)(x - 3) + f_y(3, 1)(y - 1) \quad \Rightarrow \quad z = 1(x - 3) + (-2)(y - 1)$, or $z = x - 2y - 1$.

11. $z = f(x, y) = x^2 + xy + 3y^2$, so $f_x(x, y) = 2x + y \quad \Rightarrow \quad f_x(1, 1) = 3,\ \ f_y(x, y) = x + 6y \quad \Rightarrow \quad f_y(1, 1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x - 1) + 7(y - 1)$, or $z = 3x + 7y - 5$. After zooming in, the surface and the

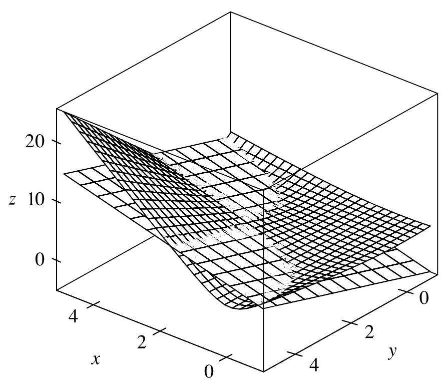
tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

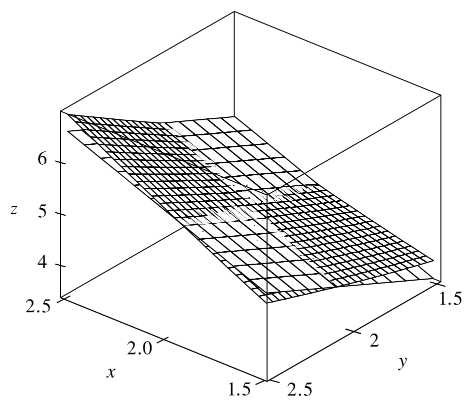




12. $z = f(x,y) = \sqrt{9 + x^2y^2} \;\Rightarrow\; f_x(x,y) = \frac{1}{2}\left(9 + x^2y^2\right)^{-1/2}(2xy^2) = xy^2/\sqrt{9 + x^2y^2}$,

$f_y(x,y) = \frac{1}{2}\left(9 + x^2y^2\right)^{-1/2}(2x^2y) = x^2y/\sqrt{9 + x^2y^2}$, so $f_x(2,2) = \frac{8}{5}$ and $f_y(2,2) = \frac{8}{5}$. Thus an equation of the tangent plane is $z - 5 = f_x(2,2)(x-2) + f_y(2,2)(y-2) \;\Rightarrow\; z - 5 = \frac{8}{5}(x-2) + \frac{8}{5}(y-2)$ or $z = \frac{8}{5}x + \frac{8}{5}y - \frac{7}{5}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





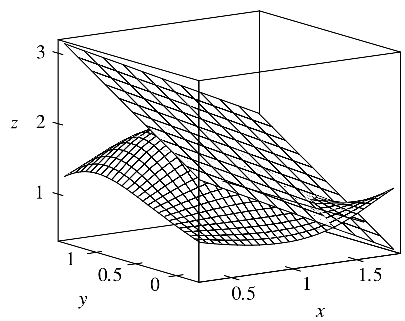
13. $f(x,y) = \dfrac{1 + \cos^2(x-y)}{1 + \cos^2(x+y)}$. A CAS gives

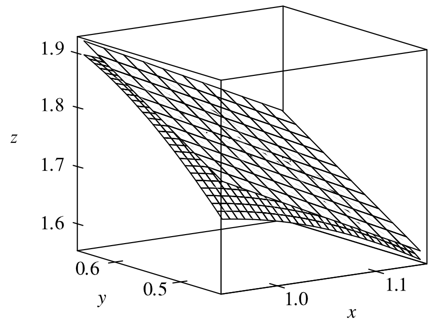
$$f_x(x,y) = -\frac{2\cos(x-y)\sin(x-y)}{1+\cos^2(x+y)} + \frac{2\left[1+\cos^2(x-y)\right]\cos(x+y)\sin(x+y)}{\left[1+\cos^2(x+y)\right]^2} \quad \text{and}$$

$f_y(x,y) = \dfrac{2\cos(x-y)\sin(x-y)}{1+\cos^2(x+y)} + \dfrac{2\left[1+\cos^2(x-y)\right]\cos(x+y)\sin(x+y)}{\left[1+\cos^2(x+y)\right]^2}$. We use the CAS to evaluate these at $(\pi/3, \pi/6)$, giving $f_x(\pi/3, \pi/6) = -\sqrt{3}/2$ and $f_y(\pi/3, \pi/6) = \sqrt{3}/2$. Substituting into Equation 2, an equation of the tangent plane is $z = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\left(y - \frac{\pi}{6}\right) + \frac{7}{4}$. The surface and tangent plane are shown in the first graph.

[continued]

After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



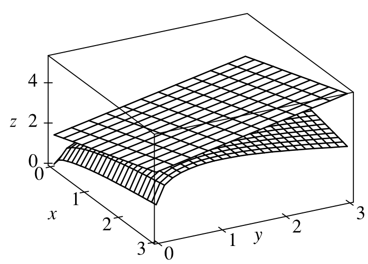


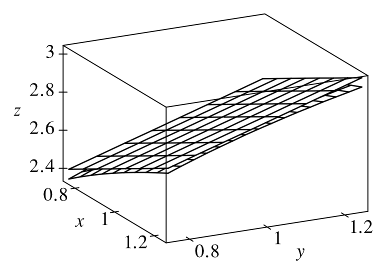
14. $f(x,y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)$. A CAS gives

$f_x(x,y) = -\frac{1}{10}ye^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{x}} + \frac{y}{2\sqrt{xy}}\right)$ and

$f_y(x,y) = -\frac{1}{10}xe^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{y}} + \frac{x}{2\sqrt{xy}}\right)$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane:

$z - 3e^{-0.1} = 0.7e^{-0.1}(x-1) + 0.7e^{-0.1}(y-1) \quad\Rightarrow\quad z = 0.7e^{-0.1}x + 0.7e^{-0.1}y + 1.6e^{-0.1}$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





15. $f(x,y) = x^3y^2$. The partial derivatives are $f_x(x,y) = 3x^2y^2$ and $f_y(x,y) = 2x^3y$, so $f_x(-2,1) = 12$ and $f_y(-2,1) = -16$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(-2,1)$. By Equation 3, the linearization of f at $(-2,1)$ is given by

$L(x,y) = f(-2,1) + f_x(-2,1)(x-(-2)) + f_y(-2,1)(y-1) = -8 + 12(x+2) - 16(y-1) = 12x - 16y + 32.$

16. $f(x,y) = y\tan x$. The partial derivatives are $f_x(x,y) = y\sec^2 x$ and $f_y = \tan x$, so $f_x\left(\frac{\pi}{4},2\right) = 4$ and $f_y\left(\frac{\pi}{4},2\right) = 1$. Both f_x and f_y are continuous for $x \neq \frac{\pi}{2} + n\pi$, so by Theorem 8, f is differentiable at $\left(\frac{\pi}{4},2\right)$. By Equation 3, the linearization of f at $\left(\frac{\pi}{4},2\right)$ is given by $L(x,y) = f\left(\frac{\pi}{4},2\right) + f_x\left(\frac{\pi}{4},2\right)\left(x-\frac{\pi}{4}\right) + f_y\left(\frac{\pi}{4},2\right)(y-2) = 2 + 4\left(x-\frac{\pi}{4}\right) + 1(y-2) = 4x + y - \pi.$

17. $f(x,y) = 1 + x\ln(xy-5)$. The partial derivatives are $f_x(x,y) = x \cdot \dfrac{1}{xy-5}(y) + \ln(xy-5) \cdot 1 = \dfrac{xy}{xy-5} + \ln(xy-5)$

and $f_y(x,y) = x \cdot \dfrac{1}{xy-5}(x) = \dfrac{x^2}{xy-5}$, so $f_x(2,3) = 6$ and $f_y(2,3) = 4$. Both f_x and f_y are continuous functions for $xy > 5$, so by Theorem 8, f is differentiable at $(2,3)$. By Equation 3, the linearization of f at $(2,3)$ is given by

$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) = 1 + 6(x-2) + 4(y-3) = 6x + 4y - 23.$

18. $f(x,y) = \sqrt{xy} = (xy)^{1/2}$. The partial derivatives are $f_x(x,y) = \frac{1}{2}(xy)^{-1/2}(y) = y/\left(2\sqrt{xy}\right)$ and

$f_y(x,y) = \frac{1}{2}(xy)^{-1/2}(x) = x/\left(2\sqrt{xy}\right)$, so $f_x(1,4) = 4/\left(2\sqrt{4}\right) = 1$ and $f_y(1,4) = 1/\left(2\sqrt{4}\right) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $xy > 0$, so f is differentiable at $(1,4)$ by Theorem 8. By Equation 3, the linearization of f at $(1,4)$ is given by $L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4) = 2 + 1(x-1) + \frac{1}{4}(y-4) = x + \frac{1}{4}y.$

19. $f(x,y) = x^2e^y$. The partial derivatives are $f_x(x,y) = 2xe^y$ and $f_y(x,y) = x^2e^y$, so $f_x(1,0) = 2$ and $f_y(1,0) = 1$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(1,0)$. By Equation 3, the linearization of f at $(1,0)$ is given by $L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + 2(x-1) + 1(y-0) = 2x + y - 1.$

20. $f(x,y) = \dfrac{1+y}{1+x} = (1+y)(1+x)^{-1}$. The partial derivatives are $f_x(x,y) = (1+y)(-1)(1+x)^{-2} = -\dfrac{1+y}{(1+x)^2}$ and

$f_y(x,y) = (1)(1+x)^{-1} = \dfrac{1}{1+x}$, so $f_x(1,3) = -1$ and $f_y(1,3) = \frac{1}{2}$. Both f_x and f_y are continuous functions for $x \neq -1$, so by Theorem 8, f is differentiable at $(1,3)$. By Equation 3, the linearization of f at $(1,3)$ is given by

$L(x,y) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) = 2 + (-1)(x-1) + \frac{1}{2}(y-3) = -x + \frac{1}{2}y + \frac{3}{2}.$

21. $f(x,y) = 4\arctan(xy)$. The partial derivatives are $f_x(x,y) = 4 \cdot \dfrac{1}{1+(xy)^2}(y) = \dfrac{4y}{1+x^2y^2}$, and

$f_y(x,y) = \dfrac{4x}{1+x^2y^2}$, so $f_x(1,1) = 2$ and $f_y(1,1) = 2$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(1,1)$. By Equation 3, the linearization of f at $(1,1)$ is given by

$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 4(\pi/4) + 2(x-1) + 2(y-1) = 2x + 2y + \pi - 4.$

22. $f(x,y) = y + \sin(x/y)$. The partial derivatives are $f_x(x,y) = (1/y)\cos(x/y)$ and $f_y(x,y) = 1 + (-x/y^2)\cos(x/y)$, so $f_x(0,3) = \frac{1}{3}$ and $f_y(0,3) = 1$. Both f_x and f_y are continuous functions for $y \neq 0$, so by Theorem 8, f is differentiable at $(0,3)$. By Equation 3, the linearization of f at $(0,3)$ is given by

$L(x,y) = f(0,3) + f_x(0,3)(x-0) + f_y(0,3)(y-3) = 3 + \frac{1}{3}(x-0) + 1(y-3) = \frac{1}{3}x + y.$

23. Let $f(x,y) = e^x\cos(xy)$. Then $f_x(x,y) = e^x[-\sin(xy)](y) + e^x\cos(xy) = e^x[\cos(xy) - y\sin(xy)]$ and $f_y(x,y) = e^x[-\sin(xy)](x) = -xe^x\sin(xy)$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0) = e^0(\cos 0 - 0) = 1$, $f_y(0,0) = 0$ and the linear approximation of f at $(0,0)$ is

$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + 1x + 0y = x + 1.$

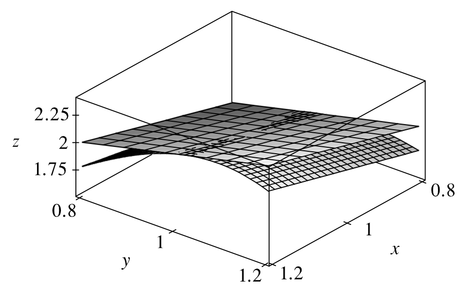
24. Let $f(x,y) = \dfrac{y-1}{x+1}$. Then $f_x(x,y) = (y-1)(-1)(x+1)^{-2} = \dfrac{1-y}{(x+1)^2}$ and $f_y(x,y) = \dfrac{1}{x+1}$. Both f_x and f_y are continuous functions for $x \neq -1$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0) = 1$, $f_y(0,0) = 1$ and the linear approximation of f at $(0,0)$ is $f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = -1 + 1x + 1y = x + y - 1$.

25. We can estimate $f(2.2, 4.9)$ using a linear approximation of f at $(2,5)$, given by $f(x,y) \approx f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5) = 6 + 1(x-2) + (-1)(y-5) = x - y + 9$. Thus $f(2.2, 4.9) \approx 2.2 - 4.9 + 9 = 6.3$.

26. $f(x,y) = 1 - xy\cos\pi y \;\Rightarrow\; f_x(x,y) = -y\cos\pi y$ and $f_y(x,y) = -x[y(-\pi\sin\pi y) + (\cos\pi y)(1)] = \pi xy\sin\pi y - x\cos\pi y$, so $f_x(1,1) = 1$, $f_y(1,1) = 1$. Then the linear approximation of f at $(1,1)$ is given by

$$\begin{aligned} f(x,y) &\approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= 2 + (1)(x-1) + (1)(y-1) = x + y \end{aligned}$$

Thus $f(1.02, 0.97) \approx 1.02 + 0.97 = 1.99$. We graph f and its tangent plane near the point $(1,1,2)$ below. Notice near $y = 1$ the surfaces are almost identical.



27. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \;\Rightarrow\; f_x(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6) = \frac{3}{7}$, $f_y(3,2,6) = \frac{2}{7}$, $f_z(3,2,6) = \frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

28. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 14.3.2. Since $f_v(40,20) = \lim\limits_{h\to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table:

$$f_v(40,20) \approx \frac{f(50,20) - f(40,20)}{10} = \frac{40-28}{10} = 1.2, \qquad f_v(40,20) \approx \frac{f(30,20) - f(40,20)}{-10} = \frac{17-28}{-10} = 1.1$$

[continued]

Averaging these values gives $f_v(40, 20) \approx 1.15$. Similarly, $f_t(40, 20) = \lim\limits_{h \to 0} \dfrac{f(40, 20 + h) - f(40, 20)}{h}$, so we use $h = 10$ and $h = -5$:

$$f_t(40, 20) \approx \frac{f(40, 30) - f(40, 20)}{10} = \frac{31 - 28}{10} = 0.3, \qquad f_t(40, 20) \approx \frac{f(40, 15) - f(40, 20)}{-5} = \frac{25 - 28}{-5} = 0.6$$

Averaging these values gives $f_t(40, 15) \approx 0.45$. The linear approximation, then, is

$$f(v, t) \approx f(40, 20) + f_v(40, 20)(v - 40) + f_t(40, 20)(t - 20) \approx 28 + 1.15(v - 40) + 0.45(t - 20)$$

When $v = 43$ and $t = 24$, we estimate $f(43, 24) \approx 28 + 1.15(43 - 40) + 0.45(24 - 20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

29. From the table, $f(94, 80) = 127$. To estimate $f_T(94, 80)$ and $f_H(94, 80)$ we follow the procedure used in Section 14.3. Since $f_T(94, 80) = \lim\limits_{h \to 0} \dfrac{f(94 + h, 80) - f(94, 80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table:

$$f_T(94, 80) \approx \frac{f(96, 80) - f(94, 80)}{2} = \frac{135 - 127}{2} = 4, \qquad f_T(94, 80) \approx \frac{f(92, 80) - f(94, 80)}{-2} = \frac{119 - 127}{-2} = 4$$

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim\limits_{h \to 0} \dfrac{f(94, 80 + h) - f(94, 80)}{h}$, so we use $h = \pm 5$:

$$f_H(94, 80) \approx \frac{f(94, 85) - f(94, 80)}{5} = \frac{132 - 127}{5} = 1, \qquad f_H(94, 80) \approx \frac{f(94, 75) - f(94, 80)}{-5} = \frac{122 - 127}{-5} = 1$$

Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T, H) &\approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \qquad [\text{or } 4T + H - 329] \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately $129°\text{F}$.

30. From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 14.3. Since $f_T(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$$f_T(-15, 50) \approx \frac{f(-10, 50) - f(-15, 50)}{5} = \frac{-22 - (-29)}{5} = 1.4$$

$$f_T(-15, 50) \approx \frac{f(-20, 50) - f(-15, 50)}{-5} = \frac{-35 - (-29)}{-5} = 1.2$$

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15, 50 + h) - f(-15, 50)}{h}$,

[continued]

so we use $h = \pm 10$:

$$f_v(-15, 50) \approx \frac{f(-15, 60) - f(-15, 50)}{10} = \frac{-30 - (-29)}{10} = -0.1$$

$$f_v(-15, 50) \approx \frac{f(-15, 40) - f(-15, 50)}{-10} = \frac{-27 - (-29)}{-10} = -0.2$$

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is $f(T, v) \approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \approx -29 + (1.3)(T + 15) - (0.15)(v - 50)$. Thus when $T = -17°\text{C}$ and $v = 55$ km/h, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately $-32.35°\text{C}$.

31. $m = p^5 q^3 \quad \Rightarrow \quad dm = \dfrac{\partial m}{\partial p}\,dp + \dfrac{\partial m}{\partial q}\,dq = 5p^4 q^3\,dp + 3p^5 q^2\,dq$

32. $z = x \ln(y^2 + 1) \quad \Rightarrow \quad dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = \ln(y^2 + 1)\,dx + \dfrac{x}{y^2 + 1}(2y)\,dy = \ln(y^2 + 1)\,dx + \dfrac{2xy}{y^2 + 1}\,dy$

33. $z = e^{-2x} \cos 2\pi t \quad \Rightarrow$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial t}dt = e^{-2x}(-2)\cos 2\pi t\,dx + e^{-2x}(-\sin 2\pi t)(2\pi)\,dt = -2e^{-2x}\cos 2\pi t\,dx - 2\pi e^{-2x}\sin 2\pi t\,dt$$

34. $u = \sqrt{x^2 + 3y^2} = (x^2 + 3y^2)^{1/2} \quad \Rightarrow$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \tfrac{1}{2}(x^2 + 3y^2)^{-1/2}(2x)\,dx + \tfrac{1}{2}(x^2 + 3y^2)^{-1/2}(6y)\,dy = \frac{x}{\sqrt{x^2 + 3y^2}}\,dx + \frac{3y}{\sqrt{x^2 + 3y^2}}\,dy$$

35. $H = x^2 y^4 + y^3 z^5 \quad \Rightarrow \quad dH = \dfrac{\partial H}{\partial x}\,dx + \dfrac{\partial H}{\partial y}\,dy + \dfrac{\partial H}{\partial z}\,dz = 2xy^4\,dx + (4x^2y^3 + 3y^2z^5)\,dy + 5y^3z^4\,dz$

36. $w = xze^{-y^2 - z^2} \quad \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz = ze^{-y^2 - z^2}\,dx + xze^{-y^2 - z^2}(-2y)\,dy + x[z \cdot e^{-y^2 - z^2}(-2z) + e^{-y^2 - z^2} \cdot 1]\,dz \\ &= ze^{-y^2 - z^2}\,dx - 2xyze^{-y^2 - z^2}\,dy + x(1 - 2z^2)e^{-y^2 - z^2}\,dz \end{aligned}$$

37. $R = \alpha\beta^2 \cos\gamma \quad \Rightarrow \quad dR = \dfrac{\partial R}{\partial \alpha}\,d\alpha + \dfrac{\partial R}{\partial \beta}\,d\beta + \dfrac{\partial R}{\partial \gamma}\,d\gamma = \beta^2 \cos\gamma\,d\alpha + 2\alpha\beta\,\cos\gamma\,d\beta - \alpha\beta^2 \sin\gamma\,d\gamma$

38. $T = \dfrac{v}{1 + uvw} \quad \Rightarrow$

$$\begin{aligned} dT &= \frac{\partial T}{\partial u}\,du + \frac{\partial T}{\partial v}\,dv + \frac{\partial T}{\partial w}\,dw \\ &= v(-1)(1 + uvw)^{-2}(vw)\,du + \frac{1(1 + uvw) - v(uw)}{(1 + uvw)^2}\,dv + v(-1)(1 + uvw)^{-2}(uv)\,dw \\ &= -\frac{v^2 w}{(1 + uvw)^2}\,du + \frac{1}{(1 + uvw)^2}\,dv - \frac{uv^2}{(1 + uvw)^2}\,dw \end{aligned}$$

39. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$,

$dz = z_x(1,2)\,dx + z_y(1,2)\,dy = (10)(0.05) + (4)(0.1) = 0.9$ while

$\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

40. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$,

$dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189$.

41. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then

the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\ \text{cm}^2$.

42. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi rh\,dr + \pi r^2\,dh$ is an estimate of the amount of metal. With

$dr = 0.05$ and $dh = 0.2$ (0.1 on top, 0.1 on bottom), we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8\ \text{cm}^3$.

43. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi rh\,dr + \pi r^2\,dh$, so put

$dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\ \text{cm}^3$.

Thus the amount of tin is about $16\ \text{cm}^3$.

44. (a) Let A be the area. Then $A = \frac{1}{2}bh$ and $\Delta A \approx dA = \dfrac{\partial A}{\partial b}\,db + \dfrac{\partial A}{\partial h}\,dh = \dfrac{1}{2}h\,db + \dfrac{1}{2}b\,dh$. We have $|\Delta h| = |\Delta b| \le \varepsilon$.

So we take $dh = db = \varepsilon$ with $b = 28$ inches, $h = 16$ inches. The maximum error in the area is

$dA = \frac{1}{2}(16)\varepsilon + \frac{1}{2}(28)\varepsilon = 22\varepsilon$ square inches.

(b) With $\varepsilon = \frac{1}{4}$, we have the estimated maximum error in the area of the triangle as $22\left(\frac{1}{4}\right) = \frac{11}{2}$ square inches.

45. (a) Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = \dfrac{\partial V}{\partial r}\,dr + \dfrac{\partial V}{\partial h}\,dh = 2\pi rh\,dr + \pi r^2\,dh$. We have

$|\Delta r| = |\Delta h| = \varepsilon$. So we take $dr = dh = \varepsilon$ with $r = 2.5$ feet and $h = 12$ feet. The maximum error in the volume is

$dV = 2\pi(2.5)(12)\varepsilon + \pi(2.5^2)\varepsilon = 66.25\pi\varepsilon\ \text{ft}^3$.

(b) We need $66.25\pi\varepsilon \le 1 \quad\Rightarrow\quad \varepsilon \lesssim 0.0048$ feet or 0.058 inches.

46. $W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$, so the differential of W is

$$\begin{aligned} dW &= \frac{\partial W}{\partial T}\,dT + \frac{\partial W}{\partial v}\,dv = \left(0.6215 + 0.3965v^{0.16}\right)dT + \left[-11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}\right]dv \\ &= \left(0.6215 + 0.3965v^{0.16}\right)dT + (-1.8192 + 0.06344T)v^{-0.84}\,dv \end{aligned}$$

Here we have $|\Delta T| \le 1$, $|\Delta v| \le 2$, so we take $dT = 1$, $dv = 2$ with $T = -11$, $v = 26$. The maximum error in the calculated

value of W is about $dW = (0.6215 + 0.3965(26)^{0.16})(1) + (-1.8192 + 0.06344(-11))(26)^{-0.84}(2) \approx 0.96$.

47. $T = \dfrac{mgR}{2r^2 + R^2}$, so the differential of T is

$$dT = \frac{\partial T}{\partial R}\,dR + \frac{\partial T}{\partial r}\,dr = \frac{(2r^2 + R^2)(mg) - mgR(2R)}{(2r^2 + R^2)^2}\,dR + \frac{(2r^2 + R^2)(0) - mgR(4r)}{(2r^2 + R^2)^2}\,dr$$
$$= \frac{mg(2r^2 - R^2)}{(2r^2 + R^2)^2}\,dR - \frac{4mgRr}{(2r^2 + R^2)^2}\,dr$$

Here we have $\Delta R = 0.1$ and $\Delta r = 0.1$, so we take $dR = 0.1$, $dr = 0.1$ with $R = 3$, $r = 0.7$. Then the change in the tension T is approximately

$$dT = \frac{mg[2(0.7)^2 - (3)^2]}{[2(0.7)^2 + (3)^2]^2}\,(0.1) - \frac{4mg(3)(0.7)}{[2(0.7)^2 + (3)^2]^2}\,(0.1)$$
$$= -\frac{0.802mg}{(9.98)^2} - \frac{0.84mg}{(9.98)^2} = -\frac{1.642}{99.6004}mg \approx -0.0165mg$$

Because the change is negative, tension decreases.

48. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\,\dfrac{T}{V}$, so

$dP = \left(\dfrac{8.31}{V}\right) dT - \dfrac{8.31 \cdot T}{V^2}\,dV = 8.31\left[-\dfrac{5}{12} - \dfrac{310}{144} \cdot \dfrac{3}{10}\right] \approx -8.83$. Thus the pressure will drop by about 8.83 kPa.

49. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$\dfrac{\partial}{\partial R_1}\left(\dfrac{1}{R}\right) = \dfrac{\partial\,[(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\,\Omega$. Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$\Delta R \approx dR = \dfrac{\partial R}{\partial R_1}\,\Delta R_1 + \dfrac{\partial R}{\partial R_2}\,\Delta R_2 + \dfrac{\partial R}{\partial R_3}\,\Delta R_3 = (0.005)R^2\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right) = (0.005)R = \frac{1}{17} \approx 0.059\,\Omega$.

50. $S = 0.1091w^{0.425}h^{0.725}$. The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425w^{0.425-1})h^{0.725}\,dw + 0.1091w^{0.425}(0.725h^{0.725-1})\,dh}{0.1091w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus the maximum percentage error is approximately 2.3%.

51. (a) $B(m,h) = m/h^2 \quad\Rightarrow\quad B_m(m,h) = 1/h^2$ and $B_h(m,h) = -2m/h^3$. Since $h > 0$, both B_m and B_h are continuous functions, so B is differentiable at $(23, 1.10)$. We have $B(23, 1.10) = 23/(1.10)^2 \approx 19.01$, $B_m(23, 1.10) = 1/(1.10)^2 \approx 0.8264$, and $B_h(23, 1.10) = -2(23)/(1.10)^3 \approx -34.56$, so the linear

approximation of B at $(23, 1.10)$ is

$B(m, h) \approx B(23, 1.10) + B_m(23, 1.10)(m - 23) + B_h(23, 1.10)(h - 1.10) \approx 19.01 + 0.8264(m - 23) - 34.56(h - 1.10)$

or $B(m, h) \approx 0.8264m - 34.56h + 38.02$.

(b) From part (a), for values near $m = 23$ and $h = 1.10$, $B(m, h) \approx 0.8264m - 34.56h + 38.02$. If m increases by 1 kg to 24 kg and h increases by 0.03 m to 1.13 m, we estimate the BMI to be $B(24, 1.13) \approx 0.8264(24) - 34.56(1.13) + 38.02 \approx 18.801$. This is very close to the actual computed BMI: $B(24, 1.13) = 24/(1.13)^2 \approx 18.796$.

52. $\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle \Rightarrow \mathbf{r}_1'(t) = \langle 3, -2t, -4 + 2t \rangle$, $\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle \Rightarrow$ $\mathbf{r}_2'(u) = \langle 2u, 6u^2, 2 \rangle$. Both curves pass through $P(2, 1, 3)$ since $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 2, 1, 3 \rangle$, so the tangent vectors $\mathbf{r}_1'(0) = \langle 3, 0, -4 \rangle$ and $\mathbf{r}_2'(1) = \langle 2, 6, 2 \rangle$ are both parallel to the tangent plane to S at P. A normal vector for the tangent plane is $\mathbf{r}_1'(0) \times \mathbf{r}_2'(1) = \langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \langle 24, -14, 18 \rangle$, so an equation of the tangent plane is $24(x - 2) - 14(y - 1) + 18(z - 3) = 0$ or $12x - 7y + 9z = 44$.

53. To show that f is continuous at (a, b) we need to show that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ or equivalently $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Since f is differentiable at (a, b), $f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0, 0)$ gives $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is continuous at (a, b).

54. (a) $\lim\limits_{h\to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim\limits_{h\to 0} \dfrac{0 - 0}{h} = 0$ and $\lim\limits_{h\to 0} \dfrac{f(0, h) - f(0, 0)}{h} = \lim\limits_{h\to 0} \dfrac{0 - 0}{h} = 0$. Thus $f_x(0, 0) = f_y(0, 0) = 0$.

To show that f isn't differentiable at $(0, 0)$ we need only show that f is not continuous at $(0, 0)$ and apply the contrapositive of Exercise 53. As $(x, y) \to (0, 0)$ along the x-axis $f(x, y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis. But as $(x, y) \to (0, 0)$ along the line $y = x$, $f(x, x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0, 0)$ along this line. Thus $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ doesn't exist, so f is discontinuous at $(0, 0)$ and thus not differentiable there.

(b) For $(x, y) \neq (0, 0)$, $f_x(x, y) = \dfrac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \dfrac{y(y^2 - x^2)}{(x^2 + y^2)^2}$. If we approach $(0, 0)$ along the y-axis, then $f_x(x, y) = f_x(0, y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x, y) \to \pm\infty$ as $(x, y) \to (0, 0)$. Thus $\lim\limits_{(x,y)\to(0,0)} f_x(x, y)$ does not exist and $f_x(x, y)$ is not continuous at $(0, 0)$. Similarly, $f_y(x, y) = \dfrac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$ for $(x, y) \neq (0, 0)$, and if we approach $(0, 0)$ along the x-axis, then $f_y(x, y) = f_x(x, 0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus $\lim\limits_{(x,y)\to(0,0)} f_y(x, y)$ does not exist and $f_y(x, y)$ is not continuous at $(0, 0)$.

APPLIED PROJECT The Speedo LZR Racer

1. $v(P,C) = \left(\dfrac{2P}{kC}\right)^{1/3} \quad \Rightarrow$

$$f(x,y) = \frac{v(P+xP, C+yC) - v(P,C)}{v(P,C)} = \frac{v(P+xP, C+yC)}{v(P,C)} - \frac{v(P,C)}{v(P,C)} = \frac{\left(\dfrac{2(P+xP)}{k(C+yC)}\right)^{1/3}}{\left(\dfrac{2P}{kC}\right)^{1/3}} - 1$$

$$= \left(\frac{2P(1+x)}{kC(1+y)} \cdot \frac{kC}{2P}\right)^{1/3} - 1 = \left(\frac{1+x}{1+y}\right)^{1/3} - 1$$

Both power and drag cannot be reduced by more than 100%, but both could be increased by any percentage, so $x \geq -1$ and $y \geq -1$. But f is undefined when $y = -1$, so the domain is $\{(x,y) \mid x \geq -1,\ y > -1\}$.

2. If x and y are small, then we can say they are near zero and we can use a linear approximation to f at $(0,0)$.

We have $f(x,y) = (1+x)^{1/3}(1+y)^{-1/3} - 1$ so the partial derivatives are

$f_x(x,y) = \frac{1}{3}(1+x)^{-2/3}(1+y)^{-1/3} = \dfrac{1}{3(1+x)^{2/3}(1+y)^{1/3}}$ and

$f_y(x,y) = -\frac{1}{3}(1+x)^{1/3}(1+y)^{-4/3} = -\dfrac{(1+x)^{1/3}}{3(1+y)^{4/3}}$. Note that f_x and f_y are continuous functions for $x > -1$, $y > -1$

so f is differentiable at $(0,0)$. Then $f_x(0,0) = \frac{1}{3}$ and $f_y(0,0) = -\frac{1}{3}$, and the linear approximation is

$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0 + \frac{1}{3}(x-0) - \frac{1}{3}(y-0) = \frac{1}{3}x - \frac{1}{3}y$. According to the linear approximation, a small fractional increase in power results in 1/3 that fractional increase in speed, and a small decrease in drag has the same effect.

3. $f_{xx}(x,y) = \dfrac{1}{3(1+y)^{1/3}} \cdot \left(-\frac{2}{3}\right)(1+x)^{-5/3} = -\dfrac{2}{9(1+x)^{5/3}(1+y)^{1/3}}$,

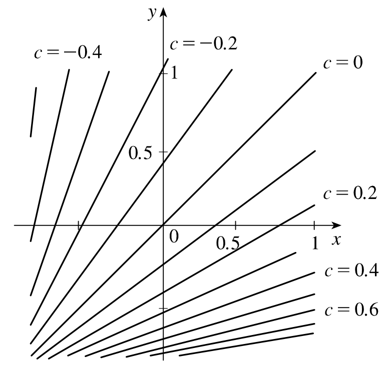
$f_{yy}(x,y) = -\frac{1}{3}(1+x)^{1/3} \cdot \left(-\frac{4}{3}\right)(1+y)^{-7/3} = \dfrac{4(1+x)^{1/3}}{9(1+y)^{7/3}}$. Because $f_x(x,y)$ is positive in the domain of f, an increase in power results in an increase in speed, but $f_{xx}(x,y)$ is negative, so as the fractional power increases, the fractional speed increases at a declining rate. (We can say that in the positive x-direction, f is increasing and concave downward.) Thus the linear approximation gives an overestimate for an increase in power. Since $f_y(x,y)$ is negative, a *decrease* in drag *increases* speed. But $f_{yy}(x,y)$ is positive, so f_y increases as y increases and f_y decreases (f_y becomes larger and larger negative) as y decreases. (In the positive y-direction, f is decreasing and concave upward.) Thus as the fractional drag decreases, the fractional speed increases at an accelerating pace and the linear approximation gives an underestimate of the increase in power. This explains why a decrease in drag is more effective than an increase in power: Reducing drag improves speed at an increasing rate while adding power improves speed at a declining rate.

4. The level curves of $f(x,y) = \left(\dfrac{1+x}{1+y}\right)^{1/3} - 1$ are

$$\left(\frac{1+x}{1+y}\right)^{1/3} - 1 = c \quad\Rightarrow\quad \frac{1+x}{1+y} = (1+c)^3 \quad\Rightarrow$$

$$y = \frac{1+x}{(1+c)^3} - 1.$$

From the level curves, we see that increasing x (from 0) by a small amount has a similar effect on the value of f as decreasing y by a small amount. However, for larger changes, a decrease in y gives greater values of f than a similar increase in x.



14.5 The Chain Rule

1. Find dz/dt using the Chain Rule: $z = x^2y + xy^2, \quad x = 3t, \quad y = t^2 \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (2xy + y^2)(3) + (x^2 + 2xy)(2t)\\ &= 3\left[2(3t)(t^2) + (t^2)^2\right] + 2t\left[(3t)^2 + 2(3t)(t^2)\right] \qquad [\text{with } x = 3t \text{ and } y = t^2]\\ &= 18t^3 + 3t^4 + 18t^3 + 12t^4 = 36t^3 + 15t^4\end{aligned}$$

Find dz/dt by substituting first: $z\,(x(t), y(t)) = (3t)^2(t^2) + (3t)(t^2)^2 = 9t^4 + 3t^5 \quad\Rightarrow\quad \dfrac{dz}{dt} = 36t^3 + 15t^4$

Yes, the two answers agree.

2. Find dz/dt using the Chain Rule: $z = xye^y, \quad x = t^2, \quad y = 5t \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = ye^y(2t) + x(ye^y + e^y)(5)\\ &= (5t)(e^{5t})(2t) + t^2(5te^{5t} + e^{5t})(5) \qquad [\text{with } x = t^2 \text{ and } y = 5t]\\ &= 10t^2e^{5t} + 25t^3e^{5t} + 5t^2e^{5t} = 15t^2e^{5t} + 25t^3e^{5t} = 5t^2e^{5t}(3+5t)\end{aligned}$$

Find dz/dt by substituting first: $z\,(x(t), y(t)) = (t^2)(5t)e^{5t} = 5t^3e^{5t} \quad\Rightarrow\quad \dfrac{dz}{dt} = 5t^3e^{5t}(5) + 15t^2e^{5t} = 5t^2e^{5t}(3+5t)$

Yes, the two answers agree.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1 \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

4. $z = \dfrac{x-y}{x+2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t} \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{(x+2y)(1) - (x-y)(1)}{(x+2y)^2}(\pi e^{\pi t}) + \frac{(x+2y)(-1) - (x-y)(2)}{(x+2y)^2}(-\pi e^{-\pi t})\\ &= \frac{3y}{(x+2y)^2}(\pi e^{\pi t}) + \frac{-3x}{(x+2y)^2}(-\pi e^{-\pi t}) = \frac{3\pi}{(x+2y)^2}\left(ye^{\pi t} + xe^{-\pi t}\right)\end{aligned}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x \cos y)\left(\tfrac{1}{2}t^{-1/2}\right) + (-\sin x \sin y)\left(-t^{-2}\right) = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y$$

6. $z = \sqrt{1+xy}, \quad x = \tan t, \quad y = \arctan t \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1+xy)^{-1/2}(y)\cdot \sec^2 t + \tfrac{1}{2}(1+xy)^{-1/2}(x)\cdot \frac{1}{1+t^2} \\ &= \frac{1}{2\sqrt{1+xy}}\left(y\sec^2 t + \frac{x}{1+t^2}\right)\end{aligned}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1-t, \quad z = 1+2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

8. $w = \ln\sqrt{x^2+y^2+z^2} = \tfrac{1}{2}\ln(x^2+y^2+z^2), \quad x = \sin t, \quad y = \cos t, \quad z = \tan t \quad \Rightarrow$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\cdot\sec^2 t \\ &= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2+y^2+z^2}\end{aligned}$$

9. First we find $\partial z/\partial s$ in two ways. $z = x^2 + y^2, \quad x = 2s+3t, \quad y = s+t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 2x(2) + 2y(1) \\ &= 2(2s+3t)(2) + 2(s+t)(1) \qquad [\text{with } x = 2s+3t \text{ and } y = s+t] \\ &= 8s + 12t + 2s + 2t = 10s + 14t\end{aligned}$$

$$z(x(s,t), y(s,t)) = (2s+3t)^2 + (s+t)^2 \quad \Rightarrow \quad \frac{\partial z}{\partial s} = 2(2s+3t)(2) + 2(s+t)(1) = 8s+12t+2s+2t = 10s+14t$$

Yes, the two answers agree. Now we find $\partial z/\partial t$ in two ways.

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2x(3) + 2y(1) \\ &= 2(2s+3t)(3) + 2(s+t)(1) \qquad [\text{with } x = 2s+3t \text{ and } y = s+t] \\ &= 12s + 18t + 2s + 2t = 14s + 20t\end{aligned}$$

$$z(x(s,t), y(s,t)) = (2s+3t)^2 + (s+t)^2 \quad \Rightarrow \quad \frac{\partial z}{\partial t} = 2(2s+3t)(3) + 2(s+t)(1) = 12s+18t+2s+2t = 14s+20t$$

Yes, the two answers agree.

10. First we find $\partial z/\partial s$ in two ways. $z = x^2 \sin y, \quad x = s^2 t, \quad y = st \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 2x\sin y(2st) + x^2\cos y(t) \\ &= 2(s^2t)\sin(st)(2st) + (s^2t)^2\cos(st)(t) \qquad [\text{with } x = s^2t \text{ and } y = st] \\ &= 4s^3t^2\sin(st) + s^4t^3\cos(st)\end{aligned}$$

[continued]

$z(x(s,t), y(s,t)) = (s^2 t)^2 \sin(st) = s^4 t^2 \sin(st) \quad \Rightarrow$

$\dfrac{\partial z}{\partial s} = s^4 t^2 \cos(st)(t) + 4s^3 t^2 \sin(st) = 4s^3 t^2 \sin(st) + s^4 t^3 \cos(st)$

Yes, the two answers agree. Now we find $\partial z/\partial t$ in two ways.

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2x\sin y(s^2) + x^2\cos y(s) \\ &= 2(s^2 t)\sin(st)(s^2) + (s^2 t)^2\cos(st)(s) \qquad [\text{with } x = s^2 t \text{ and } y = st] \\ &= 2s^4 t\sin(st) + s^5 t^2\cos(st) \end{aligned}$$

$z(x(s,t), y(s,t)) = (s^2 t)^2 \sin(st) = s^4 t^2 \sin(st) \quad \Rightarrow$

$\dfrac{\partial z}{\partial t} = 2s^4 t\sin(st) + s^4 t^2\cos(st)(s) = 2s^4 t\sin(st) + s^5 t^2 \cos(st)$

Yes, the two answers agree.

11. $z = (x-y)^5, \quad x = s^2 t, \quad y = st^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 5(x-y)^4(1)\cdot 2st + 5(x-y)^4(-1)\cdot t^2 = 5(x-y)^4\left(2st - t^2\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 5(x-y)^4(1)\cdot s^2 + 5(x-y)^4(-1)\cdot 2st = 5(x-y)^4\left(s^2 - 2st\right)$$

12. $z = \tan^{-1}(x^2+y^2), \quad x = s\ln t, \quad y = te^s \quad \Rightarrow$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{1+(x^2+y^2)^2}\cdot\ln t + \frac{2y}{1+(x^2+y^2)^2}\cdot te^s \\ &= \frac{2}{1+(x^2+y^2)^2}\left(x\ln t + yte^s\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{1+(x^2+y^2)^2}\cdot\frac{s}{t} + \frac{2y}{1+(x^2+y^2)^2}\cdot e^s \\ &= \frac{2}{1+(x^2+y^2)^2}\left(\frac{xs}{t} + ye^s\right) \end{aligned}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{3}{3x+2y}(\sin t) + \frac{2}{3x+2y}(-t\sin s) = \frac{3\sin t - 2t\sin s}{3x+2y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{3}{3x+2y}(s\cos t) + \frac{2}{3x+2y}(\cos s) = \frac{3s\cos t + 2\cos s}{3x+2y}$$

14. $z = \sqrt{x}\,e^{xy}, \quad x = 1+st, \quad y = s^2 - t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot\tfrac{1}{2}x^{-1/2}\right)(t) + \sqrt{x}\,e^{xy}(x)\,(2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot\tfrac{1}{2}x^{-1/2}\right)(s) + \sqrt{x}\,e^{xy}(x)\,(-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} = -\frac{\sin\theta}{r^2}(t) + \frac{\cos\theta}{r}(2s) = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = -\frac{\sin\theta}{r^2}(s) + \frac{\cos\theta}{r}(2t) = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}$$

16. $z = \tan(u/v), \quad u = 2s + 3t, \quad v = 3s - 2t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \sec^2(u/v)(1/v)\cdot 2 + \sec^2(u/v)(-uv^{-2})\cdot 3 \\ &= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2) \\ &= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)\end{aligned}$$

17. Let $x = g(t)$ and $y = h(t)$. Then $p(t) = f(x, y)$ and the Chain Rule (2) gives $\dfrac{dp}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$. When $t = 2$, $x = g(2) = 4$ and $y = h(2) = 5$, so $p'(2) = f_x(4,5)\,g'(2) + f_y(4,5)\,h'(2) = (2)(-3) + (8)(6) = 42$.

18. $R(s,t) = G\,(u(s,t), v(s,t)) \quad \Rightarrow \quad \dfrac{\partial R}{\partial s} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial R}{\partial t} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial t}$ by the Chain Rule (3). When $s = 1$ and $t = 2$, $u(1,2) = 5$ and $v(1,2) = 7$.

Thus $R_s(1,2) = G_u(5,7)\,u_s(1,2) + G_v(5,7)\,v_s(1,2) = (9)(4) + (-2)(2) = 32$ and

$R_t(1,2) = G_u(5,7)\,u_t(1,2) + G_v(5,7)\,v_t(1,2) = (9)(-3) + (-2)(6) = -39$.

19. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v,\ y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u,\ \dfrac{\partial x}{\partial v} = \cos v,\ \dfrac{\partial y}{\partial u} = e^u,\ \dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$\begin{aligned}g_v(0,0) &= f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0) \\ &= 2(1) + 5(0) = 2\end{aligned}$$

20. $g(r,s) = f(x(r,s), y(r,s))$ where $x = 2r - s,\ y = s^2 - 4r \quad \Rightarrow \quad \dfrac{\partial x}{\partial r} = 2,\ \dfrac{\partial x}{\partial s} = -1,\ \dfrac{\partial y}{\partial r} = -4,\ \dfrac{\partial y}{\partial s} = 2s$.

By the Chain Rule (3) $\dfrac{\partial g}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r}$. Then

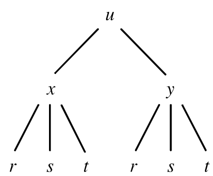
$$\begin{aligned}g_r(1,2) &= f_x(x(1,2), y(1,2))\,x_r(1,2) + f_y(x(1,2), y(1,2))\,y_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4) \\ &= 4(2) + 8(-4) = -24\end{aligned}$$

[continued]

Similarly, $\dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$. Then

$$\begin{aligned} g_s(1,2) &= f_x(x(1,2),y(1,2))\,x_s(1,2) + f_y(x(1,2),y(1,2))\,y_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4) \\ &= 4(-1) + 8(4) = 28 \end{aligned}$$

21.

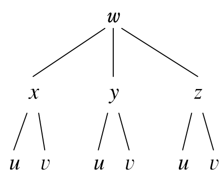


$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

22.

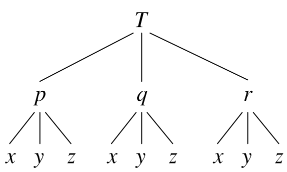


$w = f(x,y,z),\ \ x = x(u,v),\ \ y = y(u,v),\ \ z = z(u,v) \quad \Rightarrow$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

23.



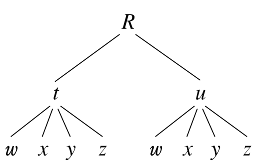
$T = F(p,q,r),\ p = p(x,y,z),\ \ q = q(x,y,z),\ r = r(x,y,z) \quad \Rightarrow$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial x},$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial z}$$

24.



$R = F(t,u),\ \ t = t(w,x,y,z),\ \ u = u(w,x,y,z) \quad \Rightarrow$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial w}, \quad \frac{\partial R}{\partial x} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial x},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial y}, \quad \frac{\partial R}{\partial z} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial z}$$

25. $z = x^4 + x^2 y,\ x = s + 2t - u,\ y = stu^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2),$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2),$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu).$$

When $s = 4$, $t = 2$, and $u = 1$ we have $x = 7$ and $y = 8$,

so $\dfrac{\partial z}{\partial s} = (1484)(1) + (49)(2) = 1582,\ \dfrac{\partial z}{\partial t} = (1484)\,(2) + (49)(4) = 3164,\ \dfrac{\partial z}{\partial u} = (1484)(-1) + (49)(16) = -700.$

26. $T = v/(2u+v) = v(2u+v)^{-1},\ u = pq\sqrt{r},\ v = p\sqrt{q}\,r \quad \Rightarrow$

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p} = [-v(2u+v)^{-2}(2)](q\sqrt{r}) + \frac{(2u+v)(1) - v(1)}{(2u+v)^2}(\sqrt{q}\,r)$$

$$= \frac{-2v}{(2u+v)^2}(q\sqrt{r}) + \frac{2u}{(2u+v)^2}(\sqrt{q}\,r),$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial q} = \frac{-2v}{(2u+v)^2}(p\sqrt{r}) + \frac{2u}{(2u+v)^2}\frac{pr}{2\sqrt{q}},$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial r} = \frac{-2v}{(2u+v)^2}\frac{pq}{2\sqrt{r}} + \frac{2u}{(2u+v)^2}(p\sqrt{q}).$$

When $p = 2$, $q = 1$, and $r = 4$ we have $u = 4$ and $v = 8$,

so $\dfrac{\partial T}{\partial p} = \left(-\frac{1}{16}\right)(2) + \left(\frac{1}{32}\right)(4) = 0,\ \dfrac{\partial T}{\partial q} = \left(-\frac{1}{16}\right)(4) + \left(\frac{1}{32}\right)(4) = -\frac{1}{8},\ \dfrac{\partial T}{\partial r} = \left(-\frac{1}{16}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{32}\right)(2) = \frac{1}{32}.$

27. $w = xy + yz + zx,\ x = r\cos\theta,\ y = r\sin\theta,\ z = r\theta \quad \Rightarrow$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r).$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$, so $\dfrac{\partial w}{\partial r} = (2+\pi)(0) + (0+\pi)(1) + (2+0)(\pi/2) = 2\pi$

and $\dfrac{\partial w}{\partial \theta} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2) = -2\pi.$

28. $P = \sqrt{u^2+v^2+w^2} = (u^2+v^2+w^2)^{1/2},\ u = xe^y,\ v = ye^x,\ w = e^{xy} \quad \Rightarrow$

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial x}$$

$$= \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2u)(e^y) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2v)(ye^x) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2w)(ye^{xy})$$

$$= \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2+v^2+w^2}},$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{u}{\sqrt{u^2+v^2+w^2}}(xe^y) + \frac{v}{\sqrt{u^2+v^2+w^2}}(e^x) + \frac{w}{\sqrt{u^2+v^2+w^2}}(xe^{xy})$$

$$= \frac{uxe^y + ve^x + wxe^{xy}}{\sqrt{u^2+v^2+w^2}}.$$

When $x = 0$ and $y = 2$ we have $u = 0$, $v = 2$, and $w = 1$, so $\dfrac{\partial P}{\partial x} = \dfrac{0+4+2}{\sqrt{5}} = \dfrac{6}{\sqrt{5}}$ and $\dfrac{\partial P}{\partial y} = \dfrac{0+2+0}{\sqrt{5}} = \dfrac{2}{\sqrt{5}}.$

29. $N = \dfrac{p+q}{p+r},\ p = u + vw,\ q = v + uw,\ r = w + uv \quad \Rightarrow$

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u} \\ &= \frac{(p+r)(1)-(p+q)(1)}{(p+r)^2}(1) + \frac{(p+r)(1)-(p+q)(0)}{(p+r)^2}(w) + \frac{(p+r)(0)-(p+q)(1)}{(p+r)^2}(v) \\ &= \frac{(r-q)+(p+r)w-(p+q)v}{(p+r)^2},\end{aligned}$$

$$\frac{\partial N}{\partial v} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial v} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial v} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial v} = \frac{r-q}{(p+r)^2}(w) + \frac{p+r}{(p+r)^2}(1) + \frac{-(p+q)}{(p+r)^2}(u) = \frac{(r-q)w+(p+r)-(p+q)u}{(p+r)^2},$$

$$\frac{\partial N}{\partial w} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial w} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial w} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial w} = \frac{r-q}{(p+r)^2}(v) + \frac{p+r}{(p+r)^2}(u) + \frac{-(p+q)}{(p+r)^2}(1) = \frac{(r-q)v+(p+r)u-(p+q)}{(p+r)^2}.$$

When $u = 2$, $v = 3$, and $w = 4$ we have $p = 14$, $q = 11$, and $r = 10$, so $\dfrac{\partial N}{\partial u} = \dfrac{-1+(24)(4)-(25)(3)}{(24)^2} = \dfrac{20}{576} = \dfrac{5}{144}$,

$\dfrac{\partial N}{\partial v} = \dfrac{(-1)(4)+24-(25)(2)}{(24)^2} = \dfrac{-30}{576} = -\dfrac{5}{96}$, and $\dfrac{\partial N}{\partial w} = \dfrac{(-1)(3)+(24)(2)-25}{(24)^2} = \dfrac{20}{576} = \dfrac{5}{144}$.

30. $u = xe^{ty},\ x = \alpha^2\beta,\ y = \beta^2\gamma,\ t = \gamma^2\alpha \quad \Rightarrow$

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha} = e^{ty}(2\alpha\beta) + xte^{ty}(0) + xye^{ty}(\gamma^2) = e^{ty}(2\alpha\beta + xy\gamma^2),$$

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta} = e^{ty}(\alpha^2) + xte^{ty}(2\beta\gamma) + xye^{ty}(0) = e^{ty}(\alpha^2 + 2xt\beta\gamma),$$

$$\frac{\partial u}{\partial \gamma} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma} = e^{ty}(0) + xte^{ty}(\beta^2) + xye^{ty}(2\gamma\alpha) = e^{ty}(xt\beta^2 + 2xy\alpha\gamma).$$

When $\alpha = -1$, $\beta = 2$, and $\gamma = 1$ we have $x = 2$, $y = 4$, and $t = -1$, so $\dfrac{\partial u}{\partial \alpha} = e^{-4}(-4+8) = 4e^{-4}$,

$\dfrac{\partial u}{\partial \beta} = e^{-4}(1-8) = -7e^{-4}$, and $\dfrac{\partial u}{\partial \gamma} = e^{-4}(-8-16) = -24e^{-4}$.

31. $y\cos x = x^2 + y^2$, so let $F(x,y) = y\cos x - x^2 - y^2 = 0$. Then by Equation 5

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}.$$

32. $\cos(xy) = 1 + \sin y$, so let $F(x,y) = \cos(xy) - 1 - \sin y = 0$. Then by Equation 5

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(xy)(y)}{-\sin(xy)(x) - \cos y} = -\frac{y\sin(xy)}{\cos y + x\sin(xy)}.$$

33. $\tan^{-1}(x^2y) = x + xy^2$, so let $F(x,y) = \tan^{-1}(x^2y) - x - xy^2 = 0$. Then

$$F_x(x,y) = \frac{1}{1+(x^2y)^2}(2xy) - 1 - y^2 = \frac{2xy}{1+x^4y^2} - 1 - y^2 = \frac{2xy - (1+y^2)(1+x^4y^2)}{1+x^4y^2},$$

$$F_y(x,y) = \frac{1}{1+(x^2y)^2}(x^2) - 2xy = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2}$$

[continued]

and $$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{[2xy - (1+y^2)(1+x^4y^2)]/(1+x^4y^2)}{[x^2 - 2xy(1+x^4y^2)]/(1+x^4y^2)} = \frac{(1+y^2)(1+x^4y^2) - 2xy}{x^2 - 2xy(1+x^4y^2)}$$

$$= \frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$$

34. $e^y \sin x = x + xy$, so let $F(x,y) = e^y \sin x - x - xy = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^y \cos x - 1 - y}{e^y \sin x - x} = \dfrac{1 + y - e^y \cos x}{e^y \sin x - x}$.

35. $x^2 + 2y^2 + 3z^2 = 1$, so let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 1 = 0$. Then by Equations 6

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}.$$

36. $x^2 - y^2 + z^2 - 2z = 4$, so let $F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4 = 0$. Then by Equations 6

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-2} = \frac{x}{1-z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z-2} = \frac{y}{z-1}.$$

37. $e^z = xyz$, so let $F(x,y,z) = e^z - xyz = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-yz}{e^z - xy} = \dfrac{yz}{e^z - xy}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}.$$

38. $yz + x \ln y = z^2$, so let $F(x,y,z) = yz + x \ln y - z^2 = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z} = \dfrac{\ln y}{2z - y}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z + (x/y)}{y - 2z} = \frac{x + yz}{2yz - y^2}.$$

39. Since x and y are each functions of t, $T(x,y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2,3)\,\dfrac{dx}{dt} + T_y(2,3)\,\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s.

40. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

41. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so

$\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate

$\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

42. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}\,1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi \text{ in}^3\text{/s}$.

43. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\,\frac{d\ell}{dt} + \ell h\,\frac{dw}{dt} + \ell w\,\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6 \text{ m}^3\text{/s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\,\frac{d\ell}{dt} + 2(\ell+h)\,\frac{dw}{dt} + 2(\ell+w)\,\frac{dh}{dt} \\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10 \text{ m}^2\text{/s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\,\dfrac{dL}{dt} = 2\ell\,\dfrac{d\ell}{dt} + 2w\,\dfrac{dw}{dt} + 2h\,\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad\Rightarrow$

$dL/dt = 0$ m/s.

44. $I = \dfrac{V}{R} \quad\Rightarrow$

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt} = \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031 \text{ A/s}$$

45. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)(320)}{400}\right] \approx -0.27$ L/s.

46. $P = 1.47L^{0.65}K^{0.35}$ and considering P, L, and K as functions of time t we have

$\dfrac{dP}{dt} = \dfrac{\partial P}{\partial L}\dfrac{dL}{dt} + \dfrac{\partial P}{\partial K}\dfrac{dK}{dt} = 1.47(0.65)L^{-0.35}K^{0.35}\dfrac{dL}{dt} + 1.47(0.35)L^{0.65}K^{-0.65}\dfrac{dK}{dt}$. We are given

that $\dfrac{dL}{dt} = -2$ and $\dfrac{dK}{dt} = 0.5$, so when $L = 30$ and $K = 8$, the rate of change of production $\dfrac{dP}{dt}$ is

$1.47(0.65)(30)^{-0.35}(8)^{0.35}(-2) + 1.47(0.35)(30)^{0.65}(8)^{-0.65}(0.5) \approx -0.596$. Thus production at that time

is decreasing at a rate of about \$596,000 per year.

47. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ [Formula 6 in Appendix D], where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$.

By the Chain Rule, $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta\cdot\dfrac{dx}{dt} + \frac{1}{2}x\sin\theta\cdot\dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta\cdot\dfrac{d\theta}{dt}$.

When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned} 0 &= \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt} \\ &= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt} \end{aligned}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048\ \text{rad/s}$.

48. $f_o = \left(\dfrac{c+v_o}{c-v_s}\right)f_s = \left(\frac{332+34}{332-40}\right)460 \approx 576.6$ Hz. v_o and v_s are functions of time t, so

$$\begin{aligned} \frac{df_o}{dt} &= \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} = \left(\frac{1}{c-v_s}\right)f_s\cdot\frac{dv_o}{dt} + \frac{c+v_o}{(c-v_s)^2}f_s\cdot\frac{dv_s}{dt} \\ &= \left(\tfrac{1}{332-40}\right)(460)\,(1.2) + \tfrac{332+34}{(332-40)^2}\,(460)\,(1.4) \approx 4.65\ \text{Hz/s} \end{aligned}$$

49. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

50. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}e^s\cos t + \dfrac{\partial u}{\partial y}e^s\sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}(-e^s\sin t) + \dfrac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus

$$\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

51. Let $u = x + at,\ v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a\,\frac{\partial}{\partial t}\left[f'(u) - g'(v)\right] = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$$

Similarly, $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$.

52. By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s\sin t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) + e^s\sin t\,\dfrac{\partial u}{\partial y} + e^s\sin t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial s} = e^s\cos t\,\frac{\partial^2 u}{\partial x^2} + e^s\sin t\,\frac{\partial^2 u}{\partial y\,\partial x} \text{ and}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial s} = e^s\sin t\,\frac{\partial^2 u}{\partial y^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}.$$

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\,\frac{\partial u}{\partial x} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial x^2} + e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s\sin t\,\frac{\partial u}{\partial y} + e^s\sin t\left(e^s\sin t\,\frac{\partial^2 u}{\partial y^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\left(-e^s\sin t\,\frac{\partial^2 u}{\partial x^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &\qquad -e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial y^2} - e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial x^2} - 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.

53. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2\\ &= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\,\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

54. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ (b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\, r\sin\theta + \dfrac{\partial z}{\partial y}\, r\cos\theta$

(c)
$$\begin{aligned}
\frac{\partial^2 z}{\partial r\,\partial\theta} &= \frac{\partial^2 z}{\partial\theta\,\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) = -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)\\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial\theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial\theta}\\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right)\\
&= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y} + r\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\\
&= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\,\frac{\partial^2 z}{\partial y\,\partial x}
\end{aligned}$$

55. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\, r\sin\theta + \dfrac{\partial z}{\partial y}\, r\cos\theta$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\
&= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right)\\
&\qquad - r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\
&= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}\\
&\qquad - \frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\
&= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}
\end{aligned}$$

56. Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x, y)$.

Thus, f is homogeneous of degree 3.

57. (a) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t}\left[t^n f(x, y)\right] \quad \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(tx,ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx,ty) \cdot \frac{\partial(ty)}{\partial t} = x\,\frac{\partial}{\partial(tx)} f(tx,ty) + y\,\frac{\partial}{\partial(ty)} f(tx,ty) = nt^{n-1} f(x,y).$$

Setting $t = 1$: $x\,\dfrac{\partial}{\partial x} f(x,y) + y\,\dfrac{\partial}{\partial y} f(x,y) = nf(x,y)$.

(b) Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial(tx)} f(tx,ty) \cdot \dfrac{\partial(tx)}{\partial t} + \dfrac{\partial}{\partial(ty)} f(tx,ty) \cdot \dfrac{\partial(ty)}{\partial t} = x\,\dfrac{\partial}{\partial(tx)} f(tx,ty) + y\,\dfrac{\partial}{\partial(ty)} f(tx,ty) = nt^{n-1} f(x,y)$ and

differentiating again with respect to t gives

$$\begin{aligned} &x\left[\frac{\partial^2}{\partial\,(tx)^2} f(tx,ty) \cdot \frac{\partial\,(tx)}{\partial t} + \frac{\partial^2}{\partial\,(ty)\,\partial\,(tx)} f(tx,ty) \cdot \frac{\partial\,(ty)}{\partial t}\right] \\ &\qquad + y\left[\frac{\partial^2}{\partial\,(tx)\,\partial\,(ty)} f(tx,ty) \cdot \frac{\partial\,(tx)}{\partial t} + \frac{\partial^2}{\partial\,(ty)^2} f(tx,ty) \cdot \frac{\partial\,(ty)}{\partial t}\right] = n(n-1)t^{n-1} f(x,y). \end{aligned}$$

Setting $t = 1$ and using the fact that $f_{yx} = f_{xy}$, we have $x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = n(n-1)f(x,y)$.

58. Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx,ty) = \frac{\partial}{\partial x}\left[t^n f(x,y)\right] \quad \Leftrightarrow$$

$$\frac{\partial}{\partial\,(tx)} f(tx,ty) \cdot \frac{\partial\,(tx)}{\partial x} + \frac{\partial}{\partial\,(ty)} f(tx,ty) \cdot \frac{\partial\,(ty)}{\partial x} = t^n\,\frac{\partial}{\partial x} f(x,y) \quad \Leftrightarrow \quad tf_x(tx,ty) = t^n f_x(x,y).$$

Thus $f_x(tx,ty) = t^{n-1} f_x(x,y)$.

59. $F(x,y,z) = 0$ is assumed to define z as a function of x and y, that is, $z = f(x,y)$. So by (6), $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ since $F_z \neq 0$.

Similarly, it is assumed that $F(x,y,z) = 0$ defines x as a function of y and z, that is $x = h(x,z)$. Then $F(h(y,z),y,z) = 0$

and by the Chain Rule, $F_x\,\dfrac{\partial x}{\partial y} + F_y\,\dfrac{\partial y}{\partial y} + F_z\,\dfrac{\partial z}{\partial y} = 0$. But $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial y}{\partial y} = 1$, so $F_x\,\dfrac{\partial x}{\partial y} + F_y = 0 \quad \Rightarrow \quad \dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$.

A similar calculation shows that $\dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$. Thus $\dfrac{\partial z}{\partial x}\,\dfrac{\partial x}{\partial y}\,\dfrac{\partial y}{\partial z} = \left(-\dfrac{F_x}{F_z}\right)\left(-\dfrac{F_y}{F_x}\right)\left(-\dfrac{F_z}{F_y}\right) = -1$.

60. Given a function defined implicitly by $F(x,y) = 0$, where F is differentiable and $F_y \neq 0$, we know that $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$. Let

$G(x,y) = -\dfrac{F_x}{F_y}$ so $\dfrac{dy}{dx} = G(x,y)$. Differentiating both sides with respect to x and using the Chain Rule gives

$\dfrac{d^2y}{dx^2} = \dfrac{\partial G}{\partial x}\,\dfrac{dx}{dx} + \dfrac{\partial G}{\partial y}\,\dfrac{dy}{dx}$ where $\dfrac{\partial G}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{F_x}{F_y}\right) = -\dfrac{F_y F_{xx} - F_x F_{yx}}{F_y^2}$, $\dfrac{\partial G}{\partial y} = \dfrac{\partial}{\partial y}\left(-\dfrac{F_x}{F_y}\right) = -\dfrac{F_y F_{xy} - F_x F_{yy}}{F_y^2}$.

Thus
$$\begin{aligned} \frac{d^2y}{dx^2} &= \left(-\frac{F_y F_{xx} - F_x F_{yx}}{F_y^2}\right)(1) + \left(-\frac{F_y F_{xy} - F_x F_{yy}}{F_y^2}\right)\left(-\frac{F_x}{F_y}\right) \\ &= -\frac{F_{xx}F_y^2 - F_{yx}F_x F_y - F_{xy}F_y F_x + F_{yy}F_x^2}{F_y^3} \end{aligned}$$

But F has continuous second derivatives, so by Clauraut's Theorem, $F_{yx} = F_{xy}$ and we have

$\dfrac{d^2y}{dx^2} = -\dfrac{F_{xx}F_y^2 - 2F_{xy}F_x F_y + F_{yy}F_x^2}{F_y^3}$ as desired.

14.6 Directional Derivatives and the Gradient Vector

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from 30°C to 27°C. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025^\circ\text{C/km}$.

3. $D_{\mathbf{u}}\, f(-20,30) = \nabla f\,(-20,30)\cdot \mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20+h,30) - f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30)-f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30)-f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40)-f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20)-f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$.

Then $D_{\mathbf{u}} f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x,y) = xy^3 - x^2 \quad\Rightarrow\quad f_x(x,y) = y^3 - 2x$ and $f_y(x,y) = 3xy^2$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/3$, then from Equation 6, $D_{\mathbf{u}} f(1,2) = f_x(1,2)\cos\left(\frac{\pi}{3}\right) + f_y(1,2)\sin\left(\frac{\pi}{3}\right) = 6\cdot\frac{1}{2} + 12\cdot\frac{\sqrt{3}}{2} = 3+6\sqrt{3}$.

5. $f(x,y) = y\cos(xy) \quad\Rightarrow\quad f_x(x,y) = y[-\sin(xy)](y) = -y^2\sin(xy)$ and

$f_y(x,y) = y[-\sin(xy)](x) + [\cos(xy)](1) = \cos(xy) - xy\sin(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/4$, then from Equation 6, $D_{\mathbf{u}} f(0,1) = f_x(0,1)\cos\left(\frac{\pi}{4}\right) + f_y(0,1)\sin\left(\frac{\pi}{4}\right) = 0\cdot\frac{\sqrt{2}}{2} + 1\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

6. $f(x,y) = \sqrt{2x+3y} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(2) = 1/\sqrt{2x+3y}$ and

$f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(3) = 3/\left(2\sqrt{2x+3y}\right)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\pi/6$, then from Equation 6, $D_{\mathbf{u}} f(3,1) = f_x(3,1)\cos\left(-\frac{\pi}{6}\right) + f_y(3,1)\sin\left(-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{6} - \frac{1}{4}$.

7. $f(x,y) = \arctan(xy) \quad \Rightarrow \quad f_x(x,y) = \dfrac{y}{1+(xy)^2}$ and $f_y(x,y) = \dfrac{x}{1+(xy)^2}$. If $\mathbf{u}$ is a unit vector in the direction $\theta = 3\pi/4$, then from Equation 6,

$$D_{\mathbf{u}}f(2,-3) = f_x(2,-3)\cos\left(\frac{3\pi}{4}\right) + f_y(2,-3)\sin\left(\frac{3\pi}{4}\right) = -\frac{3}{37}\left(-\frac{\sqrt{2}}{2}\right) + \frac{2}{37}\left(\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{74}.$$

8. $f(x,y) = x^2e^y$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2xe^y\,\mathbf{i} + x^2e^y\,\mathbf{j}$

(b) $\nabla f(3,0) = 2(3)e^0\,\mathbf{i} + 3^2e^0\,\mathbf{j} = 6\,\mathbf{i} + 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}f(3,0) = \nabla f(3,0)\cdot\mathbf{u} = (6\mathbf{i}+9\,\mathbf{j})\cdot\left(\frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}\right) = \frac{18}{5} - \frac{36}{5} = -\frac{18}{5}$.

9. $f(x,y) = x/y = xy^{-1}$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = y^{-1}\,\mathbf{i} + (-xy^{-2})\,\mathbf{j} = \dfrac{1}{y}\,\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j}$

(b) $\nabla f(2,1) = \dfrac{1}{1}\,\mathbf{i} - \dfrac{2}{1^2}\,\mathbf{j} = \mathbf{i} - 2\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(2,1) = \nabla f(2,1)\cdot\mathbf{u} = (\mathbf{i} - 2\,\mathbf{j})\;\cdot\left(\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1$.

10. $f(x,y) = x^2\ln y$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2x\ln y\,\mathbf{i} + (x^2/y)\,\mathbf{j}$

(b) $\nabla f(3,1) = 0\,\mathbf{i} + (9/1)\,\mathbf{j} = 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(3,1) = \nabla f(3,1)\cdot\mathbf{u} = 9\,\mathbf{j}\;\cdot\left(-\frac{5}{13}\,\mathbf{i} + \frac{12}{13}\,\mathbf{j}\right) = 0 + \frac{108}{13} = \frac{108}{13}$.

11. $f(x,y,z) = x^2yz - xyz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\right\rangle$

(b) $\nabla f(2,-1,1) = \langle -4+1, 4-2, -4+6\rangle = \langle -3,2,2\rangle$

(c) By Equation 14, $D_{\mathbf{u}}f(2,-1,1) = \nabla f(2,-1,1)\cdot\mathbf{u} = \langle -3,2,2\rangle\cdot\left\langle 0, \frac{4}{5}, -\frac{3}{5}\right\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \frac{2}{5}$.

12. $f(x,y,z) = y^2e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle y^2e^{xyz}(yz), y^2\cdot e^{xyz}(xz) + e^{xyz}\cdot 2y, y^2e^{xyz}(xy)\right\rangle$

$\qquad = \left\langle y^3ze^{xyz}, (xy^2z+2y)e^{xyz}, xy^3e^{xyz}\right\rangle$

(b) $\nabla f(0,1,-1) = \langle -1,2,0\rangle$

(c) $D_{\mathbf{u}}f(0,1,-1) = \nabla f(0,1,-1)\cdot\mathbf{u} = \langle -1,2,0\rangle\cdot\left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

13. $f(x,y) = e^x\sin y \quad \Rightarrow \quad \nabla f(x,y) = \langle e^x\sin y, e^x\cos y\rangle$, $\nabla f(0,\pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2+8^2}}\langle -6,8\rangle = \frac{1}{10}\langle -6,8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(0,\pi/3) = \nabla f(0,\pi/3)\cdot\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\cdot\left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4-3\sqrt{3}}{10}$.

14. $f(x,y) = \dfrac{x}{x^2+y^2} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}, \dfrac{0 - x(2y)}{(x^2+y^2)^2} \right\rangle = \left\langle \dfrac{y^2-x^2}{(x^2+y^2)^2}, -\dfrac{2xy}{(x^2+y^2)^2} \right\rangle,$

$\nabla f(1,2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$, and a unit vector in the direction of $\mathbf{v} = \langle 3,5 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{9+25}} \langle 3,5 \rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$, so

$D_{\mathbf{u}}\, f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle = \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}} = -\frac{11}{25\sqrt{34}}.$

15. $g(s,t) = s\sqrt{t} \quad \Rightarrow \quad \nabla g(s,t) = \left(\sqrt{t}\,\right)\mathbf{i} + \left(s/(2\sqrt{t}\,)\right)\mathbf{j}, \quad \nabla g(2,4) = 2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is

$\mathbf{u} = \frac{1}{\sqrt{2^2+(-1)^2}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j})$, so $D_{\mathbf{u}}\, g(2,4) = \nabla g(2,4) \cdot \mathbf{u} = (2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}) \cdot \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) = \frac{7}{2\sqrt{5}}$

or $\frac{7\sqrt{5}}{10}$.

16. $g(u,v) = u^2 e^{-v} \quad \Rightarrow \quad \nabla g(u,v) = \left(2ue^{-v}\right)\mathbf{i} + \left(-u^2 e^{-v}\right)\mathbf{j}, \quad \nabla g(3,0) = 6\,\mathbf{i} - 9\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$

is $\mathbf{u} = \frac{1}{\sqrt{3^2+4^2}}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j})$, so $D_{\mathbf{u}}\, g(3,0) = \nabla g(3,0) \cdot \mathbf{u} = (6\,\mathbf{i} - 9\,\mathbf{j}) \cdot \frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}\,(18 - 36) = -\frac{18}{5}.$

17. $f(x,y,z) = x^2 y + y^2 z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2 \right\rangle, \quad \nabla f(1,2,3) = \langle 4, 13, 4 \rangle$, and a unit

vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}} \langle 2, -1, 2 \rangle = \frac{1}{3} \langle 2, -1, 2 \rangle$, so

$D_{\mathbf{u}}\, f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u} = \langle 4, 13, 4 \rangle \cdot \frac{1}{3} \langle 2, -1, 2 \rangle = \frac{1}{3}\,(8 - 13 + 8) = \frac{3}{3} = 1.$

18. $f(x,y,z) = xy^2 \tan^{-1} z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y^2 \tan^{-1} z, 2xy \tan^{-1} z, \dfrac{xy^2}{1+z^2} \right\rangle,$

$\nabla f(2,1,1) = \left\langle 1 \cdot \frac{\pi}{4}, 4 \cdot \frac{\pi}{4}, \frac{2}{1+1} \right\rangle = \left\langle \frac{\pi}{4}, \pi, 1 \right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+1+1}} \langle 1,1,1 \rangle = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$,

so $D_{\mathbf{u}}\, f(2,1,1) = \nabla f(2,1,1) \cdot \mathbf{u} = \left\langle \frac{\pi}{4}, \pi, 1 \right\rangle \cdot \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + \pi + 1\right) = \frac{1}{\sqrt{3}}\left(\frac{5\pi}{4} + 1\right).$

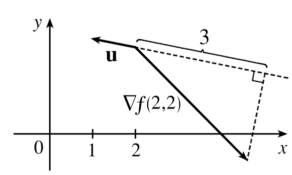
19. $h(r,s,t) = \ln(3r + 6s + 9t) \quad \Rightarrow \quad \nabla h(r,s,t) = \langle 3/(3r+6s+9t), 6/(3r+6s+9t), 9/(3r+6s+9t) \rangle,$

$\nabla h(1,1,1) = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle$, and a unit vector in the direction of $\mathbf{v} = 4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$ is

$\mathbf{u} = \frac{1}{\sqrt{16+144+36}}\,(4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}) = \frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$, so

$D_{\mathbf{u}}\, h(1,1,1) = \nabla h(1,1,1) \cdot \mathbf{u} = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle = \frac{1}{21} + \frac{2}{7} + \frac{3}{14} = \frac{23}{42}.$

20. $D_{\mathbf{u}} f(2,2) = \nabla f(2,2) \cdot \mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than 90°, the scalar projection is negative. Thus $D_{\mathbf{u}}\, f(2,2) \approx -3$.



21. $f(x,y) = x^2 y^2 - y^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 2xy^2, 2x^2 y - 3y^2 \right\rangle$, so $\nabla f(1,2) = \langle 8, -8 \rangle$. The unit vector in the

direction of $\overrightarrow{PQ} = \langle -3-1, 5-2 \rangle = \langle -4, 3 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-4)^2 + 3^2}} \langle -4, 3 \rangle = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$, so

$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle 8, -8 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = -\frac{56}{5}.$

22. $f(x,y) = \dfrac{x}{y^2} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{1}{y^2}, -\dfrac{2x}{y^3} \right\rangle$, so $\nabla f(3,-1) = \langle 1, 6 \rangle$. The unit vector in the direction of

$\overrightarrow{PQ} = \langle -2-3, 11-(-1) \rangle = \langle -5, 12 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-5)^2+12^2}} \langle -5, 12 \rangle = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$, so

$D_{\mathbf{u}} f(3,-1) = \nabla f(3,-1) \cdot \mathbf{u} = \langle 1, 6 \rangle \cdot \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle = \frac{67}{13}$.

23. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{3^2+(-4)^2}} \langle 3, -4 \rangle = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$,

so $D_{\mathbf{u}} f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

24. $f(x,y,z) = xy^2z^3 \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$, so $\nabla f(2,1,1) = \langle 1, 4, 6 \rangle$. The unit vector in the direction

of $\overrightarrow{PQ} = \langle 0-2, -3-1, 5-1 \rangle = \langle -2, -4, 4 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-2)^2+(-4)^2+4^2}} \langle -2, -4, 4 \rangle = \frac{1}{6} \langle -2, -4, 4 \rangle$, so

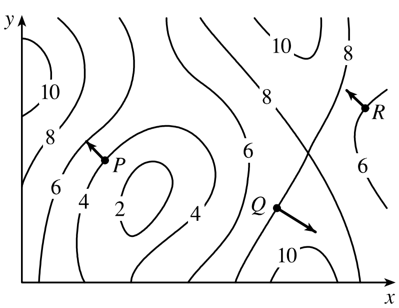
$D_{\mathbf{u}} f(2,1,1) = \nabla f(2,1,1) \cdot \mathbf{u} = \langle 1, 4, 6 \rangle \cdot \frac{1}{6} \langle -2, -4, 4 \rangle = \frac{1}{6}(-2-16+24) = 1$.

25. $f(x,y,z) = xy - xy^2z^2 \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y - y^2z^2, x - 2xyz^2, -2xy^2z \right\rangle$, so $\nabla f(2,-1,1) = \langle -2, 6, -4 \rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 1-(-1), 7-1 \rangle = \langle 3, 2, 6 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{3^2+2^2+6^2}} \langle 3, 2, 6 \rangle = \left\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \right\rangle$, so

$D_{\mathbf{u}} f(2,-1,1) = \nabla f(2,-1,1) \cdot \mathbf{u} = \langle -2, 6, -4 \rangle \cdot \left\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \right\rangle = -\frac{18}{7}$.

26.



Note that the vectors drawn at P, Q, and R are perpendicular to the curves and represent the direction of maximum increase.

27. $f(x,y) = 5xy^2 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 5y^2, 10xy \right\rangle$. Then $\nabla f(3,-2) = \langle 20, -60 \rangle$, or equivalently, $\langle 1, -3 \rangle$ is the direction of maximum rate of change, and the maximum rate of change is $|\nabla f(3,-2)| = \sqrt{20^2 + (-60)^2} = \sqrt{4000} = 20\sqrt{10}$.

28. $f(s,t) = \dfrac{s}{s^2+t^2} = s(s^2+t^2)^{-1} \quad \Rightarrow$

$$\nabla f(s,t) = \left\langle s(-1)(s^2+t^2)^{-2}(2s) + (1)(s^2+t^2)^{-1},\ s(-1)(s^2+t^2)^{-2}(2t) \right\rangle = \left\langle \frac{t^2-s^2}{(s^2+t^2)^2}, -\frac{2st}{(s^2+t^2)^2} \right\rangle.$$

Then $\nabla f(-1,1) = \left\langle 0, \frac{1}{2} \right\rangle$ is the direction of maximum rate of change, and the maximum rate of change is $|\nabla f(-1,1)| = \frac{1}{2}$.

29. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus, the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

30. $f(x,y,z) = x\ln(yz) \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \ln(yz), x \cdot \dfrac{z}{yz}, x \cdot \dfrac{y}{yz} \right\rangle = \left\langle \ln(yz), \dfrac{x}{y}, \dfrac{x}{z} \right\rangle, \quad \nabla f(1,2,\frac{1}{2}) = \langle 0, \frac{1}{2}, 2 \rangle.$

Thus, the maximum rate of change is $\left|\nabla f(1,2,\frac{1}{2})\right| = \sqrt{0 + \frac{1}{4} + 4} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$ in the direction $\langle 0, \frac{1}{2}, 2 \rangle$, or equivalently, $\langle 0, 1, 4 \rangle$.

31. $f(x,y,z) = x/(y+z) = x(y+z)^{-1} \quad \Rightarrow$

$\nabla f(x,y,z) = \langle 1/(y+z), -x(y+z)^{-2}(1), -x(y+z)^{-2}(1) \rangle = \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2} \right\rangle,$

$\nabla f(8,1,3) = \langle \frac{1}{4}, -\frac{8}{4^2}, -\frac{8}{4^2} \rangle = \langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2} \rangle$. Thus, the maximum rate of change is

$|\nabla f(8,1,3)| = \sqrt{\frac{1}{16} + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ in the direction $\langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2} \rangle$, or equivalently, $\langle 1, -2, -2 \rangle$.

32. $f(p,q,r) = \arctan(pqr) \quad \Rightarrow \quad \nabla f(p,q,r) = \left\langle \dfrac{qr}{1+(pqr)^2}, \dfrac{pr}{1+(pqr)^2}, \dfrac{pq}{1+(pqr)^2} \right\rangle, \nabla f(1,2,1) = \langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \rangle.$

Thus, the maximum rate of change is $|\nabla f(1,2,1)| = \sqrt{\frac{4}{25} + \frac{1}{25} + \frac{4}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ in the direction $\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \rangle$, or equivalently, $\langle 2, 1, 2 \rangle$.

33. (a) As in the proof of Theorem 15, $D_{\mathbf{u}} f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is -1, occurring when $\theta = \pi$, the maximum rate of decrease of $D_{\mathbf{u}} f$ is $-|\nabla f|$ occurring when $\theta = \pi$; that is, when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4 y - x^2 y^3 \quad \Rightarrow \quad \nabla f(x,y) = \langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$. The maximum rate of decrease is $-|\nabla f(2,-3)| = -|\langle 12, -92 \rangle| = -\sqrt{12^2 + (-92)^2} = -\sqrt{8608} = -4\sqrt{538}$.

34. $f(x,y) = x^2 + xy^3 \quad \Rightarrow \quad \nabla f(x,y) = \langle 2x + y^3, 3xy^2 \rangle$ so $\nabla f(2,1) = \langle 5, 6 \rangle$. If $\mathbf{u} = \langle a, b \rangle$ is a unit vector in the desired direction then $D_{\mathbf{u}} f(2,1) = 2 \quad \Leftrightarrow \quad \langle 5, 6 \rangle \cdot \langle a, b \rangle = 2 \quad \Leftrightarrow \quad 5a + 6b = 2 \quad \Leftrightarrow \quad b = \frac{1}{3} - \frac{5}{6}a$. But $a^2 + b^2 = 1 \quad \Leftrightarrow$ $a^2 + \left(\frac{1}{3} - \frac{5}{6}a\right)^2 = 1 \quad \Leftrightarrow \quad \frac{61}{36}a^2 - \frac{5}{9}a + \frac{1}{9} = 1 \quad \Leftrightarrow \quad 61a^2 - 20a - 32 = 0$. By the quadratic formula, the solutions are

$a = \dfrac{-(-20) \pm \sqrt{(-20)^2 - 4(61)(-32)}}{2(61)} = \dfrac{20 \pm \sqrt{8208}}{122} = \dfrac{10 \pm 6\sqrt{57}}{61}$. If $a = \dfrac{10 + 6\sqrt{57}}{61} \approx 0.9065$ then

$b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10 + 6\sqrt{57}}{61}\right) \approx -0.4221$, and if $a = \dfrac{10 - 6\sqrt{57}}{61} \approx -0.5787$ then $b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10 - 6\sqrt{57}}{61}\right) \approx 0.8156$.

Thus the two directions giving a directional derivative of 2 are approximately $\langle 0.9065, -0.4221 \rangle$ and $\langle -0.5787, 0.8156 \rangle$.

35. For $f(x,y) = x^2 + y^2 - 2x - 4y$, the direction of greatest rate of change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = c\,(\mathbf{i} + \mathbf{j}) \quad \Leftrightarrow \quad c = 2x - 2$ and $c = 2y - 4$. Then $2x - 2 = 2y - 4 \quad \Rightarrow \quad y = x + 1$, so at all points on the line $y = x + 1$, the direction of greatest rate of change of f is $\mathbf{i} + \mathbf{j}$.

36. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x, y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x, y) = \left\langle 0.04x, -0.003y^2\right\rangle$. $D_{\mathbf{u}}\, f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} = \langle 3.2, -10.8\rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_{\mathbf{u}}\, f(80, 60)$ is positive, the depth of the lake is increasing near $(80, 60)$ in the direction toward the buoy.

37. $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}}T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[-360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle\right]_{(1,2,2)} \cdot \mathbf{u} = -\tfrac{40}{3}\langle 1, 2, 2\rangle \cdot \tfrac{1}{\sqrt{3}}\langle 1, -1, 1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle$, and since $\langle x, y, z\rangle$ is the position vector of the point (x, y, z), the vector $-\langle x, y, z\rangle$, and thus ∇T, always points toward the origin.

38. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$, $\nabla T(2, -1, 2) = -400e^{-43}\langle 2, -3, 18\rangle$ and

$$D_{\mathbf{u}}\, T(2, -1, 2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\ {}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2, -1, 2) = 400e^{-43}\langle -2, 3, -18\rangle$ or equivalently $\langle -2, 3, -18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2 + 9y^2 + 81z^2}\ {}^\circ\mathrm{C/m}$ is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\ {}^\circ\mathrm{C/m}$.

39. $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}}\, V(3, 4, 5) = \langle 38, 6, 12\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$, or equivalently, $\langle 19, 3, 6\rangle$.

(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

40. $z = f(x, y) = 1000 - 0.005x^2 - 0.01y^2 \quad\Rightarrow\quad \nabla f(x, y) = \langle -0.01x, -0.02y\rangle$ and $\nabla f(60, 40) = \langle -0.6, -0.8\rangle$.

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and

$D_{\mathbf{u}}f(60, 40) = \nabla f\,(60, 40) \cdot \langle 0, -1\rangle = \langle -0.6, -0.8\rangle \cdot \langle 0, -1\rangle = 0.8$. Thus, if you walk due south from $(60, 40, 966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

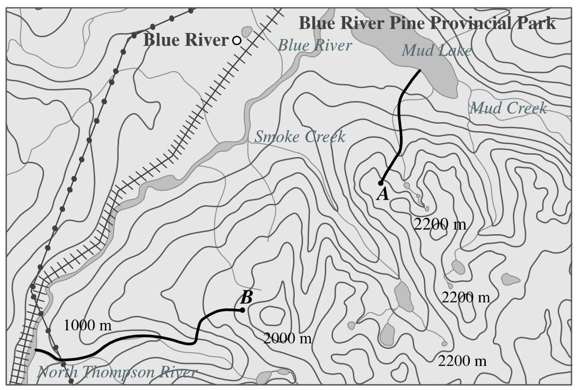
(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$ and

$D_{\mathbf{u}}f(60, 40) = \nabla f\,(60, 40) \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = \langle -0.6, -0.8\rangle \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60, 40, 966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60,40) = \langle -0.6, -0.8\rangle$ is the direction of largest slope with a rate of ascent given by $|\nabla f(60,40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by $\tan\theta = 1 \;\Rightarrow\; \theta = 45^\circ$.

41. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\, f(1,3) = f_x(1,3) = 3$ and $D_{\overrightarrow{AC}}\, f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle = \langle 3, 26\rangle$, and by definition, $D_{\overrightarrow{AD}}\, f(1,3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}\, f(1,3) = \langle 3, 26\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

42. The curves of steepest ascent or descent are perpendicular to all of the contour lines (see Figure 13) so we sketch curves beginning at A and B that head toward lower elevations, crossing each contour line at a right angle.



43. (a) $$\nabla(au+bv) = \left\langle \frac{\partial(au+bv)}{\partial x}, \frac{\partial(au+bv)}{\partial y}\right\rangle = \left\langle a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}, a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}\right\rangle = a\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle + b\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle$$
$$= a\,\nabla u + b\,\nabla v$$

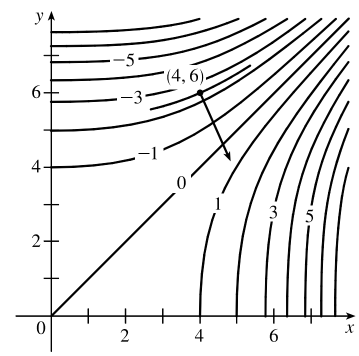
(b) $$\nabla(uv) = \left\langle v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}, v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right\rangle = v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle + u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$$

(c) $$\nabla\left(\frac{u}{v}\right) = \left\langle \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}, \frac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}\right\rangle = \frac{v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle - u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle}{v^2} = \frac{v\,\nabla u - u\,\nabla v}{v^2}$$

(d) $$\nabla u^n = \left\langle \frac{\partial(u^n)}{\partial x}, \frac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\frac{\partial u}{\partial x}, nu^{n-1}\frac{\partial u}{\partial y}\right\rangle = nu^{n-1}\,\nabla u$$

44. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing

function values, and with length equal to the maximum value of the directional derivative of f at $(4, 6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



45. $f(x, y) = x^3 + 5x^2y + y^3 \quad \Rightarrow$

$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \left\langle 3x^2 + 10xy, 5x^2 + 3y^2 \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2.$

Then

$$\begin{aligned} D_{\mathbf{u}}^2 f(x, y) &= D_{\mathbf{u}}\left[D_{\mathbf{u}} f(x, y)\right] = \nabla\left[D_{\mathbf{u}} f(x, y)\right] \cdot \mathbf{u} = \left\langle \tfrac{58}{5}x + 6y, 6x + \tfrac{24}{5}y \right\rangle \cdot \left\langle \tfrac{3}{5}, \tfrac{4}{5} \right\rangle \\ &= \tfrac{174}{25}x + \tfrac{18}{5}y + \tfrac{24}{5}x + \tfrac{96}{25}y = \tfrac{294}{25}x + \tfrac{186}{25}y \end{aligned}$$

and $D_{\mathbf{u}}^2 f(2, 1) = \frac{294}{25}(2) + \frac{186}{25}(1) = \frac{774}{25}$.

46. (a) From Equation 9 we have $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = f_x a + f_y b$ and from Exercise 39 we have

$$\begin{aligned} D_{\mathbf{u}}^2 f &= D_{\mathbf{u}}\left[D_{\mathbf{u}} f\right] = \nabla\left[D_{\mathbf{u}} f\right] \cdot \mathbf{u} \\ &= \langle f_{xx}a + f_{yx}b, f_{xy}a + f_{yy}b \rangle \cdot \langle a, b \rangle \\ &= f_{xx}a^2 + f_{yx}ab + f_{xy}ab + f_{yy}b^2 \end{aligned}$$

But $f_{yx} = f_{xy}$ by Clairaut's Theorem, so $D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$.

(b) $f(x, y) = xe^{2y} \quad \Rightarrow \quad f_x = e^{2y}$, $f_y = 2xe^{2y}$, $f_{xx} = 0$, $f_{xy} = 2e^{2y}$, $f_{yy} = 4xe^{2y}$ and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+6^2}}\langle 4, 6 \rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \langle a, b \rangle$. Then

$$\begin{aligned} D_{\mathbf{u}}^2 f &= f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2 \\ &= 0 \cdot \left(\tfrac{2}{\sqrt{13}}\right)^2 + 2 \cdot 2e^{2y}\left(\tfrac{2}{\sqrt{13}}\right)\left(\tfrac{3}{\sqrt{13}}\right) + 4xe^{2y}\left(\tfrac{3}{\sqrt{13}}\right)^2 = \tfrac{24}{13}e^{2y} + \tfrac{36}{13}xe^{2y} \end{aligned}$$

47. Let $F(x, y, z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x, y, z) = 4(x-2) \quad \Rightarrow \quad F_x(3, 3, 5) = 4$, $F_y(x, y, z) = 2(y-1) \quad \Rightarrow \quad F_y(3, 3, 5) = 4$, and $F_z(x, y, z) = 2(z-3) \quad \Rightarrow \quad F_z(3, 3, 5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3, 3, 5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \quad \Leftrightarrow$ $4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

48. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 + 1 \quad \Leftrightarrow \quad y^2 + z^2 - x = -1$ is a level surface of F.

$F_x(x,y,z) = -1 \quad \Rightarrow \quad F_x(3,1,-1) = -1, \;\; F_y(x,y,z) = 2y \quad \Rightarrow \quad F_y(3,1,-1) = 2,$ and $F_z(x,y,z) = 2z \quad \Rightarrow$

$F_z(3,1,-1) = -2$.

(a) By Equation 19, an equation of the tangent plane at $(3,1,-1)$ is $(-1)(x-3) + 2(y-1) + (-2)[z-(-1)] = 0$ or

$-x + 2y - 2z = 1$ or $x - 2y + 2z = -1$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{-1} = \dfrac{y-1}{2} = \dfrac{z-(-1)}{-2}$ or equivalently

$x - 3 = \dfrac{y-1}{-2} = \dfrac{z+1}{2}$ and parametric equations $x = 3 - t$, $y = 1 + 2t$, $z = -1 - 2t$.

49. Let $F(x,y,z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$.

(a) $\nabla F(2,2,1) = \langle 4, 8, 24 \rangle$ is a normal vector for the tangent plane at $(2,2,1)$, so an equation of the tangent plane is

$4(x-2) + 8(y-2) + 24(z-1) = 0$ or $4x + 8y + 24z = 48$ or equivalently $x + 2y + 6z = 12$.

(b) The normal line has direction $\nabla F(2,2,1) = \langle 4, 8, 24 \rangle$ or equivalently $\langle 1, 2, 6 \rangle$, so parametric equations are $x = 2 + t$,

$y = 2 + 2t$, $z = 1 + 6t$, and symmetric equations are $x - 2 = \dfrac{y-2}{2} = \dfrac{z-1}{6}$.

50. Let $F(x,y,z) = xy + yz + zx$. Then $xy + yz + zx = 5$ is a level surface of F and $\nabla F(x,y,z) = \langle y+z, x+z, x+y \rangle$.

(a) $\nabla F(1,2,1) = \langle 3, 2, 3 \rangle$ is a normal vector for the tangent plane at $(1,2,1)$, so an equation of the tangent plane

is $3(x-1) + 2(y-2) + 3(z-1) = 0$ or $3x + 2y + 3z = 10$.

(b) The normal line has direction $\langle 3, 2, 3 \rangle$, so parametric equations are $x = 1 + 3t$, $y = 2 + 2t$, $z = 1 + 3t$, and symmetric

equations are $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{3}$.

51. Let $F(x,y,z) = x + y + z - e^{xyz}$. Then $x + y + z = e^{xyz}$ is the level surface $F(x,y,z) = 0$,

and $\nabla F(x,y,z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$.

(a) $\nabla F(0,0,1) = \langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(0,0,1)$, so an equation of the tangent plane

is $1(x-0) + 1(y-0) + 1(z-1) = 0$ or $x + y + z = 1$.

(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = t$, $y = t$, $z = 1 + t$, and symmetric equations are

$x = y = z - 1$.

52. Let $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. Then $x^4 + y^4 + z^4 = 3x^2y^2z^2$ is the level surface $F(x,y,z) = 0$,

and $\nabla F(x,y,z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \rangle$.

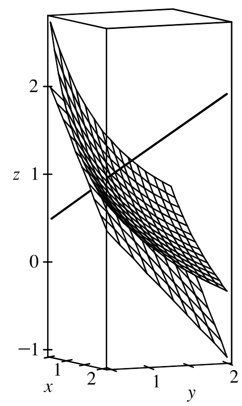
(a) $\nabla F(1,1,1) = \langle -2, -2, -2 \rangle$ or equivalently $\langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(1,1,1)$, so an equation

of the tangent plane is $1(x-1) + 1(y-1) + 1(z-1) = 0$ or $x + y + z = 3$.

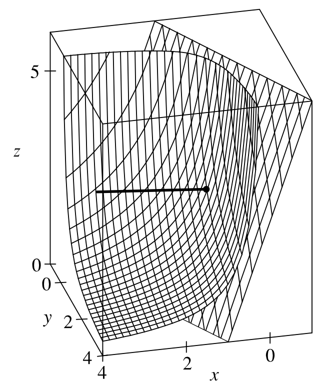
(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, and symmetric

equations are $x - 1 = y - 1 = z - 1$ or equivalently $x = y = z$.

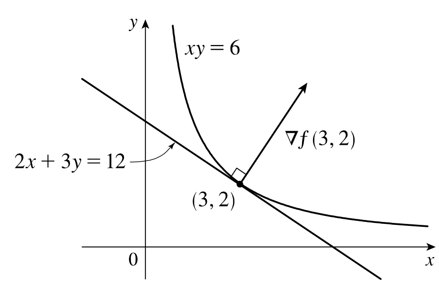
53. $F(x,y,z) = xy + yz + zx \quad\Rightarrow\quad xy + yz + zx = 3$ is the level surface $F(x,y,z) = 3$. $\nabla F(x,y,z) = \langle y+z, x+z, y+x\rangle \quad\Rightarrow$ $\nabla F(1,1,1) = \langle 2,2,2\rangle$, and an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$. The normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



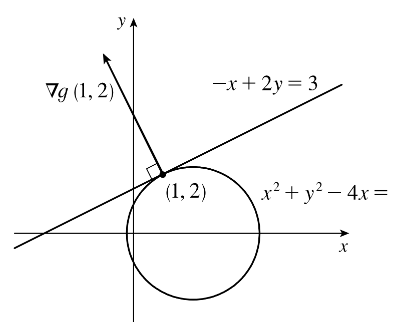
54. $F(x,y,z) = xyz \quad\Rightarrow\quad xyz = 6$ is the level surface $F(x,y,z) = 6$. $\nabla F(x,y,z) = \langle yz, xz, yx\rangle \quad\Rightarrow\quad \nabla F(1,2,3) = \langle 6,3,2\rangle$, and an equation of the tangent plane is $6x + 3y + 2z = 18$. The normal line is given by $\dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-3}{2}$ or $x = 1 + 6t,\ y = 2 + 3t,$ $z = 3 + 2t$. To graph the surface we solve for z: $z = \dfrac{6}{xy}$.



55. $f(x,y) = xy \quad\Rightarrow\quad \nabla f(x,y) = \langle y, x\rangle$ and $\nabla f(3,2) = \langle 2,3\rangle$. Since $\nabla f(3,2)$ is perpendicular to the tangent line, the tangent line has equation $\nabla f(3,2)\cdot\langle x-3, y-2\rangle = 0 \quad\Rightarrow\quad \langle 2,3\rangle\cdot\langle x-3, x-2\rangle = 0 \quad\Rightarrow$ $2(x-3) + 3(y-2) = 0 \quad\text{or}\quad 2x + 3y = 12$.



56. $g(x,y) = x^2 + y^2 - 4x \quad\Rightarrow\quad \nabla g(x,y) = \langle 2x - 4, 2y\rangle$ and $\nabla g(1,2) = \langle -2, 4\rangle$. Since $\nabla g(1,2)$ is perpendicular to the tangent line, the tangent line has equation $\nabla g(1,2)\cdot\langle x-1, y-2\rangle = 0 \quad\Rightarrow$ $\langle -2,4\rangle\cdot\langle x-1, y-2\rangle = 0 \quad\Rightarrow\quad -2(x-1) + 4(y-2) = 0 \quad\Leftrightarrow$ $-2x + 4y = 6$ or equivalently $-x + 2y = 3$.



57. $F(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Then $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is the level surface $F(x,y,z) = 1$ and

$\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2}\right\rangle$. Thus, an equation of the tangent plane at (x_0,y_0,z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence $\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

58. $F(x, y, z) = x^2/a^2 + y^2/b^2 - z^2/c^2$. Then $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ is the level surface $F(x, y, z) = 1$ and $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} - \dfrac{z_0^2}{c^2}\right) = 2$ or $\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y - \dfrac{z_0}{c^2}\,z = 1$.

59. $F(x, y, z) = x^2/a^2 + y^2/b^2 - z/c$. Then $z/c = x^2/a^2 + y^2/b^2$ is the level surface $F(x, y, z) = 0$ and $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$ or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z + z_0}{c}$.

60. Let $F(x, y, z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, 2y, 4z\rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0\rangle = k\,\langle 1, 2, 1\rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \;\Rightarrow\; x_0 = k/2$, $2y_0 = 2k \;\Rightarrow\; y_0 = k$, $4z_0 = k \;\Rightarrow\; z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so $(k/2)^2 + k^2 + 2(k/4)^2 = 1 \;\Rightarrow\; \frac{11}{8}k^2 = 1 \;\Rightarrow\; k^2 = \frac{8}{11} \;\Rightarrow\; k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points $\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.

61. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x, y, z) = x^2 - y^2 - z^2$ and $\nabla F\,(x, y, z) = \langle 2x, -2y, -2z\rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x, y, z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0\rangle = c\,\langle 1, 1, -1\rangle$ or equivalently $\langle x_0, -y_0, -z_0\rangle = k\,\langle 1, 1, -1\rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \;\Leftrightarrow\; -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

62. First note that the point $(1, 1, 2)$ is on both surfaces. The ellipsoid $3x^2 + 2y^2 + z^2 = 9$ is a level surface of $F(x, y, z) = 3x^2 + 2y^2 + z^2$ and $\nabla F(x, y, z) = \langle 6x, 4y, 2z\rangle$. A normal vector to the surface at $(1, 1, 2)$ is $\nabla F(1, 1, 2) = \langle 6, 4, 4\rangle$ and an equation of the tangent plane there is $6(x - 1) + 4(y - 1) + 4(z - 2) = 0$ or

$6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ and $\nabla G(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8\rangle$. A normal vector to the sphere at $(1, 1, 2)$ is $\nabla G(1, 1, 2) = \langle -6, -4, -4\rangle$ and the tangent plane there is $-6(x - 1) - 4(y - 1) - 4(z - 2) = 0$ or $3x + 2y + 2z = 9$. Since these tangent planes are identical, the surfaces are tangent to each other at the point $(1, 1, 2)$.

63. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. The cone $x^2 + y^2 = z^2$ is a level surface of $F(x, y, z) = x^2 + y^2 - z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, -2z\rangle$, so $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0\rangle$ is a normal vector to the cone at this point and an equation of the tangent plane there is $2x_0\,(x - x_0) + 2y_0\,(y - y_0) - 2z_0\,(z - z_0) = 0$ or $x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

64. Let (x_0, y_0, z_0) be a point on the sphere and $F(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla F(x, y, z) = \langle 2x, 2y, 2z\rangle$ and $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0\rangle$, so the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

65. Let $F(x, y, z) = x^2 + y^2 - z$. Then the paraboloid is the level surface $F(x, y, z) = 0$ and $\nabla F(x, y, z) = \langle 2x, 2y, -1\rangle$, so $\nabla F(1, 1, 2) = \langle 2, 2, -1\rangle$ is a normal vector to the surface. Thus the normal line at $(1, 1, 2)$ is given by $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 - t$. Substitution into the equation of the paraboloid $z = x^2 + y^2$ gives $2 - t = (1 + 2t)^2 + (1 + 2t)^2 \iff 2 - t = 2 + 8t + 8t^2 \iff 8t^2 + 9t = 0 \iff t(8t + 9) = 0$. Thus the line intersects the paraboloid when $t = 0$, corresponding to the given point $(1, 1, 2)$, or when $t = -\frac{9}{8}$, corresponding to the point $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$.

66. The ellipsoid $4x^2 + y^2 + 4z^2 = 12$ is a level surface of $F(x, y, z) = 4x^2 + y^2 + 4z^2$ and $\nabla F(x, y, z) = \langle 8x, 2y, 8z\rangle$, so $\nabla F(1, 2, 1) = \langle 8, 4, 8\rangle$ or equivalently $\langle 2, 1, 2\rangle$ is a normal vector to the surface. Thus, the normal line to the ellipsoid at $(1, 2, 1)$ is given by $x = 1 + 2t$, $y = 2 + t$, $z = 1 + 2t$. Substitution into the equation of the sphere gives $(1 + 2t)^2 + (2 + t)^2 + (1 + 2t)^2 = 102 \iff 6 + 12t + 9t^2 = 102 \iff 9t^2 + 12t - 96 = 0 \iff 3(t + 4)(3t - 8) = 0$. Thus, the line intersects the sphere when $t = -4$, corresponding to the point $(-7, -2, -7)$, and when $t = \frac{8}{3}$, corresponding to the point $\left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)$.

67. Let (x_0, y_0, z_0) be a point on the surface and $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. Then $\nabla F(x, y, z) = \left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}, \dfrac{1}{2\sqrt{z}} \right\rangle$ and $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{1}{2\sqrt{x_0}}, \dfrac{1}{2\sqrt{y_0}}, \dfrac{1}{2\sqrt{z_0}} \right\rangle$, so an equation of the tangent plane at the point is

$$\frac{x - x_0}{2\sqrt{x_0}} + \frac{y - y_0}{2\sqrt{y_0}} + \frac{z - z_0}{2\sqrt{z_0}} = 0 \quad\iff\quad \frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{x_0}{2\sqrt{x_0}} + \frac{y_0}{2\sqrt{y_0}} + \frac{z_0}{2\sqrt{z_0}} \quad\iff$$

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}}{2}$. But $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$, and $\sqrt{cz_0}$, respectively. (The x-intercept is found by setting $y=z=0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$, a constant.

68. The surface $xyz=1$ is a level surface of $F(x,y,z)=xyz$ and $\nabla F(x,y,z)=\langle yz,xz,xy\rangle$ is normal to the surface, so a normal vector for the tangent plane to the surface at (x_0,y_0,z_0) is $\langle y_0z_0,x_0z_0,x_0y_0\rangle$. An equation for the tangent plane there is $y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0 \;\Rightarrow\; y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0$ or $\dfrac{x}{x_0}+\dfrac{y}{y_0}+\dfrac{z}{z_0}=3$.

If (x_0,y_0,z_0) is in the first octant, then the tangent plane cuts off a pyramid in the first octant with vertices $(0,0,0)$, $(3x_0,0,0)$, $(0,3y_0,0)$, $(0,0,3z_0)$. The base in the xy-plane is a triangle with area $\frac{1}{2}(3x_0)(3y_0)$ and the height (along the z-axis) of the pyramid is $3z_0$. The volume of the pyramid for any point (x_0,y_0,z_0) on the surface $xyz=1$ in the first octant is $\frac{1}{3}\,(\text{base})\,(\text{height})=\frac{1}{3}\cdot\frac{1}{2}(3x_0)(3y_0)\cdot 3z_0=\frac{9}{2}x_0y_0z_0=\frac{9}{2}$ since $x_0y_0z_0=1$.

69. If $f(x,y,z)=z-x^2-y^2$ and $g(x,y,z)=4x^2+y^2+z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\mathbf{v}=\nabla f\times\nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z)=\langle -2x,-2y,1\rangle \;\Rightarrow\; \nabla f(-1,1,2)=\langle 2,-2,1\rangle$, and $\nabla g(x,y,z)=\langle 8x,2y,2z\rangle \;\Rightarrow$

$\nabla g(-1,1,2)=\langle -8,2,4\rangle$. Hence $\mathbf{v}=\nabla f\times\nabla g=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-2&1\\-8&2&4\end{vmatrix}=-10\,\mathbf{i}-16\,\mathbf{j}-12\,\mathbf{k}$.

Parametric equations are: $x=-1-10t,\; y=1-16t,\; z=2-12t$.

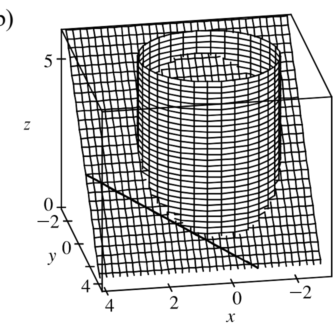
70. (a) Let $f(x,y,z)=y+z$ and $g(x,y,z)=x^2+y^2$. The plane is a level surface of f and the cylinder is a level surface of g. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\mathbf{v}=\nabla f\times\nabla g$ is parallel to the tangent line. We have

$\nabla f(x,y,z)=\langle 0,1,1\rangle \;\Rightarrow\; \nabla f(1,2,1)=\langle 0,1,1\rangle$, and

$\nabla g(x,y,z)=\langle 2x,2y,0\rangle \;\Rightarrow\; \nabla g(1,2,1)=\langle 2,4,0\rangle$. Hence

$\mathbf{v}=\nabla f\times\nabla g=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&1\\2&4&0\end{vmatrix}=-4\,\mathbf{i}+2\,\mathbf{j}-2\,\mathbf{k}$. So parametric equations of the desired tangent line are $x=1-4t,\; y=2+2t,\; z=1-2t$.

(b)



71. Parametric equations for the helix are $x=\cos\pi t,\;\; y=\sin\pi t,\;\; z=t$, and substituting into the equation of the paraboloid gives $t=\cos^2\pi t+\sin^2\pi t \;\Rightarrow\; t=1$. Thus the helix intersects the surface at the point $(\cos\pi,\sin\pi,1)=(-1,0,1)$. Here $\mathbf{r}'(t)=\langle -\pi\sin\pi t,\pi\cos\pi t,1\rangle$, so the tangent vector to the helix at that point is $\mathbf{r}'(1)=\langle -\pi\sin\pi,\pi\cos\pi,1\rangle=\langle 0,-\pi,1\rangle$.

[continued]

The paraboloid $z = x^2 + y^2 \quad \Leftrightarrow \quad x^2 + y^2 - z = 0$ is a level surface of $F(x, y, z) = x^2 + y^2 - z$ and $\nabla F(x, y, z) = \langle 2x, 2y, -1 \rangle$, so a normal vector to the tangent plane at $(-1, 0, 1)$ is $\nabla F(-1, 0, 1) = \langle -2, 0, -1 \rangle$. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(-1, 0, 1)$ is given by

$$\cos\theta = \frac{\langle 0, -\pi, 1\rangle \cdot \langle -2, 0, -1\rangle}{|\langle 0, -\pi, 1\rangle|\,|\langle -2, 0, -1\rangle|} = \frac{0 + 0 - 1}{\sqrt{0 + \pi^2 + 1}\,\sqrt{4 + 0 + 1}} = \frac{-1}{\sqrt{5(\pi^2 + 1)}} \quad \Rightarrow$$

$\theta = \cos^{-1} \dfrac{-1}{\sqrt{5(\pi^2 + 1)}} \approx 97.8^\circ$. Because $\nabla F(-1, 0, 1)$ is perpendicular to the tangent plane, the angle of intersection between the helix and the paraboloid is approximately $97.8^\circ - 90^\circ = 7.8^\circ$.

72. Parametric equations for the helix are $x = \cos(\pi t/2)$, $\;y = \sin(\pi t/2)$, $\;z = t$, and substituting into the equation of the sphere gives $\cos^2(\pi t/2) + \sin^2(\pi t/2) + t^2 = 2 \quad \Rightarrow \quad 1 + t^2 = 2 \quad \Rightarrow \quad t = \pm 1$. Thus the helix intersects the sphere at two points: $(\cos(\pi/2), \sin(\pi/2), 1) = (0, 1, 1)$, when $t = 1$, and $(\cos(-\pi/2), \sin(-\pi/2), -1) = (0, -1, -1)$, when $t = -1$. Here $\mathbf{r}'(t) = \left\langle -\frac{\pi}{2}\sin(\pi t/2), \frac{\pi}{2}\cos(\pi t/2), 1 \right\rangle$, so the tangent vector to the helix at $(0, 1, 1)$ is $\mathbf{r}'(1) = \langle -\pi/2, 0, 1 \rangle$. The sphere $x^2 + y^2 + z^2 = 2$ is a level surface of $F(x, y, z) = x^2 + y^2 + z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, 2z \rangle$, so a normal vector to the tangent plane at $(0, 1, 1)$ is $\nabla F(0, 1, 1) = \langle 0, 2, 2 \rangle$. As in Exercise 71, the angle of intersection between the helix and the sphere is the angle between the tangent vector to the helix and the tangent plane to the sphere. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(0, 1, 1)$ is given by

$$\cos\theta = \frac{\langle -\pi/2, 0, 1\rangle \cdot \langle 0, 2, 2\rangle}{|\langle -\pi/2, 0, 1\rangle|\,|\langle 0, 2, 2\rangle|} = \frac{2}{\sqrt{(\pi^2/4) + 1}\,\sqrt{8}} = \frac{2}{\sqrt{2\pi^2 + 8}} \quad \Rightarrow \quad \theta = \cos^{-1} \frac{2}{\sqrt{2\pi^2 + 8}} \approx 67.7^\circ$$

Because $\nabla F(0, 1, 1)$ is perpendicular to the tangent plane, the angle between $\mathbf{r}'(1)$ and the tangent plane is approximately $90^\circ - 67.7^\circ = 22.3^\circ$.

At $(0, -1, -1)$, $\;\mathbf{r}'(-1) = \langle \pi/2, 0, 1 \rangle$ and $\nabla F(0, -1, -1) = \langle 0, -2, -2 \rangle$, and the angle ϕ between these vectors is given by $\cos\phi = \dfrac{\langle \pi/2, 0, 1\rangle \cdot \langle 0, -2, -2\rangle}{|\langle \pi/2, 0, 1\rangle|\,|\langle 0, -2, -2\rangle|} = \dfrac{-2}{\sqrt{2\pi^2 + 8}} \quad \Rightarrow \quad \phi = \cos^{-1} \dfrac{-2}{\sqrt{2\pi^2 + 8}} \approx 112.3^\circ$. Thus the angle between the helix and the sphere at $(0, -1, -1)$ is approximately $112.3^\circ - 90^\circ = 22.3^\circ$. (By symmetry, we would expect the angles to be identical.)

73. The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad \Leftrightarrow \quad \langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0$ at $P \quad \Leftrightarrow \quad F_x G_x + F_y G_y + F_z G_z = 0$ at P.

74. Here $F(x, y, z) = x^2 + y^2 - z^2$ and $G(x, y, z) = x^2 + y^2 + z^2 - r^2$, so $z^2 = x^2 + y^2$ is a level surface of F and $x^2 + y^2 + z^2 = r^2$ is a level surface of $G \quad \Rightarrow \quad \nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x, y, z) lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin)

lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

75. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\, D_{\mathbf{u}}\, f - b\, D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\, D_{\mathbf{v}}\, f - c\, D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

76. (a) The function $f(x, y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 14.2.9.)

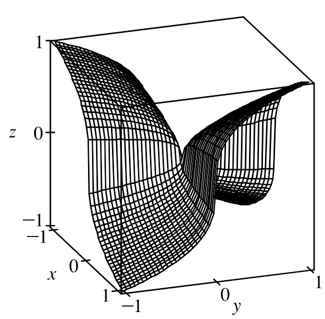
$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = 0,$$

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = 0.$$

Therefore, $f_x(0, 0)$ and $f_y(0, 0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$D_{\mathbf{u}}\, f(0, 0) = \lim\limits_{h\to 0} \dfrac{f(0 + ha, 0 + hb) - f(0, 0)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{(ha)(hb)}}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\mathbf{u}}\, f(0, 0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

77. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 14.4.7 we have $\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y\rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0)\,, f_y(x_0, y_0)\rangle = \nabla f(\mathbf{x}_0)$. Substituting into 14.4.7 gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2\rangle \cdot \langle \Delta x, \Delta y\rangle$ or $\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \dfrac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$

14.7 Maximum and Minimum Values

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. (a) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D(0,2) < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D(0,2) > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \;\Rightarrow\; 3x(x^3 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minimums at or near $(-1,\pm 1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \;\Rightarrow\; f_x(x,y) = 3 - 3x^2,\; f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \;\Rightarrow\; x = \pm 1$ and $-4y + 4y^3 = 0 \;\Rightarrow$ $y(y^2 - 1) = 0 \;\Rightarrow\; y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, 1)$, $(\pm 1, -1)$.

The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x.$

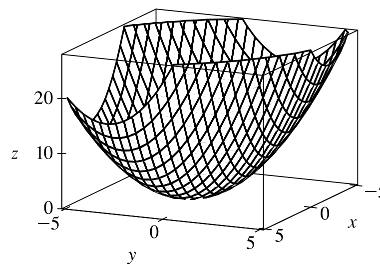
We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D > 0,\ f_{xx} < 0 \quad \Rightarrow \quad f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D > 0,\ f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D > 0,\ f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1,-1)$

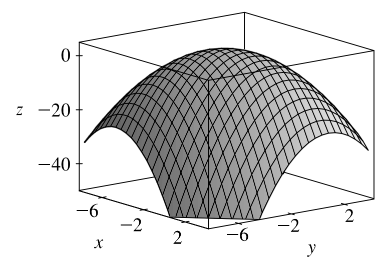
5. $f(x,y) = x^2 + xy + y^2 + y \quad \Rightarrow \quad f_x = 2x + y,\ f_y = x + 2y + 1,\ f_{xx} = 2,\ f_{xy} = 1,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = -2x$, and substitution into $f_y = x + 2y + 1 = 0$ gives $x + 2(-2x) + 1 = 0 \quad \Rightarrow \quad -3x = -1 \quad \Rightarrow \quad x = \frac{1}{3}$.

Then $y = -\frac{2}{3}$ and the only critical point is $\left(\frac{1}{3}, -\frac{2}{3}\right)$.

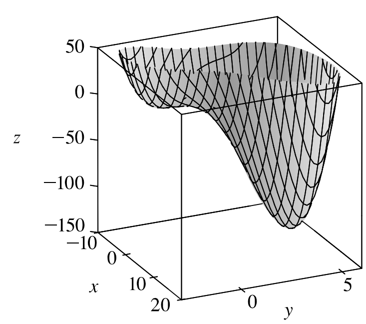
$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(2) - (1)^2 = 3$, and since $D\left(\frac{1}{3}, -\frac{2}{3}\right) = 3 > 0$ and $f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$, $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$ is a local minimum by the Second Derivatives Test.



6. $f(x,y) = xy - 2x - 2y - x^2 - y^2 \quad \Rightarrow \quad f_x = y - 2 - 2x$, $f_y = x - 2 - 2y,\ f_{xx} = -2,\ f_{xy} = 1,\ f_{yy} = -2$. Then $f_x = 0$ implies $y = 2x + 2$, and substitution into $f_y = 0$ gives $x - 2 - 2(2x + 2) = 0 \quad \Rightarrow$ $-3x = 6 \quad \Rightarrow \quad x = -2$. Then $y = -2$ and the only critical point is $(-2,-2)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - 1^2 = 3$, and since $D(-2,-2) = 3 > 0$ and $f_{xx}(-2,-2) = -2 < 0$, $f(-2,-2) = 4$ is a local maximum by the Second Derivatives Test.



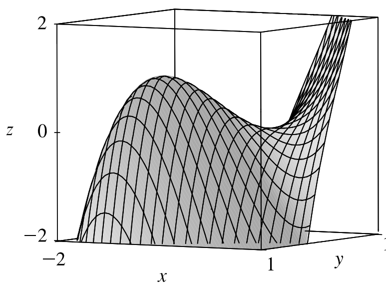
7. $f(x,y) = 2x^2 - 8xy + y^4 - 4y^3 \quad \Rightarrow \quad f_x = 4x - 8y$, $f_y = -8x + 4y^3 - 12y^2$, $f_{xx} = 4$, $f_{xy} = -8$, $f_{yy} = 12y^2 - 24y$. Then $f_x = 0$ implies $x = 2y$, and sustitution into $f_y = 0$ gives $-16y + 4y^3 - 12y^2 = 0 \quad \Rightarrow \quad 4y(y^2 - 3y - 4) = 0 \quad \Rightarrow \quad y = 0, -1, 4$. Then $x = 0, -2, 8$ and the critical points are $(0,0)$, $(-2,-1)$, and $(8,4)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$. $D(0,0) = 4(0) - 64 = -64 < 0$, so $(0,0)$ is a saddle point. $D(-2,-1) = 4(36) - 64 = 80 > 0$ and



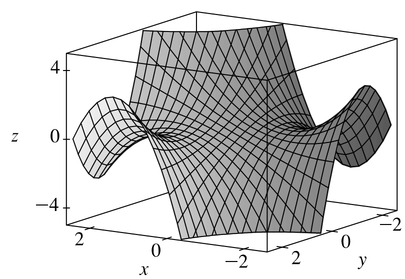
$f_{xx}(-2,-1) = 4 > 0$, so $f(-2,-1) = -3$ is a local minimum by the Second Derivatives Test.

$D(8,4) = 4(96) - 64 = 320 > 0$ and $f_{xx}(8,4) = 4 > 0$, so $f(8,4) = -128$ is a local minimum.

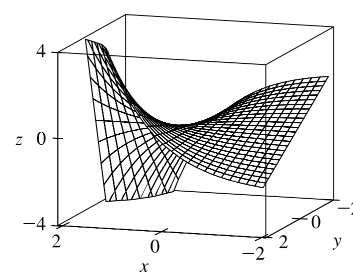
8. $f(x,y) = x^3 + y^3 + 3xy \quad \Rightarrow \quad f_x = 3x^2 + 3y,\ f_y = 3y^2 + 3x,$ $f_{xx} = 6x,\ f_{xy} = 3,\ f_{yy} = 6y$. Then $f_x = 0$ implies $y = -x^2$ and substitution into $f_y = 0$ gives $3(-x^2)^2 + 3x = 3x^4 + 3x = 0 \quad \Rightarrow$ $3x(x^3 + 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = -1$. Thus, the critical points are $(0,0)$ and $(-1,-1)$. $D(x,y) = (6x)(6y) - 3^2 = 36xy - 9$. $D(0,0) = -9 < 0$, so $(0,0)$ is a saddle point. $D(-1,-1) = 27 > 0$ and $f_{xx}(-1,-1) = -6 < 0$, so $f(-1,-1) = 1$ is a local maximum by the Second Derivatives Test.



9. $f(x,y) = (x-y)(1-xy) = x - y - x^2y + xy^2 \quad \Rightarrow \quad f_x = 1 - 2xy + y^2,\ f_y = -1 - x^2 + 2xy,\ f_{xx} = -2y,$ $f_{xy} = -2x + 2y,\ f_{yy} = 2x$. Then $f_x = 0$ implies $1 - 2xy + y^2 = 0$ and $f_y = 0$ implies $-1 - x^2 + 2xy = 0$. Adding the two equations gives $1 + y^2 - 1 - x^2 = 0 \quad \Rightarrow \quad y^2 = x^2 \quad \Rightarrow \quad y = \pm x$, but if $y = -x$ then $f_x = 0$ implies $1 + 2x^2 + x^2 = 0 \quad \Rightarrow \quad 3x^2 = -1$ which has no real solution. If $y = x$, then substitution into $f_x = 0$ gives $1 - 2x^2 + x^2 = 0 \quad \Rightarrow$ $x^2 = 1 \quad \Rightarrow \quad x = \pm 1$, so the critical points are $(1,1)$ and $(-1,-1)$. Now $D(1,1) = (-2)(2) - 0^2 = -4 < 0$ and $D(-1,-1) = (2)(-2) - 0^2 = -4 < 0$, so $(1,1)$ and $(-1,-1)$ are saddle points.

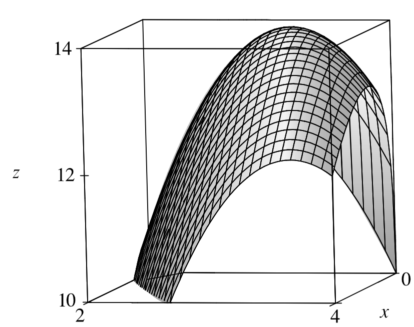


10. $f(x,y) = y(e^x - 1) \quad \Rightarrow \quad f_x = ye^x,\ f_y = e^x - 1,\ f_{xx} = ye^x,$ $f_{xy} = e^x,\ f_{yy} = 0$. Because e^x is never zero, $f_x = 0$ only when $y = 0$, and $f_y = 0$ when $e^x = 1 \quad \Rightarrow \quad x = 0$, so the only critical point is $(0,0)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (ye^x)(0) - (e^x)^2 = -e^{2x}$, and since $D(0,0) = -1 < 0$, $(0,0)$ is a saddle point.



11. $f(x,y) = y\sqrt{x} - y^2 - 2x + 7y \quad \Rightarrow \quad f_x = \frac{1}{2}yx^{-1/2} - 2,$ $f_y = \sqrt{x} - 2y + 7,\ f_{xx} = -\frac{1}{4}yx^{-3/2},\ f_{xy} = \frac{1}{2}x^{-1/2},\ f_{yy} = -2.$ Then $f_x = 0 \quad \Rightarrow \quad y = 4\sqrt{x}$ and substitution into $f_y = 0$ gives $\sqrt{x} - 2\left(4\sqrt{x}\right) + 7 = -7\sqrt{x} + 7 = 0 \quad \Rightarrow \quad x = 1$, so the only critical point is $(1,4)$.

$$D(x,y) = -\frac{1}{4}yx^{-3/2}(-2) - \left(\frac{1}{2}x^{-1/2}\right)^2 = \frac{1}{2}yx^{-3/2} - \frac{1}{4x}.$$



$D(1,4) = \frac{7}{4} > 0$, and $f_{xx}(1,4) = -1 < 0$, so $f(1,4) = 14$ is a local maximum by the Second Derivatives Test.

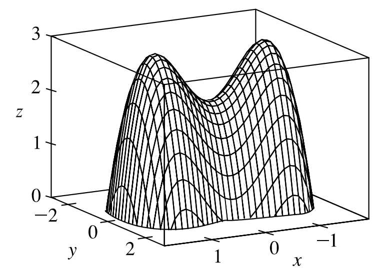
12. $f(x,y) = 2 - x^4 + 2x^2 - y^2 \quad\Rightarrow\quad f_x = -4x^3 + 4x,\ f_y = -2y,\ f_{xx} = -12x^2 + 4,\ f_{xy} = 0,\ f_{yy} = -2$. Then $f_x = 0$ implies $-4x(x^2 - 1) = 0$, so $x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $y = 0$. Thus the critical points are $(0,0)$, $(\pm 1, 0)$.

$D(0,0) = (4)(-2) - 0^2 = -8 < 0$, so $(0,0)$ is a saddle point.

$D(1,0) = D(-1,0) = (-8)(-2) - (0)^2 = 16 > 0$, and

$f_{xx}(1,0) = f_{xx}(-1,0) = -8 < 0$, so $f(1,0) = 3$ and $f(-1,0) = 3$

are local maximums.

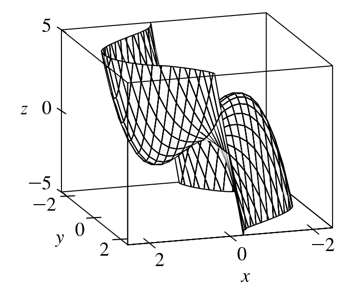


13. $f(x,y) = x^3 - 3x + 3xy^2 \quad\Rightarrow\quad f_x = 3x^2 - 3 + 3y^2,\ f_y = 6xy,\ f_{xx} = 6x,\ f_{xy} = 6y,\ f_{yy} = 6x$. Then $f_y = 0$ implies $x = 0$ or $y = 0$. If $x = 0$, substitution into $f_x = 0$ gives $3y^2 = 3 \quad\Rightarrow\quad y = \pm 1$, and if $y = 0$, substitution into $f_x = 0$ gives $x = \pm 1$. Thus the critical points are $(0, \pm 1)$ and $(\pm 1, 0)$.

$D(0, \pm 1) = 0 - 36 < 0$, so $(0, \pm 1)$ are saddle points.

$D(\pm 1, 0) = 36 - 0 > 0$, $f_{xx}(1,0) = 6 > 0$, and $f_{xx}(-1,0) = -6 < 0$,

so $f(1,0) = -2$ is a local minimum and $f(-1,0) = 2$ is a local maximum.



14. $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x \quad\Rightarrow\quad f_x = 3x^2 - 6x - 9,\ f_y = 3y^2 - 6y,\ f_{xx} = 6x - 6,\ f_{xy} = 0,\ f_{yy} = 6y - 6$. Then $f_x = 0$ implies $3(x+1)(x-3) = 0 \quad\Rightarrow\quad x = -1$ or $x = 3$, and $f_y = 0$ implies $3y(y-2) = 0 \quad\Rightarrow\quad y = 0$ or $y = 2$. Thus the critical points are $(-1,0)$, $(-1,2)$, $(3,0)$, and $(3,2)$. $\quad D(-1,2) = (-12)(6) - (0)^2 = -72 < 0$ and

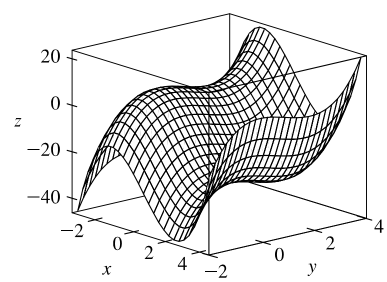
$D(3,0) = (12)(-6) - (0)^2 = -72 < 0$, so $(-1,2)$ and $(3,0)$ are

saddle points. $\quad D(-1,0) = (-12)(-6) - (0)^2 = 72 > 0$ and

$f_{xx}(-1,0) = -12 < 0$, so $f(-1,0) = 5$ is a local maximum.

$D(3,2) = (12)(6) - (0)^2 = 72 > 0$ and $f_{xx}(3,2) = 12 > 0$, so

$f(3,2) = -31$ is a local minimum.



15. $f(x,y) = x^4 - 2x^2 + y^3 - 3y \quad\Rightarrow\quad f_x = 4x^3 - 4x,\ f_y = 3y^2 - 3,\ f_{xx} = 12x^2 - 4,\ f_{xy} = 0,\ f_{yy} = 6y$. Then $f_x = 0$ implies $4x(x^2 - 1) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $3(y^2 - 1) = 0 \quad\Rightarrow\quad y = \pm 1$.

Thus there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

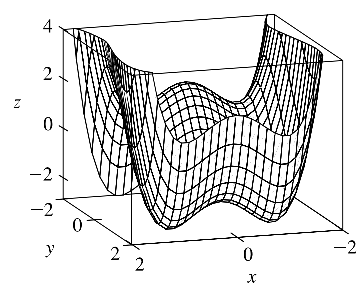
$D(0,1) = (-4)(6) - (0)^2 = -24 < 0$ and

$D(\pm 1, -1) = (8)(-6) = -48 < 0$, so $(0,1)$ and $(\pm 1, -1)$ are saddle

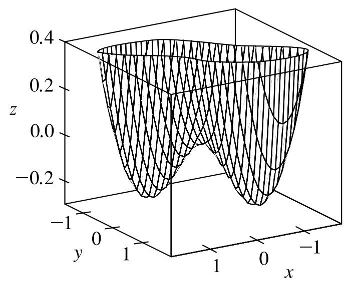
points. $\quad D(0,-1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0,-1) = -4 < 0$, so

$f(0,-1) = 2$ is a local maximum. $\quad D(\pm 1, 1) = (8)(6) = 48 > 0$ and

$f_{xx}(\pm 1, 1) = 8 > 0$, so $f(\pm 1, 1) = -3$ are local minimums.

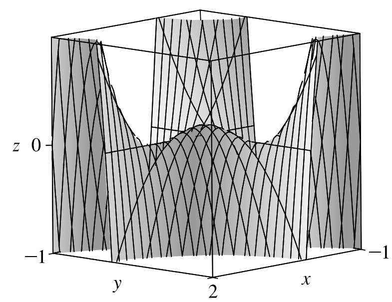


16. $f(x,y) = x^2 + y^4 + 2xy \quad \Rightarrow \quad f_x = 2x + 2y,\ f_y = 4y^3 + 2x,\ f_{xx} = 2,\ f_{xy} = 2,\ f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$, and substitution into $f_y = 4y^3 + 2x = 0$ gives $-4x^3 + 2x = 0 \quad \Rightarrow \quad 2x\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. Thus the critical points are $(0,0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Now



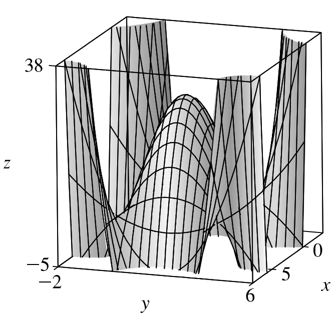
$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(12y^2) - (2)^2 = 24y^2 - 4$,
so $D(0,0) = -4 < 0$ and $(0,0)$ is a saddle point.
$D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 24\left(\frac{1}{2}\right) - 4 = 8 > 0$ and
$f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 > 0$, so $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$
and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$ are local minimums.

17. $f(x,y) = xy - x^2y - xy^2 \quad \Rightarrow \quad f_x = y - 2xy - y^2$,
$f_y = x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{xy} = 1 - 2x - 2y$, $f_{yy} = -2x$. Then
$f_x = y - 2xy - y^2 = 0 \quad \Rightarrow \quad y(1 - 2x - y) = 0 \quad \Rightarrow \quad y = 0$ or
$y = 1 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x - x^2 = 0 \quad \Rightarrow$
$x = 0$ or $x = 1$. Next, substituting $y = 1 - 2x$ into $f_y = 0$ gives
$x - x^2 - 2x(1 - 2x) = 0 \quad \Rightarrow \quad 3x^2 - x = 0 \quad \Rightarrow \quad x = 0$ or $x = \frac{1}{3}$.



Thus, the critical points are $(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.
$D(x,y) = (-2y)(-2x) - (1 - 2x - 2y)^2 = 4xy - (1 - 2x - 2y)^2$. $D(0,0) = D(1,0) = D(0,1) = -1 < 0$, so $(0,0)$, $(1,0)$, and $(0,1)$ are saddle points. $D\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$, so $f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$ is a local maximum by the Second Derivatives Test.

18. $f(x,y) = (6x - x^2)(4y - y^2) \quad \Rightarrow \quad f_x = (6 - 2x)(4y - y^2)$,
$f_y = (6x - x^2)(4 - 2y)$, $f_{xx} = -2(4y - y^2)$, $f_{xy} = (6 - 2x)(4 - 2y)$,
$f_{yy} = -2(6x - x^2)$. Then $f_x = 0 \quad \Rightarrow \quad 6 - 2x = 0 \quad \Rightarrow \quad x = 3$ or
$4y - y^2 = 0 \quad \Rightarrow \quad y = 0$ or $y = 4$. Substituting $x = 3$ into $f_y = 0 \quad \Rightarrow$
$y = 2$. Substituting $y = 0$ into $f_y = 0 \quad \Rightarrow \quad x = 0$ or $x = 6$. $y = 4 \quad \Rightarrow$
$x = 0$ or $x = 6$ also. This gives critical points $(3,2)$, $(0,0)$, $(6,0)$, $(0,4)$,

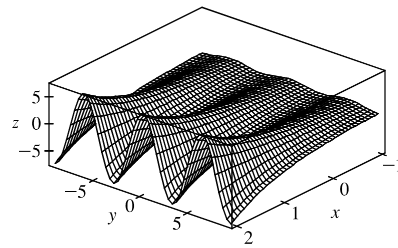


and $(6,4)$. $D(x,y) = 4(4y - y^2)(6x - x^2) - [(6 - 2x)(4 - 2y)]^2$.
$D(0,0) = D(6,0) = D(0,4) = D(6,4) = -576 < 0$, so $(0,0)$, $(6,0)$, $(0,4)$, and $(6,4)$ are saddle points.
$D(3,2) = 144 > 0$ and $f_{xx}(3,2) = -8 < 0$, so $f(3,2) = 36$ is a local maximum by the Second Derivatives Test.

19. $f(x,y) = e^x \cos y \quad \Rightarrow \quad f_x = e^x \cos y,\ f_y = -e^x \sin y.$

Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



20. $f(x,y) = (x^2+y^2)e^{-x} \quad \Rightarrow \quad f_x = (x^2+y^2)(-e^{-x}) + e^{-x}(2x) = (2x - x^2 - y^2)e^{-x},\quad f_y = 2ye^{-x},$

$f_{xx} = (2x - x^2 - y^2)(-e^{-x}) + e^{-x}(2-2x) = (x^2+y^2-4x+2)e^{-x},\quad f_{xy} = -2ye^{-x},\quad f_{yy} = 2e^{-x}$. Then $f_y = 0$

implies $y = 0$ and substituting into $f_x = 0$ gives $(2x - x^2)e^{-x} = 0 \quad \Rightarrow$

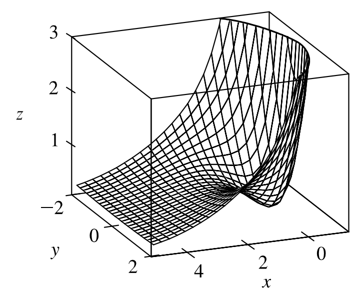
$x(2-x) = 0 \quad \Rightarrow \quad x = 0$ or $x = 2$, so the critical points are $(0,0)$ and

$(2,0)$. $\quad D(0,0) = (2)(2) - (0)^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, so

$f(0,0) = 0$ is a local minimum.

$D(2,0) = (-2e^{-2})(2e^{-2}) - (0)^2 = -4e^{-4} < 0$ so $(2,0)$ is a saddle

point.



21. $f(x,y) = y^2 - 2y\cos x \quad \Rightarrow \quad f_x = 2y\sin x,\ f_y = 2y - 2\cos x,$

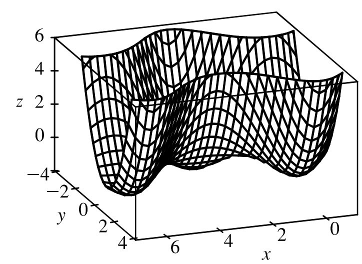
$f_{xx} = 2y\cos x,\ f_{xy} = 2\sin x,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or

$\sin x = 0 \quad \Rightarrow \quad x = 0, \pi,$ or 2π for $-1 \leq x \leq 7$. Substituting $y = 0$ into

$f_y = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$

into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$.

Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.



$D\left(\frac{\pi}{2}, 0\right) = D\left(\frac{3\pi}{2}, 0\right) = -4 < 0$ so $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are saddle points. $D(0,1) = D(\pi,-1) = D(2\pi,1) = 4 > 0$ and

$f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minimums.

22. $f(x,y) = \sin x\,\sin y \quad \Rightarrow \quad f_x = \cos x\,\sin y,\ f_y = \sin x\,\cos y,\ f_{xx} = -\sin x\,\sin y,\ f_{xy} = \cos x\,\cos y,$

$f_{yy} = -\sin x\,\sin y$. Here we have $-\pi < x < \pi$ and $-\pi < y < \pi$, so $f_x = 0$ implies $\cos x = 0$ or $\sin y = 0$. If $\cos x = 0$

then $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y = 0$ then $y = 0$. Substituting $x = \pm\frac{\pi}{2}$ into $f_y = 0$ gives $\cos y = 0 \quad \Rightarrow \quad y = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and

substituting $y = 0$ into $f_y = 0$ gives $\sin x = 0 \quad \Rightarrow \quad x = 0$. Thus the critical points are $\left(-\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, and $(0,0)$.

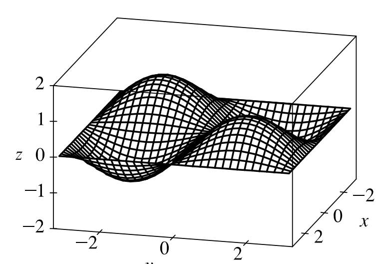
$D(0,0) = -1 < 0$ so $(0,0)$ is a saddle point.

$D\left(-\frac{\pi}{2}, \pm\frac{\pi}{2}\right) = D\left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right) = 1 > 0$ and

$f_{xx}\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1 < 0$ while

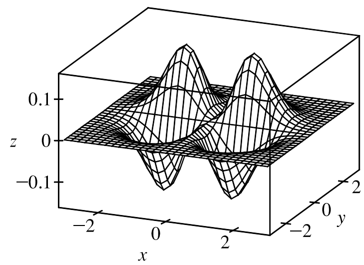
$f_{xx}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1 > 0$, so $f\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 1$

are local maximums and $f\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1$ are local minimums.



23. $f(x,y) = x^2 + 4y^2 - 4xy + 2 \quad \Rightarrow \quad f_x = 2x - 4y$, $f_y = 8y - 4x$, $f_{xx} = 2$, $f_{xy} = -4$, $f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0, \frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \geq 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0, \frac{1}{2}x_0\right) = 2$ are all local (and absolute) minimums.

24. $f(x,y) = x^2ye^{-x^2-y^2} \quad \Rightarrow$



$f_x = x^2ye^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$,

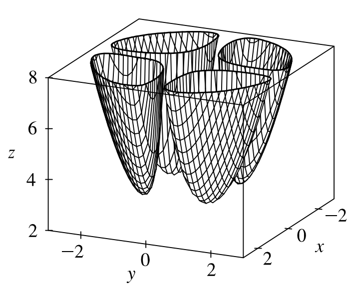
$f_y = x^2ye^{-x^2-y^2}(-2y) + x^2e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$,

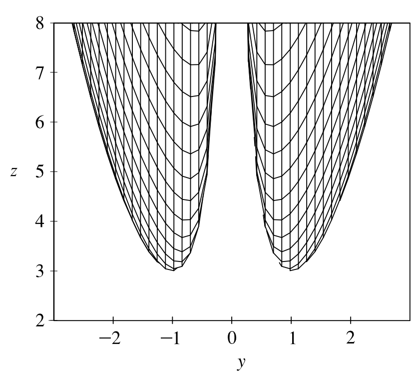
$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2-y^2}$,

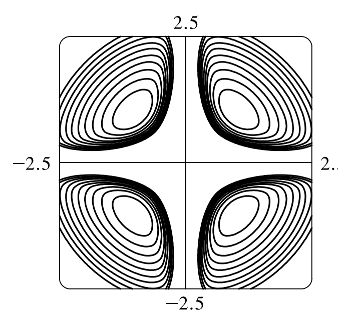
$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy} = 2x^2y(2y^2-3)e^{-x^2-y^2}$.

$f_x = 0$ implies $x = 0$, $y = 0$, or $x = \pm 1$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0, y)$ are critical points. If $y = 0$ then $f_y = 0 \quad \Rightarrow \quad x^2e^{-x^2} = 0 \quad \Rightarrow \quad x = 0$, so $(0,0)$ (already included above) is a critical point. If $x = \pm 1$ then $(1-2y^2)e^{-1-y^2} = 0 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$, so $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ are critical points. Now $D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}\,e^{-3/2} < 0$ and $D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while $f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0,y)$ we have $D(0,y) = 0$, so the Second Derivatives Test gives no information. However, if $y > 0$ then $x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0,y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0,y) = 0$, $y < 0$ are local maximum values, and $(0,0)$ is a saddle point.

25. $f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$



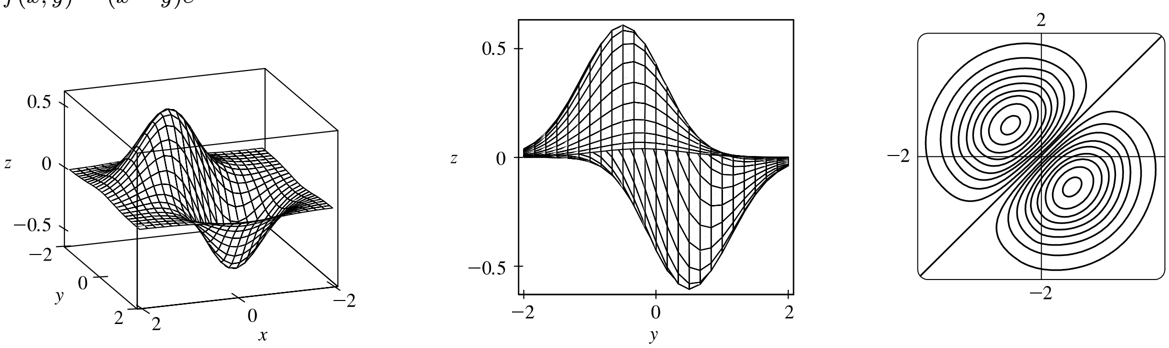




From the graphs, there appear to be local minimums of about $f(1, \pm 1) = f(-1, \pm 1) \approx 3$ (and no local maximums or saddle points). $f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{xy} = 4x^{-3}y^{-3}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$. Then

$f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1, 1)$, $(1, -1)$,$(-1, 1)$ and $(-1, -1)$. Now $D(1, \pm 1) = D(-1, \pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1, \pm 1) = f(-1, \pm 1) = 3$ are local minimums.

26. $f(x, y) = (x - y)e^{-x^2 - y^2}$

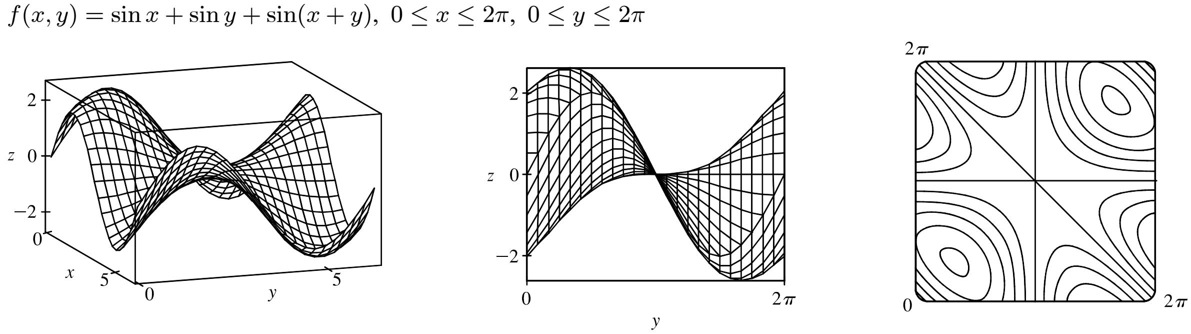


From the graphs, there appears to be a local maximum of about $f(0.5, -0.5) \approx 0.6$ and a local minimum of about $f(-0.5, 0.5) \approx -0.6$.

$f_x = (x - y)e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(1) = e^{-x^2-y^2}(1 - 2x^2 + 2xy)$,

$f_y = (x - y)e^{-x^2-y^2}(-2y) + e^{-x^2-y^2}(-1) = -e^{-x^2-y^2}(1 - 2y^2 + 2xy)$, $f_{xx} = 2e^{-x^2-y^2}(2x^3 - 3x + y - 2x^2y)$, $f_{xy} = 2e^{-x^2-y^2}(x - y + 2x^2y - 2xy^2)$, $f_{yy} = -2e^{-x^2-y^2}(2y^3 - 3y + x - 2xy^2)$. Then $f_x = 0$ implies $1 - 2x^2 + 2xy = 0$ and $f_y = 0$ implies $1 - 2y^2 + 2xy = 0$. Subtracting these two equations gives $-2x^2 + 2y^2 = 0 \quad \Rightarrow \quad y = \pm x$. If $y = x$ then substituting into $f_x = 0$ gives $1 - 2x^2 + 2x^2 = 0$, an impossibility. Substituting $y = -x$ gives $1 - 2x^2 - 2x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{4} \quad \Rightarrow \quad x = \pm\frac{1}{2}$. Thus the critical points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Now $D\left(\frac{1}{2}, -\frac{1}{2}\right) = (-3e^{-1/2})(-3e^{-1/2}) - (e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(\frac{1}{2}, -\frac{1}{2}\right) = -3e^{-1/2} < 0$, so $f\left(\frac{1}{2}, -\frac{1}{2}\right) = e^{-1/2} \approx 0.607$ is a local maximum, and $D\left(-\frac{1}{2}, \frac{1}{2}\right) = (3e^{-1/2})(3e^{-1/2}) - (-e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(-\frac{1}{2}, \frac{1}{2}\right) = 3e^{-1/2} > 0$, so $f\left(-\frac{1}{2}, \frac{1}{2}\right) = -e^{-1/2} \approx -0.607$ is a local minimum.

27. $f(x, y) = \sin x + \sin y + \sin(x + y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$



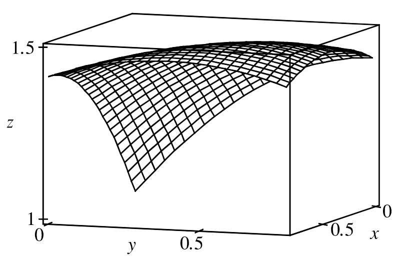
From the graphs it appears that f has a local maximum at about $(1, 1)$ with value approximately 2.6, a local minimum

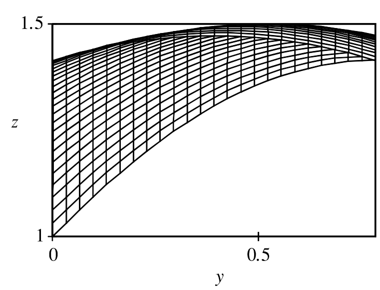
at about $(5, 5)$ with value approximately -2.6, and a saddle point at about $(3, 3)$.

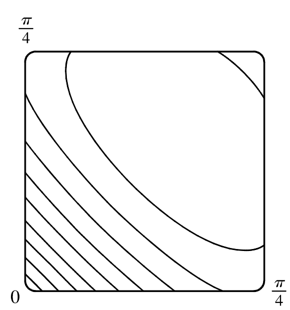
$f_x = \cos x + \cos(x + y)$, $f_y = \cos y + \cos(x + y)$, $f_{xx} = -\sin x - \sin(x + y)$, $f_{yy} = -\sin y - \sin(x + y)$, $f_{xy} = -\sin(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, giving the critical points (π, π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π, π). Now $D(x, y) = \sin x\, \sin y + \sin x\, \sin(x + y) + \sin y\, \sin(x + y)$. So $D(\pi, \pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x, x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\, \cos x = 2\sin x(1 + \cos x)$, and $f(x, x) > 0$ for $0 < x < \pi$ while $f(x, x) < 0$ for $\pi < x < 2\pi$. Thus every disk with center (π, π) contains points where f is positive as well as points where f is negative, so the graph crosses its tangent plane $(z = 0)$ there and (π, π) is a saddle point.

$D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

28. $f(x, y) = \sin x + \sin y + \cos(x + y)$, $0 \le x \le \frac{\pi}{4}$, $0 \le y \le \frac{\pi}{4}$



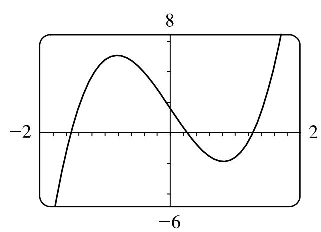


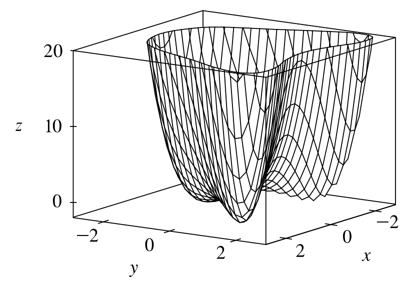


From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$.

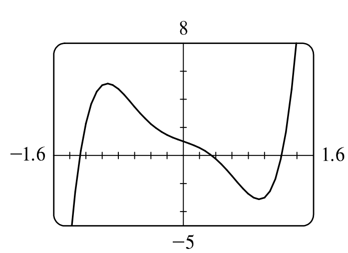
$f_x = \cos x - \sin(x + y)$, $f_y = \cos y - \sin(x + y)$, $f_{xx} = -\sin x - \cos(x + y)$, $f_{yy} = -\sin y - \cos(x + y)$, $f_{xy} = -\cos(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \ne 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

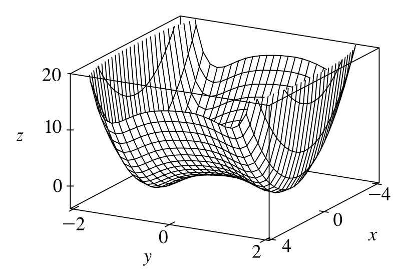
29. $f(x,y) = x^4 + y^4 - 4x^2y + 2y \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 8xy$ and $f_y(x,y) = 4y^3 - 4x^2 + 2$. $f_x = 0 \quad \Rightarrow$ $4x(x^2 - 2y) = 0$, so $x = 0$ or $x^2 = 2y$. If $x = 0$ then substitution into $f_y = 0$ gives $4y^3 = -2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt[3]{2}}$, so $\left(0, -\frac{1}{\sqrt[3]{2}}\right)$ is a critical point. Substituting $x^2 = 2y$ into $f_y = 0$ gives $4y^3 - 8y + 2 = 0$. Using a graph, solutions are approximately $y = -1.526, 0.259$, and 1.267. (Alternatively, we could have used a calculator or a CAS to find these roots.) We have $x^2 = 2y \quad \Rightarrow \quad x = \pm\sqrt{2y}$, so $y = -1.526$ gives no real-valued solution for x, but $y = 0.259 \quad \Rightarrow \quad x \approx \pm 0.720$ and $y = 1.267 \quad \Rightarrow \quad x \approx \pm 1.592$. Thus to three decimal places, the critical points are $\left(0, -\frac{1}{\sqrt[3]{2}}\right) \approx (0, -0.794)$, $(\pm 0.720, 0.259)$, and $(\pm 1.592, 1.267)$. Now since $f_{xx} = 12x^2 - 8y$, $f_{xy} = -8x$, $f_{yy} = 12y^2$, and $D = (12x^2 - 8y)(12y^2) - 64x^2$, we have $D(0, -0.794) > 0$, $f_{xx}(0, -0.794) > 0$, $D(\pm 0.720, 0.259) < 0$, $D(\pm 1.592, 1.267) > 0$, and $f_{xx}(\pm 1.592, 1.267) > 0$. Therefore $f(0, -0.794) \approx -1.191$ and $f(\pm 1.592, 1.267) \approx -1.310$ are local minimums, and $(\pm 0.720, 0.259)$ are saddle points. There is no highest point on the graph, but the lowest points are approximately $(\pm 1.592, 1.267, -1.310)$.



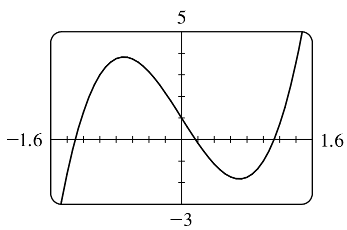


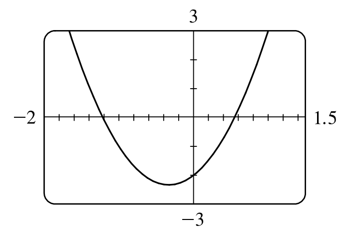
30. $f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y \quad \Rightarrow \quad f_x(x,y) = 2x$ and $f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$. $f_x = 0$ implies $x = 0$, and the graph of f_y shows that the roots of $f_y = 0$ are approximately $y = -1.273, 0.347$, and 1.211. (Alternatively, we could have found the roots of $f_y = 0$ directly, using a calculator or CAS.) So to three decimal places, the critical points are $(0, -1.273)$, $(0, 0.347)$, and $(0, 1.211)$. Now since $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 30y^4 - 24y^2 - 2$, and $D = 60y^4 - 48y^2 - 4$, we have $D(0, -1.273) > 0$, $f_{xx}(0, -1.273) > 0$, $D(0, 0.347) < 0$, $D(0, 1.211) > 0$, and $f_{xx}(0, 1.211) > 0$, so $f(0, -1.273) \approx -3.890$ and $f(0, 1.211) \approx -1.403$ are local minimums, and $(0, 0.347)$ is a saddle point. The lowest point on the graph is approximately $(0, -1.273, -3.890)$.

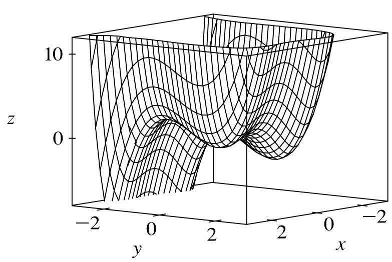




31. $f(x,y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 6x + 1$ and $f_y(x,y) = 3y^2 + 2y - 2$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -1.301$, 0.170, or 1.131, and $f_y = 0$ when $y \approx -1.215$ or 0.549. (Alternatively, we could have used a calculator or a CAS to find these roots. We could also use the quadratic formula to find the solutions of $f_y = 0$.) So, to three decimal places, f has critical points at $(-1.301, -1.215)$, $(-1.301, 0.549)$, $(0.170, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, and $(1.131, 0.549)$. Now since $f_{xx} = 12x^2 - 6$, $f_{xy} = 0$, $f_{yy} = 6y + 2$, and $D = (12x^2 - 6)(6y + 2)$, we have $D(-1.301, -1.215) < 0$, $D(-1.301, 0.549) > 0$, $f_{xx}(-1.301, 0.549) > 0$, $D(0.170, -1.215) > 0$, $f_{xx}(0.170, -1.215) < 0$, $D(0.170, 0.549) < 0$, $D(1.131, -1.215) < 0$, $D(1.131, 0.549) > 0$, and $f_{xx}(1.131, 0.549) > 0$. Therefore, to three decimal places, $f(-1.301, 0.549) \approx -3.145$ and $f(1.131, 0.549) \approx -0.701$ are local minimums, $f(0.170, -1.215) \approx 3.197$ is a local maximum, and $(-1.301, -1.215)$, $(0.170, 0.549)$, and $(1.131, -1.215)$ are saddle points. There is no highest or lowest point on the graph.







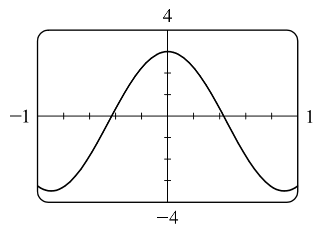
32. $f(x,y) = 20e^{-x^2-y^2} \sin 3x \cos 3y \quad \Rightarrow$

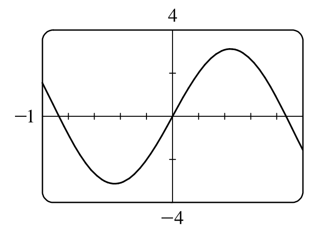
$$\begin{aligned} f_x(x,y) &= 20 \cos 3y \left[e^{-x^2-y^2}(3\cos 3x) + (\sin 3x)e^{-x^2-y^2}(-2x) \right] \\ &= 20e^{-x^2-y^2} \cos 3y \,(3\cos 3x - 2x \sin 3x) \\ f_y(x,y) &= 20 \sin 3x \left[e^{-x^2-y^2}(-3\sin 3y) + (\cos 3y)e^{-x^2-y^2}(-2y) \right] \\ &= -20e^{-x^2-y^2} \sin 3x \,(3\sin 3y + 2y \cos 3y) \end{aligned}$$

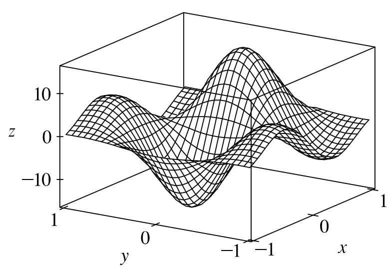
Now $f_x = 0$ implies $\cos 3y = 0$ or $3\cos 3x - 2x\sin 3x = 0$. For $|y| \leq 1$, the solutions to $\cos 3y = 0$ are $y = \pm\frac{\pi}{6} \approx \pm 0.524$. Using a graph (or a calculator or CAS), we estimate the roots of $3\cos 3x - 2x \sin 3x$ for $|x| \leq 1$ to be $x \approx \pm 0.430$. $f_y = 0$ implies $\sin 3x = 0$, so $x = 0$, or $3 \sin 3y + 2y \cos 3y = 0$. From a graph (or calculator or CAS), the roots of $3 \sin 3y + 2y \cos 3y$ between -1 and 1 are approximately 0 and ± 0.872. So to three decimal places, f has critical points at $(\pm 0.430, 0)$, $(0.430, \pm 0.872)$, $(-0.430, \pm 0.872)$, and $(0, \pm 0.524)$. Now

$$\begin{aligned} f_{xx} &= 20e^{-x^2-y^2} \cos 3y[(4x^2 - 11)\sin 3x - 12x \cos 3x] \\ f_{xy} &= -20e^{-x^2-y^2}(3\cos 3x - 2x\sin 3x)(3\sin 3y + 2y\cos 3y) \\ f_{yy} &= 20e^{-x^2-y^2} \sin 3x[(4y^2 - 11)\cos 3y - 12y \sin 3y] \end{aligned}$$

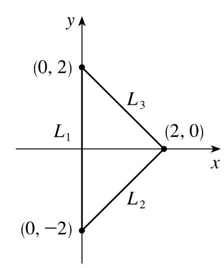
and $D = f_{xx}f_{yy} - f_{xy}^2$. Then $D(\pm 0.430, 0) > 0$, $f_{xx}(0.430, 0) < 0$, $f_{xx}(-0.430, 0) > 0$, $D(0.430, \pm 0.872) > 0$, $f_{xx}(0.430, \pm 0.872) > 0$, $D(-0.430, \pm 0.872) > 0$, $f_{xx}(-0.430, \pm 0.872) < 0$, and $D(0, \pm 0.524) < 0$, so $f(0.430, 0) \approx 15.973$ and $f(-0.430, \pm 0.872) \approx 6.459$ are local maximums, $f(-0.430, 0) \approx -15.973$ and $f(0.430, \pm 0.872) \approx -6.459$ are local minimums, and $(0, \pm 0.524)$ are saddle points. The highest point on the graph is approximately $(0.430, 0, 15.973)$ and the lowest point is approximately $(-0.430, 0, -15.973)$.







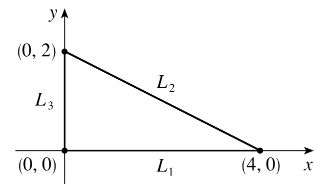
33. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 2x - 2$, $f_y = 2y$, and setting $f_x = f_y = 0$ gives $(1, 0)$ as the only critical point (which is inside D), where $f(1, 0) = -1$. Along L_1: $x = 0$ and $f(0, y) = y^2$ for $-2 \leq y \leq 2$, a quadratic function which attains its minimum at $y = 0$, where $f(0, 0) = 0$, and its maximum at $y = \pm 2$, where $f(0, \pm 2) = 4$. Along L_2: $y = x - 2$ for $0 \leq x \leq 2$, and $f(x, x - 2) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, -2) = 4$. Along L_3: $y = 2 - x$ for $0 \leq x \leq 2$, and $f(x, 2 - x) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, 2) = 4$. Thus the absolute maximum of f on D is $f(0, \pm 2) = 4$ and the absolute minimum is $f(1, 0) = -1$.



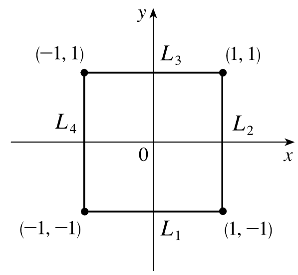
34. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = 1 - y$, $f_y = 1 - x$, and setting $f_x = f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = 1$. Along L_1: $y = 0$ and $f(x, 0) = x$ for $0 \leq x \leq 4$, an increasing function in x, so the maximum value is $f(4, 0) = 4$ and the minimum value is $f(0, 0) = 0$. Along L_2: $y = 2 - \frac{1}{2}x$ and $f\left(x, 2 - \frac{1}{2}x\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + \frac{7}{8}$ for $0 \leq x \leq 4$, a quadratic function which has a minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$, and a maximum at $x = 4$, where $f(4, 0) = 4$.

[continued]

Along L_3: $x = 0$ and $f(0, y) = y$ for $0 \le y \le 2$, an increasing function in y, so the maximum value is $f(0, 2) = 2$ and the minimum value is $f(0, 0) = 0$. Thus the absolute maximum of f on D is $f(4, 0) = 4$ and the absolute minimum is $f(0, 0) = 0$.



35. $f(x, y) = x^2 + y^2 + x^2 y + 4 \quad \Rightarrow \quad f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.



On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

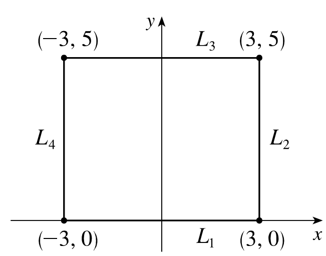
On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

36. $f(x, y) = x^2 + xy + y^2 - 6y \quad \Rightarrow \quad f_x = 2x + y$, $\ f_y = x + 2y - 6$. Then $f_x = 0$ implies $y = -2x$, and substituting into $f_y = 0$ gives $x - 4x - 6 = 0 \quad \Rightarrow \quad x = -2$, so the only critical point is $(-2, 4)$ (which is in D) where $f(-2, 4) = -12$.

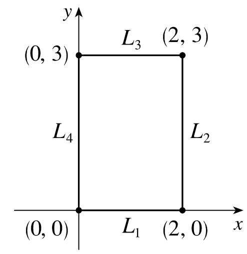
Along L_1: $y = 0$, so $f(x, 0) = x^2$, $-3 \le x \le 3$, which has a maximum value at $x = \pm 3$ where $f(\pm 3, 0) = 9$ and a minimum value at $x = 0$, where $f(0, 0) = 0$. Along L_2: $x = 3$, so $f(3, y) = 9 - 3y + y^2 = \left(y - \frac{3}{2}\right)^2 + \frac{27}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 5$ where $f(3, 5) = 19$ and a minimum value at $y = \frac{3}{2}$ where $f(3, \frac{3}{2}) = \frac{27}{4}$.

Along L_3: $y = 5$, so $f(x, 5) = x^2 + 5x - 5 = \left(x + \frac{5}{2}\right)^2 - \frac{45}{4}$, $-3 \le x \le 3$, which has a maximum value at $x = 3$ where $f(3, 5) = 19$ and a minimum value at $x = -\frac{5}{2}$, where $f\left(-\frac{5}{2}, 5\right) = -\frac{45}{4}$. Along L_4: $x = -3$, so $f(-3, y) = 9 - 9y + y^2 = \left(y - \frac{9}{2}\right)^2 - \frac{45}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 0$ where $f(-3, 0) = 9$ and a minimum value at $y = \frac{9}{2}$ where $f\left(-3, \frac{9}{2}\right) = -\frac{45}{4}$. Thus the absolute maximum of f on D is $f(3, 5) = 19$ and the absolute minimum is $f(-2, 4) = -12$.

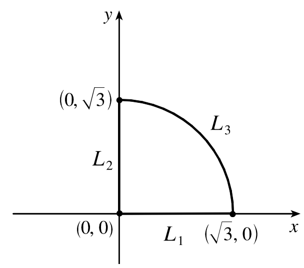


37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1 \quad \Rightarrow \quad f_x = 2x - 2$, $\ f_y = 4y - 4$. Setting $f_x = 0$ and $f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = -2$. Along L_1: $y = 0$, so $f(x, 0) = x^2 - 2x + 1 = (x - 1)^2$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0, 0) = f(2, 0) = 1$ and a minimum value at $x = 1$, where $f(1, 0) = 0$. Along L_2: $x = 2$, so $f(2, y) = 2y^2 - 4y + 1 = 2(y - 1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at

$y = 3$ where $f(2,3) = 7$ and a minimum value at $y = 1$ where $f(2,1) = -1$. Along L_3: $y = 3$, so $f(x,3) = x^2 - 2x + 7 = (x-1)^2 + 6$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,3) = f(2,3) = 7$ and a minimum value at $x = 1$, where $f(1,3) = 6$. Along L_4: $x = 0$, so $f(0,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(0,3) = 7$ and a minimum value at $y = 1$ where $f(0,1) = -1$. Thus the absolute maximum is attained at both $(0,3)$ and $(2,3)$, where $f(0,3) = f(2,3) = 7$, and the absolute minimum is $f(1,1) = -2$.



38. $f(x,y) = xy^2 \quad \Rightarrow \quad f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \quad \Leftrightarrow$ $y = 0$, there are no critical points in the interior of D. Along L_1: $y = 0$ and $f(x,0) = 0$. Along L_2: $x = 0$ and $f(0,y) = 0$. Along L_3: $y = \sqrt{3 - x^2}$, so let $g(x) = f\left(x, \sqrt{3 - x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \quad \Leftrightarrow \quad x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$



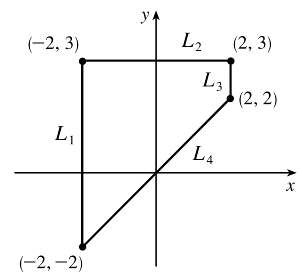
and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.

39. $f(x,y) = 2x^3 + y^4 \quad \Rightarrow \quad f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow$ $x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

40. $f(x,y) = x^3 - 3x - y^3 + 12y \quad \Rightarrow \quad f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$. Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(2,3) = 11$. Along L_3: $y = 3$ and $f(x,3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = 1$

and $x = -2$ where $f(1, 3) = f(-2, 3) = 7$. Along L_4: $y = x$ and $f(x, x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2, 2) = 18$ and a minimum at $x = -2$ where $f(-2, -2) = -18$. So the absolute maximum value of f on D is $f(2, 2) = 18$ and the minimum is $f(-2, -2) = -18$.



41. $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 \quad \Rightarrow \quad f_x(x, y) = -2(x^2 - 1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x, y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x, y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$.

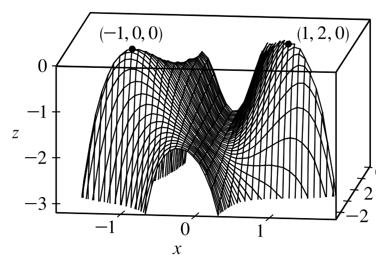
There are no critical points for $x = 0$, since $f_x(0, y) = -2$, so we set $x^2y - x - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{x+1}{x^2} \quad [x \neq 0]$,

so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2 - 1)(2x) - 2\left(x^2\,\dfrac{x+1}{x^2} - x - 1\right)\left(2x\,\dfrac{x+1}{x^2} - 1\right) = -4x(x^2 - 1)$. Therefore

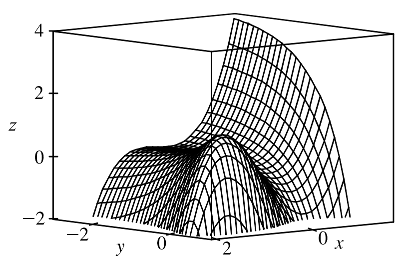
$f_x(x, y) = f_y(x, y) = 0$ at the points $(1, 2)$ and $(-1, 0)$. To classify these critical points, we calculate $f_{xx}(x, y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x, y) = -2x^4$, and $f_{xy}(x, y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$D(-1, 0) = f_{xx}(-1, 0)\,f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 = 16 > 0$,

$f_{xx}(-1, 0) = -10 < 0$, $D(1, 2) = 16 > 0$, and $f_{xx}(1, 2) = -26 < 0$, so both $(-1, 0)$ and $(1, 2)$ give local maximums.



42. $f(x, y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0$ **(1)** and $f_y = 3xe^y - 3e^{3y} = 0$ **(2)**. From **(1)** we obtain $e^y = x^2$, and then **(2)** gives $3x^3 - 3x^6 = 0 \quad \Rightarrow \quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes **(1)** impossible. So substituting $x = 1$ into **(1)** gives $y = 0$, and the only critical point is $(1, 0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1, 0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1, 0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1, 0) = 1$ is not an absolute maximum because, for instance, $f(-3, 0) = 17$. This can also be seen from the graph.

43. Let d be the distance from $(2, 0, -3)$ to any point (x, y, z) on the plane $x + y + z = 1$, so $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$ where $z = 1 - x - y$, and we minimize $d^2 = f(x, y) = (x - 2)^2 + y^2 + (4 - x - y)^2$. Then $f_x(x, y) = 2(x - 2) + 2(4 - x - y)(-1) = 4x + 2y - 12$, $f_y(x, y) = 2y + 2(4 - x - y)(-1) = 2x + 4y - 8$. Solving $4x + 2y - 12 = 0$ and $2x + 4y - 8 = 0$ simultaneously gives $x = \frac{8}{3}$, $y = \frac{2}{3}$, so the only critical point is $\left(\frac{8}{3}, \frac{2}{3}\right)$. An absolute

minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = \frac{8}{3}$, $y = \frac{2}{3}$ for which $d = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

44. Here the distance d from a point on the plane to the point $(0, 1, 1)$ is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = f(x, y) = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so

$f_x(x, y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and

$f_y(x, y) = 2(y-1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$. Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$.

This point must correspond to the minimum distance, so the point on the plane closest to $(0, 1, 1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

45. Let d be the distance from the point $(4, 2, 0)$ to any point (x, y, z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x, y)$. Then $f_x(x, y) = 2(x-4) + 2x = 4x - 8$, $f_y(x, y) = 2(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \;\Rightarrow\; x = 2$, $f_y = 0 \;\Rightarrow\; y = 1$. Thus, the only critical point is $(2, 1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4, 2, 0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

46. The distance from the origin to a point (x, y, z) on the surface is $d = \sqrt{x^2 + y^2 + z^2}$ where $y^2 = 9 + xz$, so we minimize $d^2 = x^2 + 9 + xz + z^2 = f(x, z)$. Then $f_x = 2x + z$, $f_z = x + 2z$, and $f_x = 0$, $f_z = 0 \;\Rightarrow\; x = 0$, $z = 0$, so the only critical point is $(0, 0)$. $D(0, 0) = (2)(2) - 1 = 3 > 0$ with $f_{xx}(0, 0) = 2 > 0$, so this is a minimum. Thus, $y^2 = 9 + 0 \;\Rightarrow\; y = \pm 3$ and the points on the surface closest to the origin are $(0, \pm 3, 0)$.

47. Let x, y, z be the positive numbers. Then $x + y + z = 100 \;\Rightarrow\; z = 100 - x - y$, and we want to maximize $xyz = xy(100 - x - y) = 100xy - x^2y - xy^2 = f(x, y)$ for $0 < x, y, z < 100$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y(100 - 2x - y) = 0 \;\Rightarrow\; y = 100 - 2x$ (since $y > 0$). Substituting into $f_y = 0$ gives $x[100 - x - 2(100 - 2x)] = 0 \;\Rightarrow\; 3x - 100 = 0$ (since $x > 0$) $\;\Rightarrow\; x = \frac{100}{3}$. Then $y = 100 - 2\left(\frac{100}{3}\right) = \frac{100}{3}$, and the only critical point is $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 = \frac{10{,}000}{3} > 0$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. It is also the absolute maximum (compare to the values of f as x, y, or $z \to 0$ or 100), so the numbers are $x = y = z = \frac{100}{3}$.

48. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x, y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \;\Rightarrow\; 6x = 24 \;\Rightarrow\; x = 4$ and then $y = 4$, so

the only critical point is $(4,4)$. $D(4,4) = 16 - 4 > 0$ and $f_{xx}(4,4) = 4 > 0$, so $f(4,4)$ is a local minimum. $f(4,4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

49. Center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2 + y^2 + z^2 = r^2$, so take (x, y, z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z = 2\sqrt{r^2 - x^2 - y^2}$ with volume given by

$V(x, y) = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$ for $0 < x < r$, $0 < y < r$. Then

$$V_x = (8xy) \cdot \tfrac{1}{2}(r^2 - x^2 - y^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2 - y^2} \cdot 8y = \frac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}} \text{ and } V_y = \frac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}}.$$

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$. Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \;\Rightarrow\; x^2 = y^2 \;\Rightarrow\; y = x$. Then $x^2 + 2y^2 = r^2 \;\Rightarrow\; 3x^2 = r^2 \;\Rightarrow\; x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus the only critical point is $\left(r/\sqrt{3}, r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \dfrac{8}{3\sqrt{3}}\,r^3$.

50. Let x, y, and z be the dimensions of the box. We wish to minimize surface area $= 2xy + 2xz + 2yz$, but we have volume $= xyz = 1000 \;\Rightarrow\; z = \dfrac{1000}{xy}$ so we minimize

$f(x, y) = 2xy + 2x\left(\dfrac{1000}{xy}\right) + 2y\left(\dfrac{1000}{xy}\right) = 2xy + \dfrac{2000}{y} + \dfrac{2000}{x}$. Then $f_x = 2y - \dfrac{2000}{x^2}$ and $f_y = 2x - \dfrac{2000}{y^2}$. Setting $f_x = 0$ implies $y = \dfrac{1000}{x^2}$ and substituting into $f_y = 0$ gives $x - \dfrac{x^4}{1000} = 0 \;\Rightarrow\; x^3 = 1000$ [since $x \neq 0$] $\Rightarrow\; x = 10$.

The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions $x = 10$ cm, $y = 1000/10^2 = 10$ cm, $z = 1000/10^2 = 10$ cm.

51. The volume of the box is $V = xyz$. Since one vertex is in the plane $x + 2y + 3z = 6 \;\Leftrightarrow\; z = \frac{1}{3}(6 - x - 2y)$, the volume is given by $V(x, y) = \frac{1}{3}xy(6 - x - 2y) = \frac{1}{3}(6xy - x^2y - 2xy^2)$. Now maximize V.

$V_x = \frac{1}{3}(6y - 2xy - 2y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $V_y = \frac{1}{3}x\,(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives $y = 3 - x$ and $x = 6 - 4y \;\Rightarrow\; y = 1$ and $x = 2$, so the critical point is $(2, 1)$, which geometrically must give a maximum. Thus, the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

52. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume

$f(x, y) = xy\,\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\,\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\,\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or

$3x^4 + 64x^2 - (32)^2 = 0$. Thus, $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus, the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

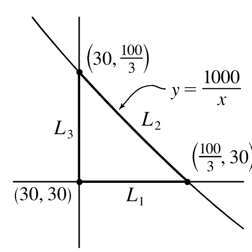
53. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus, the box is a cube with edge length $\frac{1}{12}c$.

54. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $C_x = 0$ implies $y = 2V/(5x^2)$, $C_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus, the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

55. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000$ cm^3. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus, the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

56. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad \Rightarrow \quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad \Rightarrow$

$h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad \Rightarrow \quad y = \dfrac{80{,}000}{6x^2}$ and substituting into

$h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad \Rightarrow \quad x^3 = \dfrac{80{,}000^2}{6 \cdot 64{,}000} = \dfrac{50{,}000}{3}$, so

$x = \sqrt[3]{\dfrac{50{,}000}{3}} = 10\sqrt[3]{\dfrac{50}{3}} \quad \Rightarrow \quad y = \dfrac{80}{\sqrt[3]{60}}$, and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$

which is not in D. Next we check the boundary of D.

[continued]

On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$.

On L_2: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$.

Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

57. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$$L = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad L^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}.$$

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$ $(x, y > 0)$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$. $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \Rightarrow y(L^2 - 2x^2 - y^2) = 0 \Rightarrow$ $2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \Rightarrow x(L^2 - x^2 - 2y^2) = 0 \Rightarrow$ $x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \Rightarrow$ $3x^2 = L^2 \Rightarrow x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$.

So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2\sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

58. $Y(N, P) = kNPe^{-N-P} \quad \Rightarrow \quad Y_N = kP\left[N(-e^{-N-P}) + e^{-N-P}(1)\right] = kP(1 - N)e^{-N-P}$, $Y_P = kN\left[P(-e^{-N-P}) + e^{-N-P}(1)\right] = kN(1 - P)e^{-N-P}$. Here $N \ge 0$ and $P \ge 0$, but if either $N = 0$ or $P = 0$ then

the yield is zero. Assuming that $N > 0$ and $P > 0$, $Y_N = 0$ implies $N = 1$ and $Y_P = 0$ implies $P = 1$, so the only critical point in $\{(N, P) \mid N > 0, P > 0\}$ is $(1, 1)$ where $Y(1, 1) = ke^{-2}$.

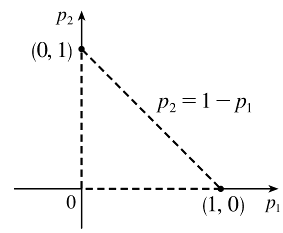
$D(N, P) = Y_{NN}Y_{PP} - (Y_{NP})^2 = \left[kP(N-2)e^{-N-P}\right]\left[kN(P-2)e^{-N-P}\right] - \left[k(1-N)(1-P)e^{-N-P}\right]^2 \quad \Rightarrow$

$D(1, 1) = \left(-ke^{-2}\right)\left(-ke^{-2}\right) - (0)^2 = k^2e^{-4} > 0$ and $Y_{NN}(1, 1) = -ke^{-2} < 0$, so $Y(1, 1) = ke^{-2}$ is a local maximum. $Y(1, 1)$ is also the absolute maximum (we have only one critical point, and $Y \to 0$ as $N \to 0$ or $P \to 0$ and $Y \to 0$ as N or P grow large) so the best yield is achieved when both the nitrogen and phosphorus levels are 1 (measured in appropriate units).

59. (a) We are given that $p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_3 = 1 - p_1 - p_2$, so

$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln (1 - p_1 - p_2)$.

(b) Because p_i is a proportion we have $0 \le p_i \le 1$, but H is undefined unless $p_1 > 0$, $p_2 > 0$, and $1 - p_1 - p_2 > 0 \quad \Leftrightarrow \quad p_1 + p_2 < 1$. This last restriction forces $p_1 < 1$ and $p_2 < 1$, so the domain of H is $\{(p_1, p_2) \mid 0 < p_1 < 1,\ p_2 < 1 - p_1\}$. It is the interior of the triangle drawn in the figure.



(c)
$$\begin{aligned} H_{p_1} &= -\left[p_1 \cdot (1/p_1) + (\ln p_1) \cdot 1\right] - \left[(1 - p_1 - p_2) \cdot (-1)/(1 - p_1 - p_2) + \ln (1 - p_1 - p_2) \cdot (-1)\right] \\ &= -1 - \ln p_1 + 1 + \ln (1 - p_1 - p_2) = \ln (1 - p_1 - p_2) - \ln p_1 \end{aligned}$$

Similarly $H_{p_2} = \ln (1 - p_1 - p_2) - \ln p_2$. Then $H_{p_1} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_1 \quad \Rightarrow \quad 1 - p_1 - p_2 = p_1 \quad \Rightarrow \quad p_2 = 1 - 2p_1$, and $H_{p_2} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_2 \quad \Rightarrow \quad p_1 = 1 - 2p_2$. Substituting, we have $p_1 = 1 - 2(1 - 2p_1) \quad \Rightarrow$ $3p_1 = 1 \quad \Rightarrow \quad p_1 = \frac{1}{3}$, and then $p_2 = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$. Thus the only critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$D(p_1, p_2) = H_{p_1p_1}H_{p_2p_2} - (H_{p_1p_2})^2 = \left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_1}\right)\left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_2}\right) - \left(\dfrac{-1}{1 - p_1 - p_2}\right)^2$, so

$D\left(\frac{1}{3}, \frac{1}{3}\right) = (-6)(-6) - (-3)^2 = 27 > 0$ and $H_{p_1p_1}\left(\frac{1}{3}, \frac{1}{3}\right) = -6 < 0$. Thus

$H\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} = -\ln \frac{1}{3} = \ln 3$ is a local maximum. Here it is also the absolute maximum, so the maximum value of H is $\ln 3$, which occurs for $p_1 = p_2 = p_3 = \frac{1}{3}$ (all three species have equal proportion in the ecosystem).

60. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving

$P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \ge 0$, $r \ge 0$, and $q + r \le 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives

$r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1 - 2q) - 2q = 0 \quad \Rightarrow \quad q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \le q \le 1$, $P(q, r) = P(q, 0) = 2q - 2q^2$, $0 \le q \le 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \le r \le 1$ we have $P(0, r) = 2r - 2r^2$, $0 \le r \le 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \le q \le 1$, $P(q, r) = P(q, 1 - q) = 2q - 2q^2$, $0 \le q \le 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q, r)$ on D is $\frac{2}{3}$.

61. Note that here the variables are m and b, and $f(m, b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then $f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - m x_i^2 - b x_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$ and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations.

Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m, b) > 0$ always and $D(m, b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

62. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1, 2, 3)$. Writing the equation of the plane as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \dfrac{abc}{6}$. But $(1, 2, 3)$ must lie on the plane, so we need $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1$ $(\star)$ and thus can think of c as a function of a and b. Then $V_a = \dfrac{b}{6}\left(c + a\,\dfrac{\partial c}{\partial a}\right)$ and $V_b = \dfrac{a}{6}\left(c + b\,\dfrac{\partial c}{\partial b}\right)$. Differentiating $(\star)$ with respect to a we get $-a^{-2} - 3c^{-2}\,\dfrac{\partial c}{\partial a} = 0 \quad \Rightarrow$ $\dfrac{\partial c}{\partial a} = \dfrac{-c^2}{3a^2}$, and differentiating $(\star)$ with respect to b gives $-2b^{-2} - 3c^{-2}\,\dfrac{\partial c}{\partial b} = 0 \quad \Rightarrow \quad \dfrac{\partial c}{\partial b} = \dfrac{-2c^2}{3b^2}$. Then $V_a = \dfrac{b}{6}\left(c + a\,\dfrac{-c^2}{3a^2}\right) = 0 \quad \Rightarrow \quad c = 3a$, and $V_b = \dfrac{a}{6}\left(c + b\,\dfrac{-2c^2}{3b^2}\right) = 0 \quad \Rightarrow \quad c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(\star)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\dfrac{x}{3} + \dfrac{y}{6} + \dfrac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

1. $Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} f_{xx}(a,b)(x-a)^2$
$$+ f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2} f_{yy}(a,b)(y-b)^2,$$

so

$$Q_x(x,y) = f_x(a,b) + \tfrac{1}{2} f_{xx}(a,b)(2)(x-a) + f_{xy}(a,b)(y-b) = f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)$$

At (a,b) we have $Q_x(a,b) = f_x(a,b) + f_{xx}(a,b)\,(a-a) + f_{xy}(a,b)(b-b) = f_x(a,b)$.

Similarly, $Q_y(x,y) = f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b) \quad \Rightarrow$

$Q_y(a,b) = f_y(a,b) + f_{xy}(a,b)(a-a) + f_{yy}(a,b)(b-b) = f_y(a,b)$.

For the second-order partial derivatives we have

$$Q_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xx}(a,b) \quad \Rightarrow \quad Q_{xx}(a,b) = f_{xx}(a,b)$$

$$Q_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xy}(a,b) \quad \Rightarrow \quad Q_{xy}(a,b) = f_{xy}(a,b)$$

$$Q_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b)\right] = f_{yy}(a,b) \quad \Rightarrow \quad Q_{yy}(a,b) = f_{yy}(a,b)$$

2. (a) First we find the partial derivatives and values that will be needed:

$$\begin{aligned} f(x,y) &= e^{-x^2-y^2} & f(0,0) &= 1 \\ f_x(x,y) &= -2xe^{-x^2-y^2} & f_x(0,0) &= 0 \\ f_y(x,y) &= -2ye^{-x^2-y^2} & f_y(0,0) &= 0 \\ f_{xx}(x,y) &= (4x^2-2)e^{-x^2-y^2} & f_{xx}(0,0) &= -2 \\ f_{xy}(x,y) &= 4xye^{-x^2-y^2} & f_{xy}(0,0) &= 0 \\ f_{yy}(x,y) &= (4y^2-2)e^{-x^2-y^2} & f_{yy}(0,0) &= -2 \end{aligned}$$

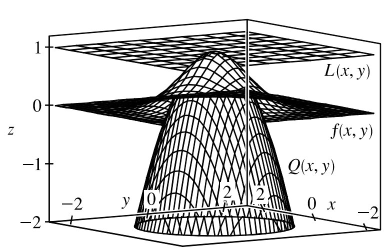
Then the first-degree Taylor polynomial of f at $(0,0)$ is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + (0)(x-0) + (0)(y-0) = 1$$

The second-degree Taylor polynomial is given by

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2} f_{xx}(0,0)(x-0)^2 \\ &\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2} f_{yy}(0,0)(y-0)^2 \\ &= 1 - x^2 - y^2 \end{aligned}$$

(b)



As we see from the graph, L approximates f well only for points (x,y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$\begin{array}{llll} f(x,y) = xe^y & f(1,0) = 1 & f_{xx}(x,y) = 0 & f_{xx}(1,0) = 0 \\ f_x(x,y) = e^y & f_x(1,0) = 1 & f_{xy}(x,y) = e^y & f_{xy}(1,0) = 1 \\ f_y(x,y) = xe^y & f_y(1,0) = 1 & f_{yy}(x,y) = xe^y & f_{yy}(1,0) = 1 \end{array}$$

Then the first-degree Taylor polynomial of f at $(1,0)$ is

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + (1)(x-1) + (1)(y-0) = x + y$$

The second-degree Taylor polynomial is given by

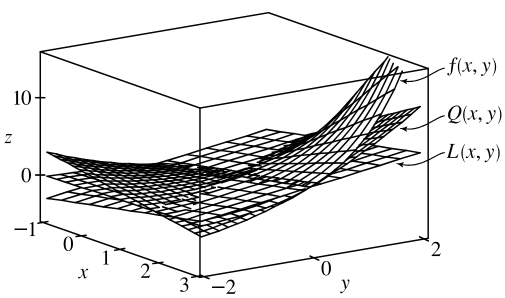
$$\begin{aligned} Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \tfrac{1}{2} f_{xx}(1,0)(x-1)^2 \\ &\qquad + f_{xy}(1,0)(x-1)(y-0) + \tfrac{1}{2} f_{yy}(1,0)(y-0)^2 \\ &= \tfrac{1}{2}y^2 + x + xy \end{aligned}$$

(b) $L(0.9, 0.1) = 0.9 + 0.1 = 1.0$

$Q(0.9, 0.1) = \frac{1}{2}(0.1)^2 + 0.9 + (0.9)(0.1) = 0.995$

$f(0.9, 0.1) = 0.9e^{0.1} \approx 0.9947$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1,0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L.

4. (a) $f(x,y) = ax^2 + bxy + cy^2 = a\left[x^2 + \dfrac{b}{a}\,xy + \dfrac{c}{a}\,y^2\right] = a\left[x^2 + \dfrac{b}{a}\,xy + \left(\dfrac{b}{2a}\,y\right)^2 - \left(\dfrac{b}{2a}\,y\right)^2 + \dfrac{c}{a}\,y^2\right]$

$$= a\left[\left(x + \frac{b}{2a}\,y\right)^2 - \frac{b^2}{4a^2}\,y^2 + \frac{c}{a}\,y^2\right] = a\left[\left(x + \frac{b}{2a}\,y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) For $D = 4ac - b^2$, from part (a) we have $f(x,y) = a\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right]$. If $D > 0$,

$\left(\dfrac{D}{4a^2}\right)y^2 \geq 0$ and $\left(x + \dfrac{b}{2a}\,y\right)^2 \geq 0$, so $\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$. Here $a > 0$, thus

$f(x,y) = a\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$. We know $f(0,0) = 0$, so $f(0,0) \leq f(x,y)$ for all (x,y), and by definition f has a local minimum at $(0,0)$.

(c) As in part (b), $\left[\left(x + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$, and since $a < 0$ we have

$f(x,y) = a\left[\left(x + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \leq 0$. Since $f(0,0) = 0$, we must have $f(0,0) \geq f(x,y)$ for all (x,y), so by definition f has a local maximum at $(0,0)$.

(d) $f(x,y) = ax^2 + bxy + cy^2$, so $f_x(x,y) = 2ax + by \;\;\Rightarrow\;\; f_x(0,0) = 0$ and $f_y(x,y) = bx + 2cy \;\;\Rightarrow\;\; f_y(0,0) = 0$. Since $f(0,0) = 0$ and f and its partial derivatives are continuous, we know from Equation 14.4.2 that the tangent plane to the graph of f at $(0,0)$ is the plane $z = 0$. Then f has a saddle point at $(0,0)$ if the graph of f crosses the tangent plane at $(0,0)$, or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the x-axis; then we have $y = 0 \;\;\Rightarrow\;\; f(x,0) = ax^2$ which has the same sign as a. We must now find at least one path to the origin where $f(x,y)$ gives values with sign opposite that of a. Since

$f(x,y) = a\left[\left(x + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right]$, if we approach the origin along the line $x = -\frac{b}{2a}\,y$, we have

$f\left(-\frac{b}{2a}\,y, y\right) = a\left[\left(-\frac{b}{2a}\,y + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] = \frac{D}{4a}\,y^2$. Since $D < 0$, these values have signs opposite that of a. Thus, f has a saddle point at $(0,0)$.

5. (a) Since the partial derivatives of f exist at $(0,0)$ and $(0,0)$ is a critical point, we know $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Then the second-degree Taylor polynomial of f at $(0,0)$ can be expressed as

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2} f_{xx}(0,0)(x-0)^2 \\ &\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2} f_{yy}(0,0)(y-0)^2 \\ &= \tfrac{1}{2} f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \tfrac{1}{2} f_{yy}(0,0)y^2 \end{aligned}$$

(b) $Q(x,y) = \frac{1}{2}\,f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}\,f_{yy}(0,0)y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2}\,f_{xx}(0,0)$, $b = f_{xy}(0,0)$, and $c = \frac{1}{2}\,f_{yy}(0,0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0,0)$. Here,

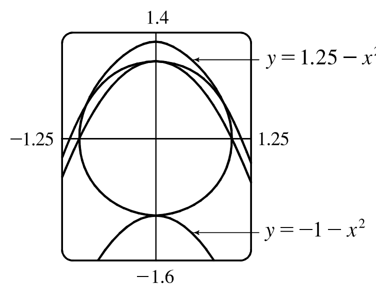
$D = 4ac - b^2 = 4\left(\frac{1}{2}\right)f_{xx}(0,0)\left(\frac{1}{2}\right)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2$, and if $D > 0$ with $a = \frac{1}{2}\,f_{xx}(0,0) > 0 \;\;\Rightarrow\;\; f_{xx}(0,0) > 0$, we know from Problem 4 that Q has a local minimum at $(0,0)$. Similarly, if $D > 0$ and $a < 0 \;\;\Rightarrow\;\; f_{xx}(0,0) < 0$, Q has a local maximum at $(0,0)$, and if $D < 0$, Q has a saddle point at $(0,0)$.

(c) Since $f(x,y) \approx Q(x,y)$ near $(0,0)$, part (b) suggests that for $D = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2$, if $D > 0$ and $f_{xx}(0,0) > 0$, f has a local minimum at $(0,0)$. If $D > 0$ and $f_{xx}(0,0) < 0$, f has a local maximum at $(0,0)$, and if $D < 0$, f has a saddle point at $(0,0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 14.7.

14.8 Lagrange Multipliers

1. At the extreme values of f, the level curves of f just touch the curve $g(x,y)=8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y)=c$ with the largest value of c which still intersects the curve $g(x,y)=8$ is approximately $c=59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y)=8$ appears to be $c=30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y)=8$ to be about 59 and the minimum to be 30.

2. (a) The values $c=\pm 1$ and $c=1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function x^2+y subject to the constraint $x^2+y^2=1$.



(b) $\nabla f=\langle 2x,1\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. So $2x=2\lambda x \Rightarrow$ either $\lambda=1$ or $x=0$. If $\lambda=1$, then $y=\frac{1}{2}$ and so $x=\pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x=0$, then $y=\pm 1$. Therefore f has possible extreme values at the points $(0,\pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)=\frac{5}{4}$ (the maximum value), $f(0,1)=1$, and $f(0,-1)=-1$ (the minimum value). These are our answers from part (a).

3. We want to find the extreme values of $f(x,y)=x^2-y^2$ subject to the constraint $g(x,y)=x^2+y^2=1$. Then $\nabla f=\lambda\nabla g \Rightarrow \langle 2x,-2y\rangle=\lambda\langle 2x,2y\rangle$, so we solve the equations $2x=2\lambda x$, $-2y=2\lambda y$, and $x^2+y^2=1$. From the first equation we have $2x(\lambda-1)=0 \Rightarrow x=0$ or $\lambda=1$. If $x=0$ then substitution into the constraint gives $y^2=1 \Rightarrow y=\pm 1$. If $\lambda=1$ then substitution into the second equation gives $-2y=2y \Rightarrow y=0$, and from the constraint we must have $x=\pm 1$. Thus the possible points for the extreme values of f are $(0,\pm 1)$ and $(\pm 1,0)$. Evaluating f at these points, we see that the maximum value of f is $f(\pm 1,0)=1$ and the minimum is $f(0,\pm 1)=-1$.

4. $f(x,y)=x^2y$, $g(x,y)=x^2+y^4=5$, and $\nabla f=\lambda\nabla g \Rightarrow \langle 2xy,x^2\rangle=\langle 2\lambda x,4\lambda y^3\rangle$, so we get the three equations $2xy=2\lambda x$, $x^2=4\lambda y^3$, and $x^2+y^4=5$. $2xy=2\lambda x \Rightarrow x=0$ or $y=\lambda$. If $x=0$, the second equation implies $y=0$ or $\lambda=0$. The point $(0,0)$ does not satisfy the constraint, so $x=\lambda=0$ and the constraint gives a possible extreme value at the point $(0,\sqrt[4]{5})$. Next, suppose $y=\lambda$. Substituting into the second equation $\Rightarrow x^2=4\lambda^4$ and substituting into the third equation gives $4\lambda^4+\lambda^4=5 \Rightarrow \lambda=\pm 1$. From the second equation with $y=\lambda=1$, we get $x=\pm 2$. From the second equation with $y=\lambda=-1$, we get $x=\pm 2$. So f also has possible extreme values at $(\pm 2,1)$ and $(\pm 2,-1)$. Evaluating f at these 5 points, we see $f(\pm 2,1)=4$ is the maximum value and $f(\pm 2,-1)=-4$ is the minimum value.

5. $f(x,y)=xy$, $g(x,y)=4x^2+y^2=8$, and $\nabla f=\lambda\nabla g \Rightarrow \langle y,x\rangle=\langle 8\lambda x,2\lambda y\rangle$, so $y=8\lambda x$, $x=2\lambda y$, and $4x^2+y^2=8$. First note that if $x=0$ then $y=0$ by the first equation, and if $y=0$ then $x=0$ by the second equation. But this contradicts the third equation, so $x\neq 0$ and $y\neq 0$. Then from the first two equations we have $\dfrac{y}{8x}=\lambda=\dfrac{x}{2y} \Rightarrow$

$2y^2 = 8x^2 \quad\Rightarrow\quad y^2 = 4x^2$, and substitution into the third equation gives $4x^2 + 4x^2 = 8 \quad\Rightarrow\quad x = \pm 1$. If $x = \pm 1$ then $y^2 = 4 \quad\Rightarrow\quad y = \pm 2$, so f has possible extreme values at $(1, \pm 2)$ and $(-1, \pm 2)$. Evaluating f at these points, we see that the maximum value is $f(1, 2) = f(-1, -2) = 2$ and the minimum is $f(1, -2) = f(-1, 2) = -2$.

6. $f(x, y) = xe^y$, $g(x, y) = x^2 + y^2 = 2$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle e^y, xe^y \rangle = \langle 2\lambda x, 2\lambda y \rangle$, so $e^y = 2\lambda x$, $xe^y = 2\lambda y$, and $x^2 + y^2 = 2$. First note that from the first equation $x \neq 0$. If $y = 0$, the second equation implies $x = 0$, so $y \neq 0$. Then from the first two equations we have $\dfrac{e^y}{2x} = \lambda = \dfrac{xe^y}{2y} \quad\Rightarrow\quad 2ye^y = 2x^2 e^y \quad\Rightarrow\quad y = x^2$, and substituting into the third equation gives $x^2 + (x^2)^2 = 2 \quad\Rightarrow\quad x^4 + x^2 - 2 = 0 \quad\Rightarrow\quad (x^2 + 2)(x^2 - 1) = 0 \quad\Rightarrow\quad x = \pm 1$. From $y = x^2$ we have $y = 1$, so f has possible extreme values at $(\pm 1, 1)$. Evaluating f at these points, we see that the maximum value is $f(1, 1) = e$ and the minimum is $f(-1, 1) = -e$.

7. $f(x, y) = 2x^2 + 6y^2$, $g(x, y) = x^4 + 3y^4 = 1$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 4x, 12y \rangle = \langle 4\lambda x^3, 12\lambda y^3 \rangle$, so we get the three equations $4x = 4\lambda x^3$, $12y = 12\lambda y^3$, and $x^4 + 3y^4 = 1$. The first equation implies that $x = 0$ or $x^2 = \dfrac{1}{\lambda}$. The second equation implies that $y = 0$ or $y^2 = \dfrac{1}{\lambda}$. Note that x and y cannot both be zero as this contradicts the third equation. If $x = 0$, the third equation implies $y = \pm \dfrac{1}{\sqrt[4]{3}}$. If $y = 0$, the third equation implies that $x = \pm 1$. Thus, f has possible extreme values at $\left(0, \pm \dfrac{1}{\sqrt[4]{3}}\right)$ and $(\pm 1, 0)$. Next, suppose $x^2 = y^2 = \dfrac{1}{\lambda}$. Then the third equation gives $\left(\dfrac{1}{\lambda}\right)^2 + 3\left(\dfrac{1}{\lambda}\right)^2 = 1 \quad\Rightarrow\quad \lambda = \pm 2$. $\lambda = -2$ results in a nonreal solution, so consider $\lambda = 2 \quad\Rightarrow\quad x = y = \pm \dfrac{1}{\sqrt{2}}$. Therefore, f also has possible extreme values at $\left(\pm \dfrac{1}{\sqrt{2}}, \pm \dfrac{1}{\sqrt{2}}\right)$ (all 4 combinations). Substituting all 8 points into f, we find the maximum value is $f\left(\pm \dfrac{1}{\sqrt{2}}, \pm \dfrac{1}{\sqrt{2}}\right) = 4$ and the minimum value is $f(\pm 1, 0) = 2$.

8. $f(x, y) = xye^{-x^2 - y^2}$, $g(x, y) = 2x - y = 0$, and $\nabla f = \lambda \nabla g \quad\Rightarrow$

$\left\langle ye^{-x^2-y^2} - 2x^2 ye^{-x^2-y^2}, xe^{-x^2-y^2} - 2xy^2 e^{-x^2-y^2} \right\rangle = \langle 2\lambda, -\lambda \rangle$, so we get the three equations

$ye^{-x^2-y^2} - 2x^2 ye^{-x^2-y^2} = 2\lambda$, $xe^{-x^2-y^2} - 2xy^2 e^{-x^2-y^2} = -\lambda$, and $2x - y = 0$. Multiplying the second equation by 2 and adding it to the first gives $2xe^{-x^2-y^2} - 4xy^2 e^{-x^2-y^2} + ye^{-x^2-y^2} - 2x^2 ye^{-x^2-y^2} = -2\lambda + 2\lambda = 0 \quad\Rightarrow$ $2x - 4xy^2 + y - 2x^2 y = 0$ (as $e^{-x^2-y^2} \neq 0$). From the third equation, $2x = y$, and substituting into the new equation, we have $2x - 4x(2x)^2 + 2x - 2x^2(2x) = 0 \quad\Rightarrow\quad 4x - 20x^3 = 0 \quad\Rightarrow\quad 4x(1 - 5x^2) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm \dfrac{1}{\sqrt{5}}$, so f has possible extreme values at $(0, 0)$, $\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$, and $\left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right)$. Substituting these into f, we see that the minimum value is $f(0, 0) = 0$ and the maximum value is $f\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right) = f\left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right) = \dfrac{2}{5e}$.

9. $f(x,y,z) = 2x + 2y + z$, $g(x,y,z) = x^2 + y^2 + z^2 = 9$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2,2,1\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$, so $2\lambda x = 2$, $2\lambda y = 2$, $2\lambda z = 1$, and $x^2 + y^2 + z^2 = 9$. The first three equations imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{1}{\lambda}$, and $z = \dfrac{1}{2\lambda}$. But substitution into the fourth equation gives $\left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{2\lambda}\right)^2 = 9 \Rightarrow \dfrac{9}{4\lambda^2} = 9 \Rightarrow \lambda = \pm\frac{1}{2}$, so f has possible extreme values at the points $(2,2,1)$ and $(-2,-2,-1)$. The maximum value of f on $x^2 + y^2 + z^2 = 9$ is $f(2,2,1) = 9$, and the minimum is $f(-2,-2,-1) = -9$.

10. $f(x,y,z) = e^{xyz}$, $g(x,y,z) = 2x^2 + y^2 + z^2 = 24$, and $\nabla f = \lambda \nabla g \Rightarrow \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle = \langle 4\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $yze^{xyz} = 4\lambda x$, $xze^{xyz} = 2\lambda y$, $xye^{xyz} = 2\lambda z$, and $2x^2 + y^2 + z^2 = 24$. If any of x, y, z, or λ is zero, then the first three equations imply that two of the variables x, y, z must be zero. If $x = y = z = 0$ it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are $(\pm 2\sqrt{3}, 0, 0)$, $(0, \pm 2\sqrt{6}, 0)$, $(0, 0, \pm 2\sqrt{6})$, all with an f-value of $e^0 = 1$. If none of x, y, z, λ is zero then from the first three equations we have $\dfrac{4\lambda x}{yz} = e^{xyz} = \dfrac{2\lambda y}{xz} = \dfrac{2\lambda z}{xy} \Rightarrow \dfrac{2x}{yz} = \dfrac{y}{xz} = \dfrac{z}{xy}$. This gives $2x^2 z = y^2 z \Rightarrow 2x^2 = y^2$ and $xy^2 = xz^2 \Rightarrow$ $y^2 = z^2$. Substituting into the fourth equation, we have $y^2 + y^2 + y^2 = 24 \Rightarrow y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$, so $x^2 = 4 \Rightarrow x = \pm 2$ and $z^2 = y^2 \Rightarrow z = \pm 2\sqrt{2}$, giving possible points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (all combinations). The value of f is e^{16} when all coordinates are positive or exactly two are negative, and the value is e^{-16} when all are negative or exactly one of the coordinates is negative. Thus the maximum of f subject to the constraint is e^{16} and the minimum is e^{-16}.

11. $f(x,y,z) = xy^2z$, $g(x,y,z) = x^2 + y^2 + z^2 = 4$, and $\nabla f = \lambda \nabla g \Rightarrow \langle y^2 z, 2xyz, xy^2\rangle = \lambda\langle 2x, 2y, 2z\rangle$. Then $y^2 z = 2\lambda x$, $2xyz = 2\lambda y$, $xy^2 = 2\lambda z$, and $x^2 + y^2 + z^2 = 4$.

Case 1: If $\lambda = 0$, then the first equation implies that $y = 0$ or $z = 0$. If $y = 0$, then any values of x and z satisfy the first three equations, so from the fourth equation all points $(x, 0, z)$ such that $x^2 + z^2 = 4$ are possible points. If $z = 0$ then from the third equation $x = 0$ or $y = 0$, and from the fourth equation, the possible points are $(0, \pm 2, 0)$, $(\pm 2, 0, 0)$. The f-value in all these cases is 0.

Case 2: If $\lambda \neq 0$ but any one of x, y, z is zero, the first three equations imply that all three coordinates must be zero, contradicting the fourth equation. Thus if $\lambda \neq 0$, none of x, y, z is zero and from the first three equations we have $\dfrac{y^2 z}{2x} = \lambda = xz = \dfrac{xy^2}{2z}$. This gives $y^2 z = 2x^2 z \Rightarrow y^2 = 2x^2$ and $2y^2 z^2 = 2x^2 y^2 \Rightarrow z^2 = x^2$. Substituting into the fourth equation, we have $x^2 + 2x^2 + x^2 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, so $y = \pm\sqrt{2}$ and $z = \pm 1$, giving possible points $(\pm 1, \pm\sqrt{2}, \pm 1)$ (all combinations). The value of f is 2 when x and z are the same sign and -2 when they are opposite. Thus the maximum of f subject to the constraint is $f(1, \pm\sqrt{2}, 1) = f(-1, \pm\sqrt{2}, -1) = 2$ and the minimum is $f(1, \pm\sqrt{2}, -1) = f(-1, \pm\sqrt{2}, 1) = -2$.

12. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2 + y^2 + z^2 + xy = 12$, and $\nabla f = \lambda \nabla g \;\;\Rightarrow$

$\langle 2x, 2y, 2z\rangle = \langle \lambda(2x+y), \lambda(2y+x), \lambda 2z\rangle$, so $(\mathbf{1})\;\; 2x = \lambda(2x+y)$, $(\mathbf{2})\;\; 2y = \lambda(2y+x)$, $(\mathbf{3})\;\; 2z = \lambda 2z$, and

$(\mathbf{4})\;\; x^2 + y^2 + z^2 + xy = 12$. First note that $\lambda = 0 \;\;\Rightarrow\;\; x = y = z = 0$, which contradicts $(\mathbf{4})$, so assume $\lambda \neq 0$. Then

$(\mathbf{3})$ implies $\lambda = 1$ or $z = 0$. If $\lambda = 1$, then $(\mathbf{1})$ and $(\mathbf{2})$ imply $x = y = 0 \;\;\Rightarrow\;\; 0^2 + 0^2 + z^2 + 0 = 12 \;\;\Rightarrow\;\; z = \pm\sqrt{12}$. If

$z = 0$, $y(\mathbf{1}) - x(\mathbf{2}) \;\;\Rightarrow\;\; 0 = \lambda y^2 - \lambda x^2 \;\;\Rightarrow\;\; x = y$ or $x = -y$. Substituting $x = y$ into $(\mathbf{4}) \;\;\Rightarrow\;\; y = x = \pm 2$.

Substituting $x = -y$ into $(\mathbf{4}) \;\;\Rightarrow\;\; y = \pm\sqrt{12}$. Thus, f has possible extreme values at $\left(0, 0, \pm\sqrt{12}\,\right)$,

$(2, 2, 0)$, $(-2, -2, 0)$, $\left(\sqrt{12}, -\sqrt{12}, 0\right)$, and $\left(-\sqrt{12}, \sqrt{12}, 0\right)$. Evaluating f at these points, we see that the maximum value

is $f\left(\sqrt{12}, -\sqrt{12}, 0\right) = f\left(-\sqrt{12}, \sqrt{12}, 0\right) = 24$ and the minimum is $f(2, 2, 0) = f(-2, -2, 0) = 8$.

13. $f(x,y,z) = x^2 + y^2 + z^2,\;\; g(x,y,z) = x^4 + y^4 + z^4 = 1 \;\;\Rightarrow\;\; \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \left\langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3\right\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f-value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f-value of 1.

Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

14. $f(x,y,z) = x^4 + y^4 + z^4,\;\; g(x,y,z) = x^2 + y^2 + z^2 = 1 \;\;\Rightarrow\;\; \nabla f = \left\langle 4x^3, 4y^3, 4z^3\right\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ giving 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

15. $f(x,y,z,t) = x + y + z + t,\;\; g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \;\;\Rightarrow\;\; \langle 1,1,1,1\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t\rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

16. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n,\;\; g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \;\;\Rightarrow$

$\langle 1, 1, \ldots, 1\rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n\rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$.

[continued]

But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

17. If the two numbers are x and y, we want to minimize $f(x,y) = x + y$, $x > 0$, $y > 0$ subject to $g(x,y) = xy = 100$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 1, 1 \rangle = \langle \lambda y, \lambda x \rangle$, so $1 = \lambda y$, $1 = \lambda x$, and $xy = 100$. The first two equations imply $y = \dfrac{1}{\lambda} = x$ and substitution into the third equation gives $\dfrac{1}{\lambda^2} = 100 \quad \Rightarrow \quad \lambda = \pm \dfrac{1}{10}$. Since $x > 0$ and $y > 0$, we have $\lambda = \dfrac{1}{10}$ and hence, $x = y = \dfrac{1}{\lambda} = 10$. Thus, the minimum value of f is $f(10, 10) = 20$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Therefore, the two numbers are 10 and 10.

18. If x and y are the dimensions of the rectangle in meters, we want to minimize $f(x,y) = 2x + 2y$, $x > 0$, $y > 0$ subject to $g(x,y) = xy = 1000$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2, 2 \rangle = \langle \lambda y, \lambda x \rangle$, so $2 = \lambda y$, $2 = \lambda x$, and $xy = 1000$. The first two equations imply $y = \dfrac{2}{\lambda} = x$ and substitution into the third equation gives $\dfrac{4}{\lambda^2} = 1000 \quad \Rightarrow \quad \lambda = \pm \dfrac{1}{5\sqrt{10}}$. Since $x > 0$ and $y > 0$, we have $\lambda = \dfrac{1}{5\sqrt{10}}$ and hence, $x = y = \dfrac{2}{\lambda} = 10\sqrt{10}$. Thus, the minimum value of f is $f\left(10\sqrt{10}, 10\sqrt{10}\right) = 40\sqrt{10}$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Therefore, the dimensions of the rectangle that minimize the perimeter are $10\sqrt{10}$ m by $10\sqrt{10}$ m.

19. Let x and y be the dimensions of the rectangle in meters. Then the perimeter constraint is given by $2x + 2y = 100 \quad \Rightarrow \quad x + y = 50$. We want to maximize $f(x,y) = xy$, $x > 0$, $y > 0$ subject to $g(x,y) = x + y = 50$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle y, x \rangle = \langle \lambda, \lambda \rangle$, so $y = \lambda$, $x = \lambda$, and $x + y = 50$. Substituting the first two equations into the third gives $2\lambda = 50 \quad \Rightarrow \quad \lambda = 25$. Thus, the area is maximized when $x = y = 25$ meters and $f(25, 25) = 625\ \text{m}^2$.

20. Let x, y, and z be the dimensions of the box. The box has a square bottom, so $x = y$. We want to minimize $f(x,z) = x^2 + 4xz$ subject to $g(x,z) = x^2 z = 32{,}000$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x + 4z, 4x \rangle = \left\langle 2\lambda xz, \lambda x^2 \right\rangle$, so $2x + 4z = 2\lambda xz$, $4x = \lambda x^2$, and $x^2 z = 32{,}000$. The second equation implies $x = 0$ or $\lambda x = 4$. Note that $x = 0$ results in a zero volume. So let $\lambda = 4/x$. Substitution into the first equation gives $2x + 4z = 8z \quad \Rightarrow \quad x = 2z$, and substituting $2z$ for x into the third equation gives $4z^3 = 32{,}000 \quad \Rightarrow \quad z = 20 \quad \Rightarrow \quad x = 40$. Therefore, the dimensions of the box that will minimize the surface area are 40 cm by 40 cm by 20 cm.

21. The distance d from any point (x,y) in the xy-plane to the origin is given by $d = \sqrt{x^2 + y^2}$. We will minimize $d^2 = f(x,y) = x^2 + y^2$ subject to $g(x,y) = -2x + y = 3$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x, 2y \rangle = \langle -2\lambda, \lambda \rangle$, so $2x = -2\lambda$, $2y = \lambda$, and $-2x + y = 3$. The first equation implies $x = -\lambda$, and the second, $y = \dfrac{\lambda}{2}$. Then substitution into the third

equation gives $2\lambda + \dfrac{\lambda}{2} = 3 \quad\Rightarrow\quad \lambda = \frac{6}{5} \quad\Rightarrow\quad x = -\frac{6}{5}$ and $y = \frac{3}{5}$. Thus, the point on the line $y = 2x + 3$ that is closest to the origin is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

22. Let r and h be the radius and height of a right circular cylinder. We want to maximize $V(r, h) = \pi r^2 h$ subject to $g(r, h) = h + 2\pi r = 108$. Then $\nabla V = \lambda \nabla g \quad\Rightarrow\quad \langle 2\pi r h, \pi r^2 \rangle = \langle 2\pi\lambda, \lambda\rangle$, so $2\pi r h = 2\pi\lambda$, $\pi r^2 = \lambda$, and $h + 2\pi r = 108$. The first equation implies $rh = \lambda$ and substitution into the second gives $\pi r^2 = rh \quad\Rightarrow\quad r = 0$ or $h = \pi r$. $r \neq 0$ (else, $V = 0$), so substitute $h = \pi r$ into the third equation. Then $\pi r + 2\pi r = 108 \quad\Rightarrow\quad \pi r = 36 \quad\Rightarrow\quad h = 36$. Therefore, the dimensions that maximize the volume are $r = \dfrac{36}{\pi}$ inches and $h = 36$ inches for a volume of $V = \dfrac{46{,}656}{\pi}$ in^3.

23. $f(x, y) = x^2 + y^2,\; g(x, y) = xy = 1$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 2x, 2y\rangle = \langle \lambda y, \lambda x\rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \quad\Rightarrow\quad \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \quad\Rightarrow\quad y^2 = x^2 \quad\Rightarrow\quad y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1, 1)$ and $(-1, -1)$. Here there is no maximum value, since the constraint $xy = 1 \quad\Leftrightarrow\quad y = 1/x$ allows x or y to become arbitrarily large, and hence $f(x, y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1, 1) = f(-1, -1) = 2$.

24. $f(x, y, z) = x^2 + 2y^2 + 3z^2,\; g(x, y) = x + 2y + 3z = 10$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 2x, 4y, 6z\rangle = \langle \lambda, 2\lambda, 3\lambda\rangle$, so $2x = \lambda$, $4y = 2\lambda$, $6z = 3\lambda$, and $x + 2y + 3z = 10$. From the first three equations we have $2x = \lambda = 2y = 2z \quad\Rightarrow\quad x = y = z$, and substituting into the fourth equation gives $x + 2x + 3x = 10 \quad\Rightarrow\quad x = \frac{5}{3} = y = z$. Thus the only possible point for an extreme value of f is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$. Notice here that the constraint $x + 2y + 3z = 10$ allows any of $|x|$, $|y|$, or $|z|$ to be arbitrarily large, and hence $f(x, y, z) = x^2 + 2y^2 + 3z^2$ can be made arbitrarily large. So f has no maximum value subject to the constraint. The minimum value is $f\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) = 6\left(\frac{5}{3}\right)^2 = \frac{50}{3}$.

25. $f(x, y) = e^{xy}, g(x, y) = x^3 + y^3 = 16$. Then $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle ye^{xy}, xe^{xy}\rangle = \langle 3\lambda x^2, 3\lambda y^2\rangle$, so $ye^{xy} = 3\lambda x^2$, $xe^{xy} = 3\lambda y^2$, and $x^3 + y^3 = 16$. Multiplying the first equation by x and the second by y gives $xye^{xy} = 3\lambda x^3$ and $xye^{xy} = 3\lambda y^3 \quad\Rightarrow\quad 3\lambda x^3 = 3\lambda y^3 \quad\Rightarrow\quad x = y$ and substituting x for y into the third equation, we have $x^3 + x^3 = 16 \quad\Rightarrow\quad x = y = 2$. Therefore, f has an extreme value at the point $(2, 2)$ and evaluating f at that point we see $f(2, 2) = e^4$. Notice from the constraint that $y^3 = 16 - x^3$ and y^3 becomes increasingly negative as x^3 becomes arbitrarily large (similarly, x^3 can be increasingly negative while y^3 is arbitrarily large). Thus, for any small value $\varepsilon > 0$, we can find x and y that satisfy the constraint such that $0 < e^{xy} < \varepsilon$. Thus, f has no minimum value subject to the constraint and $f(2, 2) = e^4$ is a maximum.

26. $f(x, y, z) = 4x + 2y + z$, $g(x, y, z) = x^2 + y + z^2 = 1$. Then $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 4, 2, 1\rangle = \langle 2\lambda x, \lambda, 2\lambda z\rangle$, so $4 = 2\lambda x$, $2 = \lambda$, $1 = 2\lambda z$, and $x^2 + y + z^2 = 1$. As $\lambda = 2$ by equation two, we have $x = 1$ and $z = \frac{1}{4}$ by equations one and three, respectively. Substituting these values into equation four gives $1^2 + y + \left(\frac{1}{4}\right)^2 = 1 \quad\Rightarrow\quad y = -\frac{1}{16}$. Thus, f has an extreme value at $\left(1, -\frac{1}{16}, \frac{1}{4}\right)$. Notice that $y = 1 - (x^2 + z^2) < 0$ for $x^2 + z^2 > 1$, and substituting into f, we see that

$f(x,y,z) = 4x + 2 + z - 2(x^2 + z^2)$ can decrease without bound for values of (x,y,z) that satisfy the constraint. Thus, f has no minimum value subject to the constraint and $f\left(1, -\frac{1}{16}, \frac{1}{4}\right) = \frac{33}{8}$ is a maximum.

27. $f(x,y) = x^2 + y^2 + 4x - 4y$. For the interior of the region, we find the critical points: $f_x = 2x + 4$, $f_y = 2y - 4$, so the only critical point is $(-2,2)$ (which is inside the region) and $f(-2,2) = -8$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + y^2 = 9$, so $\nabla f = \lambda \nabla g \Rightarrow \langle 2x+4, 2y-4 \rangle = \langle 2\lambda x, 2\lambda y \rangle$. Thus $2x + 4 = 2\lambda x$ and $2y - 4 = 2\lambda y$. Adding the two equations gives $2x + 2y = 2\lambda x + 2\lambda y \Rightarrow x + y = \lambda(x+y) \Rightarrow (x+y)(\lambda - 1) = 0$, so $x + y = 0 \Rightarrow y = -x$ or $\lambda - 1 = 0 \Rightarrow \lambda = 1$. But $\lambda = 1$ leads to a contradition in $2x + 4 = 2\lambda x$, so $y = -x$ and $x^2 + y^2 = 9$ implies $2y^2 = 9 \Rightarrow y = \pm\frac{3}{\sqrt{2}}$. We have $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2} \approx 25.97$ and $f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - 12\sqrt{2} \approx -7.97$, so the maximum value of f on the disk $x^2 + y^2 \le 9$ is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2}$ and the minimum is $f(-2,2) = -8$.

28. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \le 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

29. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \Rightarrow x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maximums on the region and the latter are the minimums.

30. $f(x,y,z) = z$, $g(x,y,z) = x^2 + y^2 - z^2 = 0$, $h(x,y,z) = x + y + z = 24$, and $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow \langle 0,0,1 \rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z \rangle + \langle \mu, \mu, \mu \rangle$. Then $0 = 2\lambda x + \mu$, $0 = 2\lambda y + \mu$, $1 = -2\lambda z + \mu$, $x^2 + y^2 - z^2 = 0$, and $x + y + z = 24$. From the first two equations we have $-2\lambda x = \mu = -2\lambda y \Rightarrow \lambda = 0$ or $x = y$. But $\lambda = 0 \Rightarrow \mu = 0$ which contradicts the third equation, so $x = y$ and substitution into the last equation gives $z = 24 - 2x$. From the fourth equation we have $x^2 + x^2 - (24 - 2x)^2 = 0 \Rightarrow -2x^2 + 96x - 576 = 0 \Rightarrow x^2 - 48x + 288 = 0 \Rightarrow$

$$x = \frac{48 \pm \sqrt{1152}}{2} = 24 \pm 12\sqrt{2} = y.$$

Now $z = 24 - 2x$, so the possible points are $\left(24 + 12\sqrt{2}, 24 + 12\sqrt{2}, -24 - 24\sqrt{2}\right)$ and $\left(24 - 12\sqrt{2}, 24 - 12\sqrt{2}, -24 + 24\sqrt{2}\right)$. The maximum of f subject to the constraints is

$f\left(24-12\sqrt{2}, 24-12\sqrt{2}, -24+24\sqrt{2}\right) = -24+24\sqrt{2} \approx 9.94$ and the minimum is

$f\left(24+12\sqrt{2}, 24+12\sqrt{2}, -24-24\sqrt{2}\right) = -24-24\sqrt{2} \approx -57.94$.

31. $f(x,y,z) = x+y+z,\ g(x,y,z) = x^2+z^2 = 2,\ h(x,y,z) = x+y = 1$, and $\nabla f = \lambda\nabla g + \mu\nabla h \quad\Rightarrow$ $\langle 1,1,1\rangle = \langle 2\lambda x, 0, 2\lambda z\rangle + \langle \mu,\mu,0\rangle$. Then $1 = 2\lambda x + \mu,\quad 1 = \mu,\quad 1 = 2\lambda z,\quad x^2+z^2 = 2,\quad$ and $\quad x+y = 1$. Substituting $\mu = 1$ into the first equation gives $\lambda = 0$ or $x = 0$. But $\lambda = 0$ contradicts $1 = 2\lambda z$, so $x = 0$. Then $x+y=1 \quad\Rightarrow\quad y = 1$ and $\ x^2+z^2 = 2 \quad\Rightarrow\quad z = \pm\sqrt{2}$, so the possible points are $\left(0,1,\pm\sqrt{2}\,\right)$. The maximum value of f subject to the constraints is $f(0,1,\sqrt{2}\,) = 1+\sqrt{2} \approx 2.41$ and the minimum is $f(0,1,-\sqrt{2}\,) = 1-\sqrt{2} \approx -0.41$.

Note: Since $x+y=1$ is one of the constraints, we could have solved the problem by solving $f(x,z) = 1+z$ subject to $x^2+z^2 = 2$.

32. $f(x,y,z) = x^2+y^2+z^2,\ g(x,y,z) = x-y = 1,\ h(x,y,z) = y^2-z^2 = 1 \quad\Rightarrow\quad \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle \lambda, -\lambda, 0\rangle$, and $\mu\nabla h = \langle 0, 2\mu y, -2\mu z\rangle$. Then $2x = \lambda$, $2y = -\lambda + 2\mu y$, and $2z = -2\mu z \quad\Rightarrow\quad z = 0$ or $\mu = -1$. If $z = 0$ then $y^2 - z^2 = 1$ implies $y^2 = 1 \quad\Rightarrow\quad y = \pm 1$. If $y = 1$, $x - y = 1$ implies $x = 2$, and if $y = -1$ we have $x = 0$, so possible points are $(2,1,0)$ and $(0,-1,0)$. If $\mu = -1$ then $2y = -\lambda + 2\mu y$ implies $4y = -\lambda$, but $\lambda = 2x$ so $4y = -2x \quad\Rightarrow\quad x = -2y$ and $x - y = 1$ implies $-3y = 1 \quad\Rightarrow\quad y = -\frac{1}{3}$. But then $y^2 - z^2 = 1$ implies $z^2 = -\frac{8}{9}$, an impossibility. Thus the maximum value of f subject to the constraints is $f(2,1,0) = 5$ and the minimum is $f(0,-1,0) = 1$.

Note: Since $x - y = 1 \quad\Rightarrow\quad x = y+1$ is one of the constraints we could have solved the problem by solving $f(y,z) = (y+1)^2 + y^2 + z^2$ subject to $y^2 - z^2 = 1$.

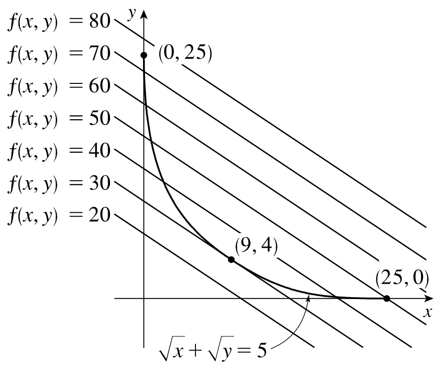
33. $f(x,y,z) = yz + xy,\ g(x,y,z) = xy = 1,\ h(x,y,z) = y^2 + z^2 = 1 \quad\Rightarrow\quad \nabla f = \langle y, x+z, y\rangle$, $\lambda\nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu\nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ $[y \neq 0$ since $g(x,y,z) = 1]$, $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

34. (a) $f(x,y) = 2x + 3y,\ g(x,y) = \sqrt{x} + \sqrt{y} = 5 \quad\Rightarrow\quad \nabla f = \langle 2, 3\rangle = \lambda\nabla g = \lambda\left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}\right\rangle$. Then

$2 = \dfrac{\lambda}{2\sqrt{x}}$ and $3 = \dfrac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x} = \lambda = 6\sqrt{y} \quad\Rightarrow\quad \sqrt{y} = \frac{2}{3}\sqrt{x}$. With $\sqrt{x} + \sqrt{y} = 5$ we have $\sqrt{x} + \frac{2}{3}\sqrt{x} = 5 \quad\Rightarrow$ $\sqrt{x} = 3 \quad\Rightarrow\quad x = 9$. Substituting into $\sqrt{y} = \frac{2}{3}\sqrt{x}$ gives $\sqrt{y} = 2$ or $y = 4$. Thus the only possible extreme value subject to the constraint is $f(9,4) = 30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25,0) = 50$ which is larger than the result of part (a).

(c) 

We can see from the level curves of f that the maximum occurs at the left endpoint $(0, 25)$ of the constraint curve g. The maximum value is $f(0, 25) = 75$.

(d) Here ∇g does not exist if $x = 0$ or $y = 0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

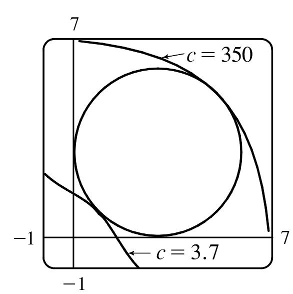
(e) Here $f(9, 4)$ is the absolute *minimum* of f subject to g.

35. (a) $f(x,y) = x,\;\; g(x,y) = y^2 + x^4 - x^3 = 0 \;\;\Rightarrow\;\; \nabla f = \langle 1, 0\rangle = \lambda \nabla g = \lambda \left\langle 4x^3 - 3x^2, 2y \right\rangle$. Then $1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \neq 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \;\;\Rightarrow\;\; x^3(x-1) = 0 \;\;\Rightarrow\;\; x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1, 0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \;\;\Leftrightarrow\;\; y^2 = x^3 - x^4$. The left side is nonnegative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore the minimum possible value for $f(x, y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0,0) = \lambda \langle 0 - 0, 0\rangle = \langle 0, 0\rangle$ and $\nabla f(0,0) = \langle 1, 0\rangle$, so $\nabla f(0,0) \neq \lambda \nabla g(0,0)$ for all values of λ.

(c) Here $\nabla g(0,0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

36. (a) The graphs of $f(x,y) = 3.7$ and $f(x,y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x,y) = x^3 + y^3 + 3xy$ subject to the constraint $(x-3)^2 + (y-3)^2 = 9$.



(b) Let $g(x,y) = (x-3)^2 + (y-3)^2$. We calculate $f_x(x,y) = 3x^2 + 3y$, $f_y(x,y) = 3y^2 + 3x$, $g_x(x,y) = 2x - 6$, and $g_y(x,y) = 2y - 6$, and use a CAS to search for solutions to the equations $g(x,y) = (x-3)^2 + (y-3)^2 = 9$, $f_x = \lambda g_x$, and $f_y = \lambda g_y$. The solutions are $(x,y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879)$ and $(x,y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121)$. These give $f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673$ and $f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33$, in accordance with part (a).

37. $P(L,K)=bL^{\alpha}K^{1-\alpha},\ g(L,K)=mL+nK=p \quad\Rightarrow\quad \nabla P=\left\langle \alpha bL^{\alpha-1}K^{1-\alpha},(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle,\ \lambda\nabla g=\langle \lambda m,\lambda n\rangle$.

Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^{\alpha}=\lambda n$ and $mL+nK=p$, so $\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^{\alpha}(L/K)^{1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$. Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

38. $C(L,K)=mL+nK,\ g(L,K)=bL^{\alpha}K^{1-\alpha}=Q \quad\Rightarrow\quad \nabla C=\langle m,n\rangle,\ \lambda\nabla g=\left\langle \lambda\alpha bL^{\alpha-1}K^{1-\alpha},\lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle$.

Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha}=\dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^{\alpha}$ and $bL^{\alpha}K^{1-\alpha}=Q \quad\Rightarrow\quad \dfrac{n\alpha}{m(1-\alpha)}=\left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^{\alpha} \quad\Rightarrow$

$L=\dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha}=Q$. Hence $K=\dfrac{Q}{b\left(n\alpha/[m(1-\alpha)]\right)^{\alpha}}=\dfrac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$

and $L=\dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

39. Let the sides of the rectangle be x and y. Then $f(x,y)=xy$, $g(x,y)=2x+2y=p \quad\Rightarrow\quad \nabla f(x,y)=\langle y,x\rangle$, $\lambda\nabla g=\langle 2\lambda,2\lambda\rangle$. Then $\lambda=\frac{1}{2}y=\frac{1}{2}x$ implies $x=y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

40. We maximize $A^2=f(x,y,z)=s(s-x)(s-y)(s-z)$ subject to $g(x,y,z)=x+y+z$. Then $\nabla f=\langle -s(s-y)(s-z),-s(s-x)(s-z),-s(s-x)(s-y)\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Thus $(s-y)(s-z)=(s-x)(s-z)$ **(1)**, and $(s-x)(s-z)=(s-x)(s-y)$ **(2)**. **(1)** implies $x=y$ while **(2)** implies $y=z$, so $x=y=z=p/3$ and the triangle with maximum area is equilateral.

41. The distance from $(2,0,-3)$ to a point (x,y,z) on the plane is $d=\sqrt{(x-2)^2+y^2+(z+3)^2}$, so we seek to minimize $d^2=f(x,y,z)=(x-2)^2+y^2+(z+3)^2$ subject to the constraint that (x,y,z) lies on the plane $x+y+z=1$, that is, that $g(x,y,z)=x+y+z=1$. Then $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle 2(x-2),2y,2(z+3)\rangle=\langle\lambda,\lambda,\lambda\rangle$, so $x=(\lambda+4)/2$, $y=\lambda/2$, $z=(\lambda-6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda}{2}+\dfrac{\lambda-6}{2}=1 \quad\Rightarrow\quad 3\lambda-2=2 \quad\Rightarrow$ $\lambda=\frac{4}{3}$, so $x=\frac{8}{3}$, $y=\frac{2}{3}$, and $z=-\frac{7}{3}$. This must correspond to a minimum, so the shortest distance is

$$d=\sqrt{\left(\tfrac{8}{3}-2\right)^2+\left(\tfrac{2}{3}\right)^2+\left(-\tfrac{7}{3}+3\right)^2}=\sqrt{\tfrac{4}{3}}=\tfrac{2}{\sqrt{3}}.$$

42. The distance from $(0,1,1)$ to a point (x,y,z) on the plane is $d=\sqrt{x^2+(y-1)^2+(z-1)^2}$, so we minimize $d^2=f(x,y,z)=x^2+(y-1)^2+(z-1)^2$ subject to the constraint that (x,y,z) lies on the plane $x-2y+3z=6$, that is, $g(x,y,z)=x-2y+3z=6$. Then $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle 2x,2(y-1),2(z-1)\rangle=\langle\lambda,-2\lambda,3\lambda\rangle$, so $x=\lambda/2$, $y=1-\lambda$, $z=(3\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda}{2}-2(1-\lambda)+3\cdot\dfrac{3\lambda+2}{2}=6 \quad\Rightarrow\quad \lambda=\frac{5}{7}$, so $x=\frac{5}{14}$, $y=\frac{2}{7}$, and $z=\frac{29}{14}$. This must correspond to a minimum, so the point on the plane closest to the point $(0,1,1)$ is $\left(\frac{5}{14},\frac{2}{7},\frac{29}{14}\right)$.

43. Let $f(x,y,z) = d^2 = (x-4)^2 + (y-2)^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-4), 2(y-2), 2z\rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z\rangle$, so $x - 4 = \lambda x$, $y - 2 = \lambda y$, and $z = -\lambda z$. From the last equation we have $z + \lambda z = 0 \Rightarrow z(1+\lambda) = 0$, so either $z = 0$ or $\lambda = -1$. But from the constraint equation we have $z = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$ which is not possible from the first two equations. So $\lambda = -1$ and $x - 4 = \lambda x \Rightarrow x = 2$, $y - 2 = \lambda y \Rightarrow y = 1$, and $x^2 + y^2 - z^2 = 0 \Rightarrow 4 + 1 - z^2 = 0 \Rightarrow z = \pm\sqrt{5}$. This must correspond to a minimum, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

44. Let $f(x,y,z) = d^2 = x^2 + y^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = y^2 - xz = 9$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z\rangle = \langle -\lambda z, 2\lambda y, -\lambda x\rangle$, so $2x = -\lambda z$, $y = \lambda y$, and $2z = -\lambda x$. If $x = 0$ then the last equation implies $z = 0$, and from the constraint $y^2 - xz = 9$ we have $y = \pm 3$. If $x \neq 0$, then the first and third equations give $\lambda = -2x/z = -2z/x \Rightarrow x^2 = z^2$. From the second equation we have $y = 0$ or $\lambda = 1$. If $y = 0$ then $y^2 - xz = 9 \Rightarrow z = -9/x$ and $x^2 = z^2 \Rightarrow x^2 = 81/x^2 \Rightarrow x = \pm 3$. Since $z = -9/x$, $x = 3 \Rightarrow z = -3$ and $x = -3 \Rightarrow z = 3$. If $\lambda = 1$, then $2x = -z$ and $2z = -x$ which implies $x = z = 0$, contradicting the assumption that $x \neq 0$. Thus the possible points are $(0, \pm 3, 0)$, $(3, 0, -3)$, $(-3, 0, 3)$. We have $f(0, \pm 3, 0) = 9$ and $f(3, 0, -3) = f(-3, 0, 3) = 18$, so the points on the surface that are closest to the origin are $(0, \pm 3, 0)$.

45. Maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = x + y + z = 100$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy\rangle = \lambda \langle 1, 1, 1\rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

46. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x + y + z = 12$ with $x > 0$, $y > 0$, $z > 0$. Then $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z\rangle = \lambda\langle 1, 1, 1\rangle \Rightarrow 2x = \lambda$, $2y = \lambda$, $2z = \lambda \Rightarrow x = y = z$, so $x + y + z = 12 \Rightarrow 3x = 12 \Rightarrow x = 4 = y = z$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Thus the three numbers are 4, 4, and 4.

47. If the dimensions are $2x$, $2y$, and $2z$, then maximize $f(x,y,z) = (2x)(2y)(2z) = 8xyz$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = r^2$ $(x > 0, y > 0, z > 0)$. Then $\nabla f = \lambda \nabla g \Rightarrow \langle 8yz, 8xz, 8xy\rangle = \lambda \langle 2x, 2y, 2z\rangle \Rightarrow 8yz = 2\lambda x$, $8xz = 2\lambda y$, and $8xy = 2\lambda z$, so $\lambda = \dfrac{4yz}{x} = \dfrac{4xz}{y} = \dfrac{4xy}{z}$. This gives $x^2 z = y^2 z \Rightarrow x^2 = y^2$ (since $z \neq 0$) and $xy^2 = xz^2 \Rightarrow z^2 = y^2$, so $x^2 = y^2 = z^2 \Rightarrow x = y = z$, and substituting into the constraint equation gives $3x^2 = r^2 \Rightarrow x = r/\sqrt{3} = y = z$. Thus the largest volume of such a box is

$$f\left(\tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}\right) = 8\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}r^3.$$

48. If the dimensions of the box are x, y, and z, then minimize $f(x,y,z) = 2xy + 2xz + 2yz$ subject to $g(x,y,z) = xyz = 1000$ $(x > 0, y > 0, z > 0)$. Then $\nabla f = \lambda \nabla g \Rightarrow$

$\langle 2y+2z, 2x+2z, 2x+2y\rangle = \lambda\langle yz, xz, xy\rangle \Rightarrow 2y+2z=\lambda yz,\ 2x+2z=\lambda xz,\ 2x+2y=\lambda xy$. Solving for λ in each equation gives $\lambda = \dfrac{2}{z}+\dfrac{2}{y}=\dfrac{2}{z}+\dfrac{2}{x}=\dfrac{2}{y}+\dfrac{2}{x} \Rightarrow x=y=z$. From $xyz=1000$ we have $x^3=1000 \Rightarrow x=10$ and the dimensions of the box are $x=y=z=10$ cm.

49. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=x+2y+3z=6$. $\nabla f=\lambda\nabla g \Rightarrow \langle yz,xz,xy\rangle=\lambda\langle 1,2,3\rangle$. Then $\lambda=yz=\frac{1}{2}xz=\frac{1}{3}xy$ implies $x=2y$, $z=\frac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\frac{2}{3}$ and the volume is $V=\frac{4}{3}$.

50. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=xy+yz+xz=32$. $\nabla f=\lambda\nabla g \Rightarrow$ $\langle yz,xz,xy\rangle=\lambda\langle y+z,x+z,x+y\rangle$. Then $\lambda(y+z)=yz$ **(1)**, $\lambda(x+z)=xz$ **(2)**, and $\lambda(x+y)=xy$ **(3)**. And **(1)** minus **(2)** implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then **(1)** implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly **(2)** minus **(3)** implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\frac{8}{\sqrt{6}}$ cm.

51. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=4(x+y+z)=c$. $\nabla f=\lambda\nabla g \Rightarrow \langle yz,xz,xy\rangle=\lambda\langle 4,4,4\rangle$. Then $yz=4\lambda$, $xz=4\lambda$, and $xy=4\lambda$. Multiplying by x, y, and z, respectively, gives us $xyz=4\lambda x=4\lambda y=4\lambda z$, so $x=y=z$. Substituting y and z for x in g gives us $4(3x)=c \Rightarrow x=y=z=\frac{1}{12}c$ are the dimensions of the cube giving the maximum volume.

52. Maximize $C(x,y,z)=5xy+2xz+2yz$ subject to $g(x,y,z)=xyz=V$. $\nabla C=\lambda\nabla g \Rightarrow$ $\langle 5y+2z,5x+2z,2x+2y\rangle=\lambda\langle yz,xz,xy\rangle$. Then $\lambda yz=5y+2z$ **(1)**, $\lambda xz=5x+2z$ **(2)**, $\lambda xy=2(x+y)$ **(3)**, and $xyz=V$ **(4)**. Now **(1)** $-$ **(2)** implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0, so $x=y$. Then twice **(2)** minus five times **(3)** together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives $z=\frac{5}{2}y$ [again $\lambda\neq 2/y$ or else **(3)** implies $y=0$]. Hence $\frac{5}{2}y^3=V$ and the dimensions which minimize cost are $x=y=\sqrt[3]{\frac{2}{5}V}$ units, $z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

53. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x,y,z)=xyz$ $[x,y,z>0]$ subject to the constraint $L=\sqrt{x^2+y^2+z^2}$ or $g(x,y,z)=x^2+y^2+z^2=L^2$. $\nabla f=\lambda\nabla g \Rightarrow$ $\langle yz,xz,xy\rangle=\lambda\langle 2x,2y,2z\rangle$, so $yz=2\lambda x \Rightarrow \lambda=\dfrac{yz}{2x}$, $xz=2\lambda y \Rightarrow \lambda=\dfrac{xz}{2y}$, and $xy=2\lambda z \Rightarrow \lambda=\dfrac{xy}{2z}$. Thus $\lambda=\dfrac{yz}{2x}=\dfrac{xz}{2y} \Rightarrow x^2=y^2$ [since $z\neq 0$] $\Rightarrow x=y$ and $\lambda=\dfrac{yz}{2x}=\dfrac{xy}{2z} \Rightarrow x=z$ [since $y\neq 0$]. Substituting into the constraint equation gives $x^2+x^2+x^2=L^2 \Rightarrow x^2=L^2/3 \Rightarrow x=L/\sqrt{3}=y=z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3=L^3/\left(3\sqrt{3}\right)$.

54. Let the dimensions of the box be x, y, and z. Then we wish to maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = 2x + 2y + z = 108 \quad \Rightarrow \quad \nabla f = \langle yz, xz, xy \rangle$ and $\lambda \nabla g = \langle 2\lambda, 2\lambda, \lambda \rangle$. Now $yz = 2\lambda$, $xz = 2\lambda$, and $xy = \lambda$, with $x \neq 0$, $y \neq 0$, $z \neq 0$. Then $yz = xz \quad \Rightarrow \quad y = x$ **(1)** and $2xy = xz \quad \Rightarrow \quad 2y = z$ **(2)**. Substituting **(1)** and **(2)** into the constraint, we get $2y + 2y + 2y = 108 \quad \Rightarrow \quad y = x = 18$, and hence $z = 36$. Thus, the dimensions of the box that will give the largest volume and still meet USPS guidelines are 18 in by 18 in by 36 in.

55. If r and h are the radius and the height of the silo, respectively, we need to maximize $V(r,h) = \pi r^2 h + \frac{2}{3}\pi r^3$ subject to $g(r,h) = 2\pi rh + \pi r^2 + (4\pi r^2)/2 = 2\pi rh + 3\pi r^2 = S$. Then $\nabla V = \lambda \nabla g \quad \Rightarrow$ $\langle 2\pi rh + 2\pi r^2, \pi r^2 \rangle = \langle 2\lambda\pi h + 6\lambda\pi r, 2\lambda\pi r \rangle$, so the three equations are $2\pi rh + 2\pi r^2 = 2\lambda\pi h + 6\lambda\pi r$, $\pi r^2 = 2\lambda\pi r$, and $2\pi rh + 3\pi r^2 = S$. The second equation implies $r = 2\lambda$ $[r \neq 0]$. Substituting $r = 2\lambda$ into the first equation gives $2\pi(2\lambda)h + 2\pi(2\lambda)^2 = 2\lambda\pi h + 6\lambda\pi(2\lambda) \quad \Rightarrow \quad 4\pi\lambda h + 8\pi\lambda^2 = 2\lambda\pi h + 12\pi\lambda^2 \quad \Rightarrow \quad 2\pi\lambda h = 4\pi\lambda^2 \quad \Rightarrow \quad h = 2\lambda$. Thus, $r = 2\lambda = h$, and the volume of the silo is maximized, subject to a given surface area, when the radius and height are equal.

56. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $2xy + 2yz + 2xz = 1500$ and its total edge length is $4x + 4y + 4z = 200$. We find the extreme values of $f(x,y,z)$ subject to the constraints $g(x,y,z) = xy + yz + xz = 750$ and $h(x,y,z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So $yz = \lambda(y+z) + \mu$ **(1)**, $xz = \lambda(x+z) + \mu$ **(2)**, and $xy = \lambda(x+y) + \mu$ **(3)**. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then **(1)** minus **(2)** implies $z\,(y - x) = \lambda(y - x)$ or $\lambda = z$, and **(1)** minus **(3)** implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}(50 \mp 10\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10})\right)$. Thus the minimum of f is $f\left(\frac{1}{3}(50 - 10\sqrt{3}), \frac{1}{3}(50 + 5\sqrt{10}), \frac{1}{3}(50 + 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 - 2500\sqrt{10})$, and its maximum is $f\left(\frac{1}{3}(50 + 10\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 + 2500\sqrt{10})$.

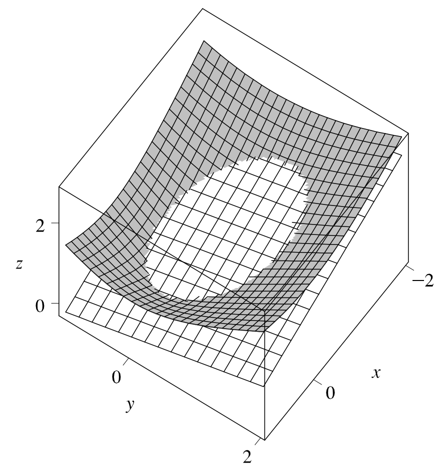
Note: If either y or z is the distinct side, then symmetry gives the same result.

57. We need to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x,y,z) = x + y + 2z = 2$ and $h(x,y,z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**. From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The

last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

58. (a) After plotting $z = \sqrt{x^2 + y^2}$, the top half of the cone, and the plane $z = (5 - 4x + 3y)/8$ we see the ellipse formed by the intersection of the surfaces. The ellipse can be plotted explicitly using cylindrical coordinates (see Section 15.7): The cone is given by $z = r$, and the plane is $4r\cos\theta - 3r\sin\theta + 8z = 5$. Substituting $z = r$ into the plane equation gives $4r\cos\theta - 3r\sin\theta + 8r = 5 \quad\Rightarrow\quad r = \dfrac{5}{4\cos\theta - 3\sin\theta + 8}$.

Since $z = r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta = t,\ r = z = \dfrac{5}{4\cos t - 3\sin t + 8},\ 0 \le t \le 2\pi$.



(b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints $g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad\Rightarrow\quad \langle 0, 0, 1\rangle = \lambda\langle 4, -3, 8\rangle + \mu\langle 2x, 2y, -2z\rangle$, so we need $4\lambda + 2\mu x = 0 \quad\Rightarrow\quad x = -\frac{2\lambda}{\mu}$ **(1)**, $-3\lambda + 2\mu y = 0 \quad\Rightarrow\quad y = \frac{3\lambda}{2\mu}$ **(2)**, $8\lambda - 2\mu z = 1 \quad\Rightarrow\quad z = \frac{8\lambda - 1}{2\mu}$ **(3)**, $4x - 3y + 8z = 5$ **(4)**, and $x^2 + y^2 = z^2$ **(5)**. [Note that $\mu \neq 0$, else $\lambda = 0$ from **(1)**, but substitution into **(3)** gives a contradiction.]

Substituting **(1)**, **(2)**, and **(3)** into **(4)** gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \quad\Rightarrow\quad \mu = \frac{39\lambda - 8}{10}$ and into **(5)** gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \quad\Rightarrow\quad 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \quad\Rightarrow\quad 39\lambda^2 - 16\lambda + 1 = 0 \quad\Rightarrow\quad \lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$.

If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}$, $y = -\frac{3}{13}$, $z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}$, $y = 1$, $z = \frac{5}{3}$. Thus the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.

59. $f(x, y, z) = ye^{x-z},\ g(x, y, z) = 9x^2 + 4y^2 + 36z^2 = 36,\ h(x, y, z) = xy + yz = 1$.

$\nabla f = \lambda\nabla g + \mu\nabla h \quad\Rightarrow\quad \left\langle ye^{x-z}, e^{x-z}, -ye^{x-z}\right\rangle = \lambda\langle 18x, 8y, 72z\rangle + \mu\langle y, x + z, y\rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x + z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$\begin{array}{lllll}
x \approx 0.222444, & y \approx -2.157012, & z \approx -0.686049, & \lambda \approx -0.200401, & \mu \approx 2.108584 \\
x \approx -1.951921, & y \approx -0.545867, & z \approx 0.119973, & \lambda \approx 0.003141, & \mu \approx -0.076238 \\
x \approx 0.155142, & y \approx 0.904622, & z \approx 0.950293, & \lambda \approx -0.012447, & \mu \approx 0.489938 \\
x \approx 1.138731, & y \approx 1.768057, & z \approx -0.573138, & \lambda \approx 0.317141, & \mu \approx 1.862675
\end{array}$$

[continued]

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$,

$f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$,

$f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the minimum is approximately -5.3506.

60. $f(x,y,z) = x + y + z$, $g(x,y,z) = x^2 - y^2 - z = 0$, $h(x,y,z) = x^2 + z^2 = 4$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 1,1,1 \rangle = \lambda \langle 2x, -2y, -1 \rangle + \mu \langle 2x, 0, 2z \rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$, $1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$\begin{array}{lllll} x \approx -1.652878, & y \approx -1.964194, & z \approx -1.126052, & \lambda \approx 0.254557, & \mu \approx -0.557060 \\ x \approx -1.502800, & y \approx 0.968872, & z \approx 1.319694, & \lambda \approx -0.516064, & \mu \approx 0.183352 \\ x \approx -0.992513, & y \approx 1.649677, & z \approx -1.736352, & \lambda \approx -0.303090, & \mu \approx -0.200682 \\ x \approx 1.895178, & y \approx 1.718347, & z \approx 0.638984, & \lambda \approx -0.290977, & \mu \approx 0.554805 \end{array}$$

Substituting these values into f gives $f(-1.652878, -1.964194, -1.126052) \approx -4.7431$,

$f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$,

$f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus the maximum is approximately 4.2525, and the minimum is approximately -4.7431.

61. $f(x,y) = 3x^2 + y^2$, $g(x,y) = x^2 + y^2 - 4y = 0$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 6x, 2y \rangle = \langle 2\lambda x, \lambda(2y - 4) \rangle$, so the three equations are $6x = 2\lambda x$, $2y = \lambda(2y - 4)$, and $x^2 + y^2 - 4y = 0$. The first equation implies $x = 0$ or $\lambda = 3$. If $x = 0$, the third equation implies $y = 0$ or $y = 4$. If $\lambda = 3$, the second equation implies $y = 3$ and substitution into the third equation gives $x = \pm\sqrt{3}$. Thus, f has possible extreme values at $(0,0)$, $(0,4)$, $(\pm\sqrt{3}, 3)$. Evaluating f at these points we see that the maximum value is $f(\pm\sqrt{3}, 3) = 18$ and the minimum value is $f(0,0) = 0$.

The minimum value of f occurs at $(0,0)$. Substituting $y = 0$ into the second equation gives $2(0) = \lambda(2(0) - 4)$, which is true only if $\lambda = 0$. Thus, the minimum value corresponds to $\lambda = 0$.

62. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum\limits_{i=1}^n x_i y_i$, $g(x_1, \ldots, x_n) = \sum\limits_{i=1}^n x_i^2$, and $h(x_1, \ldots, x_n) = \sum\limits_{i=1}^n y_i^2$. Then

$\nabla f = \nabla \sum\limits_{i=1}^n x_i y_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n \rangle$, $\nabla g = \nabla \sum\limits_{i=1}^n x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0 \rangle$ and

$\nabla h = \nabla \sum\limits_{i=1}^n y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n \rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \quad \Leftrightarrow \quad y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$,

$1 \le i \le n$. Then $1 = \sum\limits_{i=1}^n y_i^2 = \sum\limits_{i=1}^n 4\lambda^2 x_i^2 = 4\lambda^2 \sum\limits_{i=1}^n x_i^2 = 4\lambda^2 \quad \Rightarrow \quad \lambda = \pm\frac{1}{2}$. If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right)x_i = x_i$,

$1 \le i \le n$. Thus $\sum\limits_{i=1}^n x_i y_i = \sum\limits_{i=1}^n x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum\limits_{i=1}^n x_i y_i = -1$. Similarly we get

$\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \le i \le n$, and $\sum\limits_{i=1}^n x_i y_i = \pm 1$. Thus the maximum value of $\sum\limits_{i=1}^n x_i y_i$ is 1.

(b) Here we assume $\sum\limits_{i=1}^{n} a_i^2 \neq 0$ and $\sum\limits_{i=1}^{n} b_i^2 \neq 0$. (If $\sum\limits_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.)

$x_i = \dfrac{a_i}{\sqrt{\sum a_j^2}} \quad \Rightarrow \quad \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_j^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_j^2}} \quad \Rightarrow \quad \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_j^2} = 1$. Therefore, from part (a),

$\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \leq 1 \quad \Leftrightarrow \quad \sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.

63. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to

$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$\nabla f = \left\langle \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \ldots, \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda \rangle$, so we need to solve the system of equations

$$\begin{aligned}
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1 \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2 \\
&\vdots \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n
\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is

$f\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right) = \sqrt[n]{\dfrac{c}{n} \cdot \dfrac{c}{n} \cdot \cdots \cdot \dfrac{c}{n}} = \dfrac{c}{n}$.

(b) From part (a), $\dfrac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\dfrac{c}{n}$, but this can occur only at the point $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \dfrac{c}{n}$.

APPLIED PROJECT Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c \ln\left(1 - \dfrac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the

second stage is $\Delta V_2 = -c\ln\left(1 - \dfrac{(1-S)M_2}{M_3 + A + M_2}\right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c\ln\left(1 - \dfrac{(1-S)M_3}{A + M_3}\right)$. Since the rocket started from rest, the final velocity attained is

$$\begin{aligned}
v_f &= \Delta V_1 + \Delta V_2 + \Delta V_3 \\
&= -c\ln\left(1 - \frac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right) + (-c)\ln\left(1 - \frac{(1-S)M_2}{M_3 + A + M_2}\right) + (-c)\ln\left(1 - \frac{(1-S)M_3}{A + M_3}\right) \\
&= -c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A - (1-S)M_1}{M_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A - (1-S)M_2}{M_2 + M_3 + A}\right)\right. \\
&\qquad\qquad \left. + \ln\left(\frac{M_3 + A - (1-S)M_3}{M_3 + A}\right)\right] \\
&= c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]
\end{aligned}$$

2. Define $N_1 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A}$, and $N_3 = \dfrac{M_3 + A}{SM_3 + A}$. Then

$$\begin{aligned}
\frac{(1-S)N_1}{1 - SN_1} &= \frac{(1-S)\,\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}}{1 - S\,\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}} = \frac{(1-S)(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - S(M_1 + M_2 + M_3 + A)} \\
&= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{(1-S)(M_2 + M_3 + A)} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A}
\end{aligned}$$

as desired.

Similarly,

$$\frac{(1-S)N_2}{1 - SN_2} = \frac{(1-S)(M_2 + M_3 + A)}{SM_2 + M_3 + A - S(M_2 + M_3 + A)} = \frac{(1-S)(M_2 + M_3 + A)}{(1-S)(M_3 + A)} = \frac{M_2 + M_3 + A}{M_3 + A}$$

and
$$\frac{(1-S)N_3}{1 - SN_3} = \frac{(1-S)(M_3 + A)}{SM_3 + A - S(M_3 + A)} = \frac{(1-S)(M_3 + A)}{(1-S)(A)} = \frac{M_3 + A}{A}$$

Then

$$\begin{aligned}
\frac{M + A}{A} &= \frac{M_1 + M_2 + M_3 + A}{A} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} \cdot \frac{M_2 + M_3 + A}{M_3 + A} \cdot \frac{M_3 + A}{A} \\
&= \frac{(1-S)N_1}{1 - SN_1} \cdot \frac{(1-S)N_2}{1 - SN_2} \cdot \frac{(1-S)N_3}{1 - SN_3} = \frac{(1-S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}
\end{aligned}$$

3. Since $A > 0$, $M + A$ and consequently $\dfrac{M + A}{A}$ is minimized for the same values as M. $\ln x$ is a strictly increasing function, so $\ln\left(\dfrac{M + A}{A}\right)$ must give a minimum for the same values as $\dfrac{M + A}{A}$ and hence M. We then wish to minimize $\ln\left(\dfrac{M + A}{A}\right)$ subject to the constraint $c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. From Problem 2,

$$\ln\left(\frac{M+A}{A}\right) = \ln\left(\frac{(1-S)^3\,N_1N_2N_3}{(1-SN_1)\,(1-SN_2)\,(1-SN_3)}\right)$$
$$= 3\ln(1-S) + \ln N_1 + \ln N_2 + \ln N_3 - \ln(1-SN_1) - \ln(1-SN_2) - \ln(1-SN_3)$$

Using the method of Lagrange multipliers, we need to solve $\nabla\left[\ln\left(\frac{M+A}{A}\right)\right] = \lambda\nabla[c(\ln N_1 + \ln N_2 + \ln N_3)]$ with $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$ in terms of N_1, N_2, and N_3. The resulting system is

$$\frac{1}{N_1} + \frac{S}{1-SN_1} = \lambda\frac{c}{N_1} \qquad \frac{1}{N_2} + \frac{S}{1-SN_2} = \lambda\frac{c}{N_2} \qquad \frac{1}{N_3} + \frac{S}{1-SN_3} = \lambda\frac{c}{N_3}$$

$$c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$$

One approach to solving the system is isolating $c\lambda$ in the first three equations which gives

$$1 + \frac{SN_1}{1-SN_1} = c\lambda = 1 + \frac{SN_2}{1-SN_2} = 1 + \frac{SN_3}{1-SN_3} \quad\Rightarrow\quad \frac{N_1}{1-SN_1} = \frac{N_2}{1-SN_2} = \frac{N_3}{1-SN_3} \quad\Rightarrow$$

$N_1 = N_2 = N_3$ (Verify!). This says the fourth equation can be expressed as $c(\ln N_1 + \ln N_1 + \ln N_1) = v_f \quad\Rightarrow$ $3c\ln N_1 = v_f \quad\Rightarrow\quad \ln N_1 = \dfrac{v_f}{3c}$. Thus the minimum mass M of the rocket engine is attained for $N_1 = N_2 = N_3 = e^{v_f/(3c)}$.

4. Using the previous results, $\dfrac{M+A}{A} = \dfrac{(1-S)^3N_1N_2N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} = \dfrac{(1-S)^3\left[e^{v_f/(3c)}\right]^3}{\left[1-Se^{v_f/(3c)}\right]^3} = \dfrac{(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3}$.

Then $M = \dfrac{A(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3} - A$.

5. (a) From Problem 4, $M = \dfrac{A(1-0.2)^3e^{(17{,}500/6000)}}{\left(1-0.2e^{[17{,}500/(3\cdot 6000)]}\right)^3} - A \approx 90.4A - A = 89.4A$.

(b) First, $N_3 = \dfrac{M_3+A}{SM_3+A} \quad\Rightarrow\quad e^{[17{,}500/(3\cdot 6000)]} = \dfrac{M_3+A}{0.2M_3+A} \quad\Rightarrow\quad M_3 = \dfrac{A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 3.49A$.

Then $N_2 = \dfrac{M_2+M_3+A}{SM_2+M_3+A} = \dfrac{M_2+3.49A+A}{0.2M_2+3.49A+A} \quad\Rightarrow\quad M_2 = \dfrac{4.49A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 15.67A$ and

$$N_3 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A} = \frac{M_1+15.67A+3.49A+A}{0.2M_1+15.67A+3.49A+A} \quad\Rightarrow\quad M_1 = \frac{20.16A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 70.36A.$$

6. As in Problem 5, $N_3 = \dfrac{M_3+A}{SM_3+A} \quad\Rightarrow\quad e^{24{,}700/(3\cdot 6000)} = \dfrac{M_3+A}{0.2M_3+A} \quad\Rightarrow\quad M_3 = \dfrac{A(1-e^{247/180})}{0.2e^{247/180}-1} \approx 13.9A$,

$N_2 = \dfrac{M_2+M_3+A}{SM_2+M_3+A} = \dfrac{M_2+13.9A+A}{0.2M_2+13.9A+A} \quad\Rightarrow\quad M_2 = \dfrac{14.9A(1-e^{247/180})}{0.2e^{247/180}-1} \approx 208A$, and

$$N_3 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A} = \frac{M_1+208A+13.9A+A}{0.2M_1+208A+13.9A+A} \quad\Rightarrow\quad M_1 = \frac{222.9A(1-e^{247/180})}{0.2e^{247/180}-1} \approx 3110A.$$

Here $A = 500$, so the mass of each stage of the rocket engine is approximately $M_1 = 3110(500) = 1{,}550{,}000$ lb, $M_2 = 208(500) = 104{,}000$ lb, and $M_3 = 13.9(500) = 6950$ lb.

APPLIED PROJECT Hydro-Turbine Optimization

1. We wish to maximize the total energy production for a given total flow, so we can say Q_T is fixed and we want to maximize $KW_1 + KW_2 + KW_3$. Notice each KW_i has a constant factor $\left(170 - 1.6 \cdot 10^{-6} Q_T^2\right)$, so to simplify the computations we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2, Q_3) &= \frac{KW_1 + KW_2 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right) \\ &\qquad + \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2, Q_3) = Q_1 + Q_2 + Q_3 = Q_T$. So first we find the values of Q_1, Q_2, Q_3 where $\nabla f(Q_1, Q_2, Q_3) = \lambda \nabla g(Q_1, Q_2, Q_3)$ and $Q_1 + Q_2 + Q_3 = Q_T$ which is equivalent to solving the system

$$\begin{aligned} 0.1277 - 2(4.08 \cdot 10^{-5})Q_1 &= \lambda \\ 0.1358 - 2(4.69 \cdot 10^{-5})Q_2 &= \lambda \\ 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 &= \lambda \\ Q_1 + Q_2 + Q_3 &= Q_T \end{aligned}$$

Comparing the first and third equations, we have $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \quad \Rightarrow$ $Q_1 = -126.2255 + 0.9412Q_3$. From the second and third equations, $0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \quad \Rightarrow \quad Q_2 = -23.4542 + 0.8188Q_3$. Substituting into $Q_1 + Q_2 + Q_3 = Q_T$ gives $(-126.2255 + 0.9412Q_3) + (-23.4542 + 0.8188Q_3) + Q_3 = Q_T \quad \Rightarrow$ $2.76Q_3 = Q_T + 149.6797 \quad \Rightarrow \quad Q_3 = 0.3623Q_T + 54.23$. Then $Q_1 = -126.2255 + 0.9412Q_3 = -126.2255 + 0.9412(0.3623Q_T + 54.23) = 0.3410Q_T - 75.18$ and $Q_2 = -23.4542 + 0.8188(0.3623Q_T + 54.23) = 0.2967Q_T + 20.95$. As long as we maintain $250 \leq Q_1 \leq 1110$, $250 \leq Q_2 \leq 1110$, and $250 \leq Q_3 \leq 1225$, we can reason from the nature of the functions KW_i that these values give a maximum of f, and hence a maximum energy production, and not a minimum.

2. From Problem 1, the value of Q_1 that maximizes energy production is $0.3410Q_T - 75.18$, but since $250 \leq Q_1 \leq 1110$, we must have $250 \leq 0.3410Q_T - 75.18 \leq 1110 \quad \Rightarrow \quad 325.18 \leq 0.3410Q_T \leq 1185.18 \quad \Rightarrow \quad 953.6 \leq Q_T \leq 3475.6$. Similarly, $250 \leq Q_2 \leq 1110 \quad \Rightarrow \quad 250 \leq 0.2967Q_T + 20.95 \leq 1110 \quad \Rightarrow \quad 772.0 \leq Q_T \leq 3670.5$, and $250 \leq Q_3 \leq 1225 \quad \Rightarrow \quad 250 \leq 0.3623Q_T + 54.23 \leq 1225 \quad \Rightarrow \quad 540.4 \leq Q_T \leq 3231.5$. Consolidating these results, we see that the values from Problem 1 are applicable only for $953.6 \leq Q_T \leq 3231.5$.

3. If $Q_T = 2500$, the results from Problem 1 show that the maximum energy production occurs for

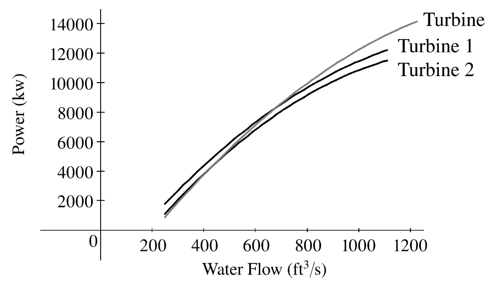
$$\begin{aligned} Q_1 &= 0.3410Q_T - 75.18 = 0.3410(2500) - 75.18 = 777.3 \\ Q_2 &= 0.2967Q_T + 20.95 = 0.2967(2500) + 20.95 = 762.7 \\ Q_3 &= 0.3623Q_T + 54.23 = 0.3623(2500) + 54.23 = 960.0 \end{aligned}$$

[continued]

The energy produced for these values is $KW_1 + KW_2 + KW_3 \approx 8915.2 + 8285.1 + 11{,}211.3 \approx 28{,}411.6$.

We compute the energy production for a nearby distribution, $Q_1 = 770$, $Q_2 = 760$, and $Q_3 = 970$: $KW_1 + KW_2 + KW_3 \approx 8839.8 + 8257.4 + 11{,}313.5 = 28{,}410.7$. For another example, we take $Q_1 = 780$, $Q_2 = 765$, and $Q_3 = 955$: $KW_1 + KW_2 + KW_3 \approx 8942.9 + 8308.8 + 11{,}159.7 = 28{,}411.4$. These distributions are both close to the distribution from Problem 1 and both give slightly lower energy productions, suggesting that $Q_1 = 777.3$, $Q_2 = 762.7$, and $Q_3 = 960.0$ is indeed the optimal distribution.

4. First we graph each power function in its domain if all of the flow is directed to that turbine (so $Q_i = Q_T$). If we use only one turbine, the graph indicates that for a water flow of 1000 ft^3/s, Turbine 3 produces the most power, approximately 12,200 kW. In comparison, if we use all three turbines, the results of Problem 1 with $Q_T = 1000$ give $Q_1 = 265.8$, $Q_2 = 317.7$, and $Q_3 = 416.5$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 8397.4$ kW. Here, using only one turbine produces significantly more energy! If the flow is only 600 ft^3/s, we do not have the option of using all three turbines, as the domain restrictions require a minimum of 250 ft^3/s in each turbine. We can use just one turbine, then, and from the graph Turbine 1 produces the most energy for a water flow of 600 ft^3.



5. If we examine the graph from Problem 4, we see that for water flows above approximately 450 ft^3/s, Turbine 2 produces the least amount of power. Therefore it seems reasonable to assume that we should distribute the incoming flow of 1500 ft^3/s between Turbines 1 and 3. (This can be verified by computing the power produced with the other pairs of turbines for comparison.) So now we wish to maximize $KW_1 + KW_3$ subject to the constraint $Q_1 + Q_3 = Q_T$ where $Q_T = 1500$.

As in Problem 1, we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_3) &= \frac{KW_1 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right) \end{aligned}$$

subject to the constraint $g(Q_1, Q_3) = Q_1 + Q_3 = Q_T$.

Then we solve $\nabla f(Q_1, Q_3) = \lambda \nabla g(Q_1, Q_3) \quad \Rightarrow \quad 0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = \lambda$ and $0.1380 - 2(3.84 \cdot 10^{-5})Q_3 = \lambda$, thus $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \quad \Rightarrow$ $Q_1 = -126.2255 + 0.9412Q_3$. Substituting into $Q_1 + Q_3 = Q_T$ gives $-126.2255 + 0.9412Q_3 + Q_3 = 1500 \quad \Rightarrow$ $Q_3 \approx 837.7$, and then $Q_1 = Q_T - Q_3 \approx 1500 - 837.7 = 662.3$. So we should apportion approximately 662.3 ft^3/s to Turbine 1 and the remaining 837.7 ft^3/s to Turbine 3. The resulting energy production is $KW_1 + KW_3 \approx 7952.1 + 10{,}256.2 = 18{,}208.3$ kW. (We can verify that this is indeed a maximum energy production by checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 1500$ we get $Q_1 = 436.3$, $Q_2 = 466.0$,

and $Q_3 = 597.7$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx$ 16,538.7 kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of 3400 ft^3/s is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but 3400 is less than the maximum combined capacity of 3445 ft^3/s, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of 1225 and distribute the remaining 2175 ft^3/s flow between Turbines 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 2175$. We can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2) &= \frac{KW_1 + KW_2}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \quad \Rightarrow$
$0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = \lambda$ and $0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = \lambda$, thus
$0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1358 - 2\left(4.69 \cdot 10^{-5}\right)Q_2 \quad \Rightarrow \quad Q_1 = -99.2647 + 1.1495Q_2$. Substituting into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \quad \Rightarrow \quad Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft^3/s, and the remaining 1065 ft^3/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

14 Review

TRUE-FALSE QUIZ

1. True. $f_y(a, b) = \lim\limits_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ from Equation 14.3.3. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a, b) = \lim\limits_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y \, \partial x}$.

4. True. From Equation 14.6.14 we get $D_{\mathbf{k}}\, f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 14.2.3.

6. False. See Exercise 14.4.54(a).

7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 14.7.2, $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \langle 0, 0 \rangle = \mathbf{0}$.

8. False. If f is not continuous at $(2, 5)$, then we can have $\lim_{(x,y)\to(2,5)} f(x, y) \neq f(2, 5)$. (See Example 14.2.8.)

9. False. $\nabla f(x, y) = \langle 0, 1/y \rangle$.

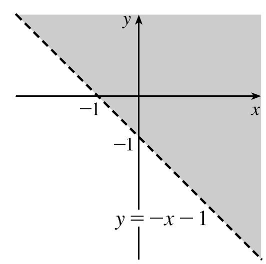
10. True. This is equivalent to part (c) of the Second Derivatives Test (14.7.3).

11. True. $\nabla f = \langle \cos x, \cos y \rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos \theta| \leq 1$, so $|\nabla f| \leq \sqrt{2}$. Now $D_{\mathbf{u}}\, f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f|\, |\mathbf{u}| \cos \theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}\, f(x, y)| \leq \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.

12. False. See Exercise 14.7.41.

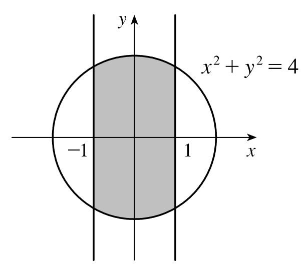
EXERCISES

1. $f(x, y) = \ln(x + y + 1)$ is defined only when $x + y + 1 > 0 \quad\Leftrightarrow\quad y > -x - 1$, so the domain of f is $\{(x, y) \mid y > -x - 1\}$, all those points above the line $y = -x - 1$.

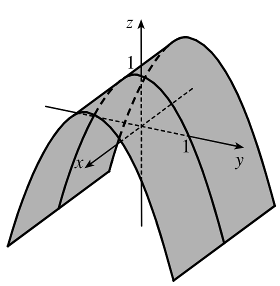


2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$.

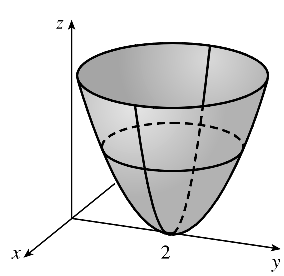
$\sqrt{4 - x^2 - y^2}$ is defined only when $4 - x^2 - y^2 \geq 0 \quad\Leftrightarrow\quad x^2 + y^2 \leq 4$, and $\sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0 \quad\Leftrightarrow\quad -1 \leq x \leq 1$, so the domain of f is $\left\{(x, y) \mid -1 \leq x \leq 1, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\,\right\}$, which consists of those points on or inside the circle $x^2 + y^2 = 4$ for $-1 \leq x \leq 1$.



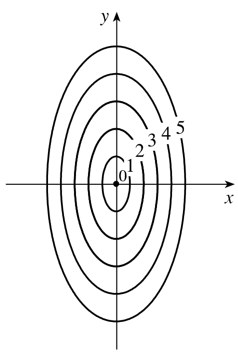
3. $z = f(x, y) = 1 - y^2$, a parabolic cylinder



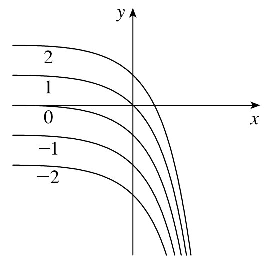
4. $z = f(x, y) = x^2 + (y - 2)^2$, a circular paraboloid with vertex $(0, 2, 0)$ and axis parallel to the z-axis



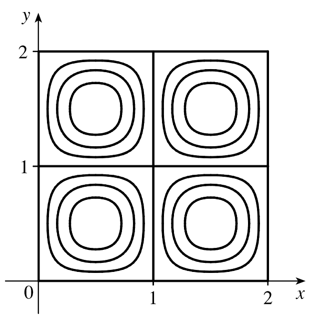
5. The level curves are $\sqrt{4x^2 + y^2} = k$ or $4x^2 + y^2 = k^2$, $k \geq 0$, a family of ellipses.



6. The level curves are $e^x + y = k$ or $y = -e^x + k$, a family of exponential curves.



7.



8. (a) The point $(3, 2)$ lies partway between the level curves with z-values 50 and 60, and it appears that $(3, 2)$ is about the same distance from either level curve. So we estimate that $f(3, 2) \approx 55$.

(b) At the point $(3, 2)$, if we fix y at $y = 2$ and allow x to vary, the level curves indicate that the z-values decrease as x increases, so $f_x(3, 2)$ is negative. In other words, if we start at $(3, 2)$ and move right (in the positive x-direction), the contours show that our path along the surface $z = f(x, y)$ is descending.

(c) Both $f_y(2, 1)$ and $f_y(2, 2)$ are positive, because if we start from either point and move in the positive y-direction, the contour map indicates that the path is ascending. But the level curves are closer together in the y-direction at $(2, 1)$ than at $(2, 2)$, so the path is steeper (the z-values increase more rapidly) at $(2, 1)$ and hence $f_y(2, 1) > f_y(2, 2)$.

9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.

10. $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$. As $(x, y) \to (0, 0)$ along the x-axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \to 0$ along this line. As $(x, y) \to (0, 0)$ along the line $x = y$, $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so $f(x, y) \to \frac{2}{3}$. Thus, the limit doesn't exist.

11. (a) $T_x(6, 4) = \displaystyle\lim_{h \to 0} \frac{T(6 + h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6, 4) \approx \dfrac{T(8, 4) - T(6, 4)}{2} = \dfrac{86 - 80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4)-T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5°\text{C/m}$. Similarly, $T_y\,(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h)-T(6,4)}{h}$, which we can approximate with $h = \pm 2$:

$T_y(6,4) \approx \dfrac{T(6,6)-T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$, $T_y(6,4) \approx \dfrac{T(6,2)-T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0°\text{C/m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 14.6.9, $D_{\mathbf{u}}\,T(6,4) = \nabla T(6,4)\cdot \mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}\,T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\text{C/m}$.

Alternatively, we can use Definition 14.6.2: $D_{\mathbf{u}}\,T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(8,6)-T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$,

$D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(4,2)-T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have $D_{\mathbf{u}}\,T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\text{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[T_x(x,y)] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h)-T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h)-T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6)-T(6,6)}{2} = \dfrac{80\,-\,75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6)-T(6,6)}{-2} = \dfrac{68\,-\,75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \dfrac{T(8,2)-T_x(6,2)}{2} = \dfrac{90\,-\,87}{2} = 1.5$, $T_x(6,2) \approx \dfrac{T(4,2)-T(6,2)}{-2} = \dfrac{74\,-\,87}{-2} = 6.5$.

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \dfrac{T_x(6,6)-T_x(6,4)}{2} = \dfrac{3.0\,-\,3.5}{2} = -0.25$, $T_{xy}(6,4) \approx \dfrac{T_x(6,2)-T_x(6,4)}{-2} = \dfrac{4.0\,-\,3.5}{-2} = -0.25$.

Averaging these values, we have $T_{xy}\,(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T\,(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5,3.8)$, we can use the linear approximation to estimate $T(5,3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1°\text{C}$.

13. $f(x,y) = (5y^3 + 2x^2y)^8 \quad \Rightarrow \quad f_x = 8(5y^3+2x^2y)^7(4xy) = 32xy(5y^3+2x^2y)^7$,

$f_y = 8(5y^3+2x^2y)^7(15y^2+2x^2) = (16x^2+120y^2)(5y^3+2x^2y)^7$

14. $g(u,v) = \dfrac{u+2v}{u^2+v^2} \quad \Rightarrow \quad g_u = \dfrac{(u^2+v^2)(1)-(u+2v)(2u)}{(u^2+v^2)^2} = \dfrac{v^2-u^2-4uv}{(u^2+v^2)^2}$,

$$g_v = \frac{(u^2+v^2)(2)-(u+2v)(2v)}{(u^2+v^2)^2} = \frac{2u^2-2v^2-2uv}{(u^2+v^2)^2}$$

15. $F(\alpha,\beta) = \alpha^2 \ln(\alpha^2+\beta^2) \quad \Rightarrow \quad F_\alpha = \alpha^2 \cdot \dfrac{1}{\alpha^2+\beta^2}\,(2\alpha) + \ln(\alpha^2+\beta^2)\cdot 2\alpha = \dfrac{2\alpha^3}{\alpha^2+\beta^2} + 2\alpha\ln(\alpha^2+\beta^2)$,

$$F_\beta = \alpha^2 \cdot \frac{1}{\alpha^2+\beta^2}\,(2\beta) = \frac{2\alpha^2\beta}{\alpha^2+\beta^2}$$

16. $G(x,y,z) = e^{xz}\sin(y/z) \quad \Rightarrow \quad G_x = ze^{xz}\sin(y/z),\ G_y = e^{xz}\cos(y/z)(1/z) = (e^{xz}/z)\cos(y/z)$,

$G_z = e^{xz}\cdot\cos(y/z)(-y/z^2) + \sin(y/z)\cdot xe^{xz} = e^{xz}\left[x\sin(y/z) - (y/z^2)\cos(y/z)\right]$

17. $S(u,v,w) = u\arctan(v\sqrt{w}\,) \quad \Rightarrow \quad S_u = \arctan(v\sqrt{w}\,),\ S_v = u\cdot\dfrac{1}{1+(v\sqrt{w})^2}\,(\sqrt{w}\,) = \dfrac{u\sqrt{w}}{1+v^2w}$,

$$S_w = u\cdot\frac{1}{1+(v\sqrt{w})^2}\left(v\cdot\tfrac{1}{2}w^{-1/2}\right) = \frac{uv}{2\sqrt{w}\,(1+v^2w)}$$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34-0.01T)(S-35) + 0.016D \quad \Rightarrow$

$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35-35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y) = 4x^3 - xy^2 \quad \Rightarrow \quad f_x = 12x^2 - y^2,\ f_y = -2xy,\ f_{xx} = 24x,\ f_{yy} = -2x,\ f_{xy} = f_{yx} = -2y$

20. $z = xe^{-2y} \quad \Rightarrow \quad z_x = e^{-2y},\ z_y = -2xe^{-2y},\ z_{xx} = 0,\ z_{yy} = 4xe^{-2y},\ z_{xy} = z_{yx} = -2e^{-2y}$

21. $f(x,y,z) = x^ky^lz^m \quad \Rightarrow \quad f_x = kx^{k-1}y^lz^m,\ f_y = lx^ky^{l-1}z^m,\ f_z = mx^ky^lz^{m-1},\ f_{xx} = k(k-1)x^{k-2}y^lz^m$,

$f_{yy} = l(l-1)x^ky^{l-2}z^m,\ f_{zz} = m(m-1)x^ky^lz^{m-2},\ f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m, f_{xz} = f_{zx} = kmx^{k-1}y^lz^{m-1}$,

$f_{yz} = f_{zy} = lmx^ky^{l-1}z^{m-1}$

22. $v = r\cos(s+2t) \quad \Rightarrow \quad v_r = \cos(s+2t),\ v_s = -r\sin(s+2t),\ v_t = -2r\sin(s+2t),\ v_{rr} = 0,\ v_{ss} = -r\cos(s+2t)$,

$v_{tt} = -4r\cos(s+2t),\ v_{rs} = v_{sr} = -\sin(s+2t),\ v_{rt} = v_{tr} = -2\sin(s+2t),\ v_{st} = v_{ts} = -2r\cos(s+2t)$

23. $z = xy + xe^{y/x} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = y - \dfrac{y}{x}e^{y/x} + e^{y/x},\ \dfrac{\partial z}{\partial y} = x + e^{y/x}$ and

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y - \frac{y}{x}e^{y/x} + e^{y/x}\right) + y\left(x + e^{y/x}\right) = xy - ye^{y/x} + xe^{y/x} + xy + ye^{y/x} = xy + xy + xe^{y/x} = xy + z.$$

24. $z = \sin(x + \sin t) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \cos(x + \sin t),\ \dfrac{\partial z}{\partial t} = \cos(x + \sin t)\cos t,$

$\dfrac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin t)\cos t,\ \dfrac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$ and

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \cos(x + \sin t)\left[-\sin(x + \sin t)\cos t\right] = \cos(x + \sin t)(\cos t)\left[-\sin(x + \sin t)\right] = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}.$$

25. (a) $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$. $z_x = 6x + 2 \quad \Rightarrow \quad z_x(1, -2) = 8$ and $z_y = -2y \quad \Rightarrow \quad z_y(1, -2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$, or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1 \rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are $\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$.

26. (a) $z = e^x \cos y$, $(0, 0, 1)$. $z_x = e^x \cos y \quad \Rightarrow \quad z_x(0, 0) = 1$ and $z_y = -e^x \sin y \quad \Rightarrow \quad z_y(0, 0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x - 0) + 0(y - 0)$, or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1 \rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1 - t$, and symmetric equations are $x = 1 - z$, $y = 0$.

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2, -1, 1) = 4$, $F_y(2, -1, 1) = -4$, $F_z(2, -1, 1) = -6$. From Equation 14.6.19, an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$. Parametric equations are $x = 2 + 4t$, $y = -1 - 4t$, $z = 1 - 6t$.

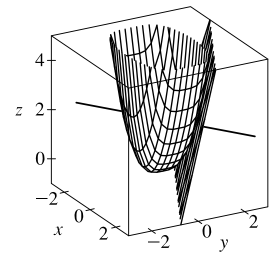
28. (a) Let $F(x, y, z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2$. From Equation 14.6.19, an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or, equivalently, $x + y + z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$ or, equivalently, $x = y = z$. Parametric equations are $x = 1 + 2t$, $y = 1 + 2t$, $z = 1 + 2t$.

29. (a) Let $F(x, y, z) = x + 2y + 3z - \sin(xyz)$. Then $F_x = 1 - yz\cos(xyz)$, $F_y = 2 - xz\cos(xyz)$, $F_z = 3 - xy\cos(xyz)$, so $F_x(2, -1, 0) = 1$, $F_y(2, -1, 0) = 2$, $F_z(2, -1, 0) = 5$. From Equation 14.6.19, an equation of the tangent plane is $1(x - 2) + 2(y + 1) + 5(z - 0) = 0$ or $x + 2y + 5z = 0$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z}{5}$ or $x - 2 = \dfrac{y+1}{2} = \dfrac{z}{5}$. Parametric equations are $x = 2 + t$, $y = -1 + 2t$, $z = 5t$.

30. Let $f(x, y) = x^2 + y^4$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 4y^3$, so $f_x(1, 1) = 2$, $f_y(1, 1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x - 1) + 4(y - 1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1 \rangle$ so the normal line is given by $\dfrac{x-1}{2} = \dfrac{y-1}{4} = \dfrac{z-2}{-1}$ or $x = 1 + 2t,\ y = 1 + 4t,$ $z = 2 - t$.



31. The hyperboloid is a level surface of the function $F(x, y, z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0 \rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1 \rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0 \rangle = k \langle 2, 2, 1 \rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \quad \Rightarrow \quad k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \quad \Rightarrow \quad k^2 = 4 \quad \Rightarrow \quad k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

32. $u = \ln(1 + se^{2t}) \quad \Rightarrow \quad du = \dfrac{\partial u}{\partial s}\,ds + \dfrac{\partial u}{\partial t}\,dt = \dfrac{e^{2t}}{1 + se^{2t}}\,ds + \dfrac{2se^{2t}}{1 + se^{2t}}\,dt$

33. $f(x, y, z) = x^3\sqrt{y^2 + z^2} \quad \Rightarrow \quad f_x(x, y, z) = 3x^2\sqrt{y^2 + z^2},\ f_y(x, y, z) = \dfrac{yx^3}{\sqrt{y^2 + z^2}},\ f_z(x, y, z) = \dfrac{zx^3}{\sqrt{y^2 + z^2}},$

so $f(2, 3, 4) = 8(5) = 40$, $f_x(2, 3, 4) = 3(4)\sqrt{25} = 60$, $f_y(2, 3, 4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2, 3, 4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2, 3, 4)$ is

$$\begin{aligned} f(x, y, z) &\approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) \\ &= 40 + 60(x - 2) + \tfrac{24}{5}(y - 3) + \tfrac{32}{5}(z - 4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus, the maximum error in the calculated area is about $dA = \frac{1}{2}(12)(0.002) + \frac{1}{2}(5)(0.002) = 0.017\ \text{m}^2$ or $170\ \text{cm}^2$.

(b) $z = \sqrt{x^2 + y^2}$, $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus, the maximum error in the calculated hypotenuse length, with $x = 5$, $y = 12$, and $z = \sqrt{5^2 + 12^2} = 13$, is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $u = x^2y^3 + z^4 \quad \Rightarrow \quad \dfrac{du}{dp} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dp} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dp} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dp} = 2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

36. $v = x^2\sin y + ye^{xy} \quad \Rightarrow \quad \dfrac{\partial v}{\partial s} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial s} = \left(2x\sin y + y^2e^{xy}\right)(1) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(t)$.

$s = 0, t = 1 \quad \Rightarrow \quad x = 2, y = 0$, so $\dfrac{\partial v}{\partial s} = 0 + (4 + 1)(1) = 5$.

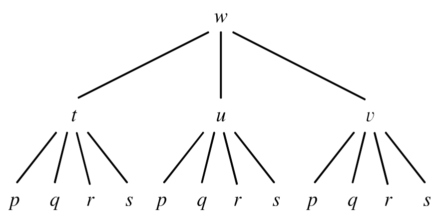
$\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial t} = \left(2x\sin y + y^2e^{xy}\right)(2) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(s) = 0 + 0 = 0$.

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p} \qquad \frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $z = y + f(x^2 - y^2) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = 2xf'(x^2 - y^2),\quad \dfrac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2) \quad \left[\text{where } f' = \dfrac{df}{d(x^2 - y^2)}\right]$. Then

$y\,\dfrac{\partial z}{\partial x} + x\,\dfrac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x.$

40. $A = \frac{1}{2}xy\sin\theta$ [Formula 6 in Appendix D], $dx/dt = 3,\ dy/dt = -2,\ d\theta/dt = 0.05$, and

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\,\dfrac{dx}{dt} + (x\sin\theta)\,\dfrac{dy}{dt} + (xy\cos\theta)\,\dfrac{d\theta}{dt}\right]$. So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$,

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8\ \text{in}^2/\text{s}.$

41. $z = f(u,v)$, $u = xy$, and $v = y/x \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\,y + \dfrac{\partial z}{\partial v}\,\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}\,y + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\,\partial v}\,y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\dfrac{\partial z}{\partial y} = x\,\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\,\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\,\partial v}\,x\right) = x^2\,\frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Thus

$$\begin{aligned}x^2\,\frac{\partial^2 z}{\partial x^2} - y^2\,\frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\ &= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\,\frac{\partial^2 z}{\partial u\,\partial v} = 2v\,\frac{\partial z}{\partial v} - 4uv\,\frac{\partial^2 z}{\partial u\,\partial v}\end{aligned}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

42. $\cos(xyz) = 1 + x^2y^2 + z^2$, so let $F(x, y, z) = 1 + x^2y^2 + z^2 - \cos(xyz) = 0$. Then by Equations 14.5.6 we have $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xy^2 + \sin(xyz) \cdot yz}{2z + \sin(xyz) \cdot xy} = -\dfrac{2xy^2 + yz\sin(xyz)}{2z + xy\sin(xyz)}$,

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2x^2y + \sin(xyz) \cdot xz}{2z + \sin(xyz) \cdot xy} = -\dfrac{2x^2y + xz\sin(xyz)}{2z + xy\sin(xyz)}$.

43. $f(x, y, z) = x^2e^{yz^2} \quad \Rightarrow \quad \nabla f = \langle f_x, f_y, f_z \rangle = \left\langle 2xe^{yz^2}, x^2e^{yz^2} \cdot z^2, x^2e^{yz^2} \cdot 2yz \right\rangle = \left\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \right\rangle$

44. (a) By Theorem 14.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta$ (see the proof of Theorem 14.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta = \frac{1}{2}\,|\nabla f| \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Leftrightarrow$ $\theta = \frac{\pi}{3}$.

45. $f(x, y) = x^2e^{-y} \quad \Rightarrow \quad \nabla f = \langle 2xe^{-y}, -x^2e^{-y} \rangle$, $\nabla f(-2, 0) = \langle -4, -4 \rangle$. The direction is given by $\langle 2 - (-2), -3 - 0 \rangle = \langle 4, -3 \rangle$, so $\mathbf{u} = \frac{1}{\sqrt{4^2 + (-3)^2}}\langle 4, -3 \rangle = \frac{1}{5}\langle 4, -3 \rangle$. Thus,

$D_{\mathbf{u}}\, f(-2, 0) = \nabla f(-2, 0) \cdot \mathbf{u} = \langle -4, -4 \rangle \cdot \frac{1}{5}\langle 4, -3 \rangle = \frac{1}{5}(-16 + 12) = -\frac{4}{5}$.

46. $f(x, y, z) = x^2y + x\sqrt{1 + z} \quad \Rightarrow \quad \nabla f = \left\langle 2xy + \sqrt{1 + z}, x^2, x/\big(2\sqrt{1 + z}\,\big) \right\rangle$, $\nabla f(1, 2, 3) = \left\langle 6, 1, \frac{1}{4} \right\rangle$. The direction is given by $2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so $\mathbf{u} = \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}}\,\langle 2, 1, -2 \rangle = \frac{1}{3}\,\langle 2, 1, -2 \rangle$. Thus,

$D_{\mathbf{u}}\, f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \mathbf{u} = \left\langle 6, 1, \frac{1}{4} \right\rangle \cdot \frac{1}{3}\,\langle 2, 1, -2 \rangle = \frac{1}{3}\left(12 + 1 - \frac{1}{2}\right) = \frac{25}{6}$.

47. $f(x, y) = x^2y + \sqrt{y} \quad \Rightarrow \quad \nabla f = \left\langle 2xy, x^2 + 1/\left(2\sqrt{y}\,\right) \right\rangle$, $\nabla f(2, 1) = \left\langle 4, \frac{9}{2} \right\rangle$. Thus, the maximum rate of change of f at $(2, 1)$ is $|\nabla f(2, 1)| = \left|\left\langle 4, \frac{9}{2} \right\rangle\right| = \frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2} \right\rangle$.

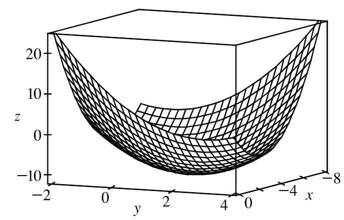
48. $f(x, y, z) = ze^{xy}$. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy} \rangle$, $\nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle$ is the direction of most rapid increase while the rate is $|\langle 2, 0, 1 \rangle| = \sqrt{5}$.

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50 - 45}{8} = \frac{5}{8} = 0.625$ knot/mi.

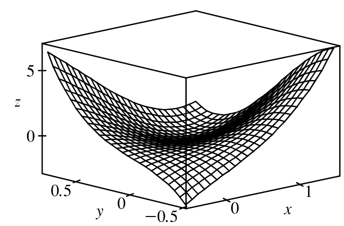
50. The surfaces are $f(x, y, z) = z - 2x^2 + y^2 = 0$ and $g(x, y, z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2, 2, 4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x, y, z) = \langle -4x, 2y, 1 \rangle \;\Rightarrow$ $\nabla f(-2, 2, 4) = \langle 8, 4, 1 \rangle$, $\nabla g(x, y, z) = \langle 0, 0, 1 \rangle \;\Rightarrow\; \nabla g\langle -2, 2, 4 \rangle = \langle 0, 0, 1 \rangle$. Hence

$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}.$$ Thus, parametric equations are: $x = -2 + 4t,\ y = 2 - 8t,\ z = 4$.

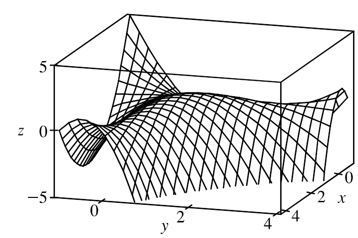
51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10 \;\Rightarrow\; f_x = 2x - y + 9$, $f_y = -x + 2y - 6,\ f_{xx} = 2 = f_{yy},\ f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4, 1) > 0$, $D(-4, 1) = 3 > 0$, so $f(-4, 1) = -11$ is a local minimum.



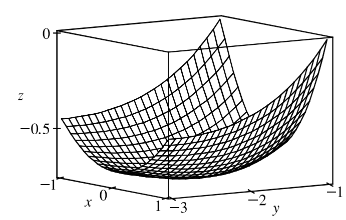
52. $f(x, y) = x^3 - 6xy + 8y^3 \;\Rightarrow\; f_x = 3x^2 - 6y,\ f_y = -6x + 24y^2,\ f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x\left(x^3 - 1\right) = 0$, so the critical points are $(0, 0)$, $\left(1, \frac{1}{2}\right)$. $D(0, 0) = -36 < 0$ so $(0, 0)$ is a saddle point while $f_{xx}\left(1, \frac{1}{2}\right) = 6 > 0$ and $D\left(1, \frac{1}{2}\right) = 108 > 0$ so $f\left(1, \frac{1}{2}\right) = -1$ is a local minimum.



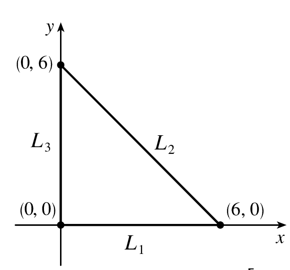
53. $f(x, y) = 3xy - x^2y - xy^2 \;\Rightarrow\; f_x = 3y - 2xy - y^2,\ f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x,\ f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(1, 1)$. $D(0, 0) = D(3, 0) = D(0, 3) = -9 < 0$ so $(0, 0)$, $(3, 0)$, and $(0, 3)$ are saddle points. $D(1, 1) = 3 > 0$ and $f_{xx}(1, 1) = -2 < 0$, so $f(1, 1) = 1$ is a local maximum.



54. $f(x, y) = (x^2 + y)e^{y/2} \;\Rightarrow\; f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4 + x^2 + y)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0, -2) > 0$, $D(0, -2) = e^{-2} - 0 > 0$ so $f(0, -2) = -2/e$ is a local minimum.



55. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2, 0)$ isn't inside D. Thus the only critical point inside D is $(1, 2)$ and $f(1, 2) = 4$. Secondly we consider the boundary of D.

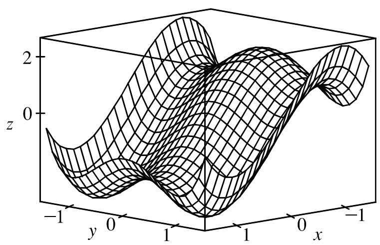


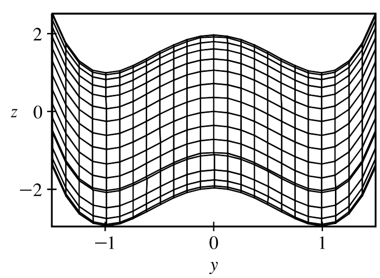
[continued]

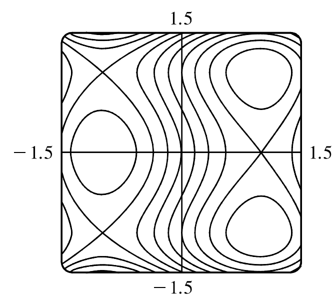
On L_1: $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2: $x = -y + 6$ and $f(-y+6, y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3: $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

56. $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$



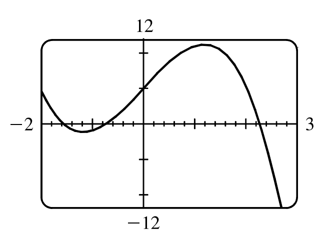


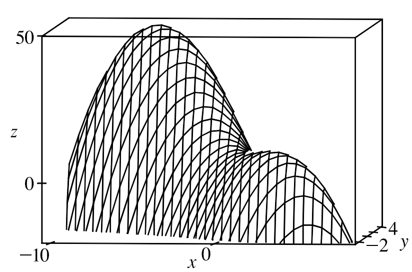


From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minimums $f(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad \Leftrightarrow \quad x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \quad \Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm 1) = 48 > 0$ and $f_{xx}(1,\pm 1) = 6 > 0$ indicating local minimums $f(1,\pm 1) = -3$; and $D(-1,\pm 1) = -48$ and $D(1,0) = -24$, indicating saddle points at $(-1,\pm 1)$ and $(1,0)$.

58. $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \quad \Rightarrow \quad f_x(x,y) = -4x - 8y$, $f_y(x,y) = 10 - 8x - 4y^3$. Now $f_x(x,y) = 0 \quad \Rightarrow$ $x = -2x$, and substituting this into $f_y(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \quad \Leftrightarrow \quad 5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542$, -0.717, or 2.260. (Alternatively, we could have found the

solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maximums $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x, y) = x^2 y,\ g(x, y) = x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $y = \lambda$. If $x = 0$ then $x^2 + y^2 = 1$ gives $y = \pm 1$ and we have possible points $(0, \pm 1)$ where $f\,(0, \pm 1) = 0$. If $y = \lambda$ then $x^2 = 2\lambda y$ implies $x^2 = 2y^2$ and substitution into $x^2 + y^2 = 1$ gives $3y^2 = 1 \quad \Rightarrow \quad y = \pm \frac{1}{\sqrt{3}}$ and $x = \pm \sqrt{\frac{2}{3}}$. The corresponding possible points are $\left(\pm \sqrt{\frac{2}{3}}, \pm \frac{1}{\sqrt{3}} \right)$. The absolute maximum is $f\left(\pm \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}} \right) = \frac{2}{3\sqrt{3}}$ while the absolute minimum is $f\left(\pm \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}} \right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x, y) = 1/x + 1/y,\ g(x, y) = 1/x^2 + 1/y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\sqrt{2} \right)$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f\left(\sqrt{2}, \sqrt{2} \right) = \sqrt{2}$ and the absolute minimum is $f\left(-\sqrt{2}, -\sqrt{2} \right) = -\sqrt{2}$.

61. $f(x, y, z) = xyz,\ g(x, y, z) = x^2 + y^2 + z^2 = 3. \quad \nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \quad \Rightarrow \quad 2y^2 z = 2x^2 z \quad \Rightarrow$ $y^2 = x^2$, and similarly $2yz^2 = 2x^2 y \quad \Rightarrow \quad z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \quad \Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1, 1, \pm 1)$, $(1, -1, \pm 1)$, $(-1, 1, \pm 1)$, $(-1, -1, \pm 1)$. The absolute maximum is $f(1, 1, 1) = f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = 1$, and the absolute minimum is $f(1, 1, -1) = f(1, -1, 1) = f(-1, 1, 1) = f(-1, -1, -1) = -1$.

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2,\ g(x, y, z) = x + y + z = 1,\ h(x, y, z) = x - y + 2z = 2 \quad \Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and $2x = \lambda + \mu$ **(1)**, $\quad 4y = \lambda - \mu$ **(2)**, $\quad 6z = \lambda + 2\mu$ **(3)**, $x + y + z = 1$ **(4)**, $\quad x - y + 2z = 2$ **(5)**. Then six times **(1)** plus three times **(2)** plus two times **(3)** implies $12(x + y + z) = 11\lambda + 7\mu$, so **(4)** gives $11\lambda + 7\mu = 12$. Also six times **(1)** minus three times **(2)** plus four times **(3)** implies $12(x - y + 2z) = 7\lambda + 17\mu$, so **(5)** gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into **(1)**, **(2)**, and **(3)** implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23} \right) = \frac{33}{23}$. Now since $(0, 0, 1)$ satisfies both constraints and $f(0, 0, 1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23} \right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

63. $f(x,y,z) = x^2 + y^2 + z^2,\ g(x,y,z) = xy^2z^3 = 2 \quad\Rightarrow\quad \nabla f = \langle 2x, 2y, 2z\rangle = \lambda\nabla g = \langle \lambda y^2z^3, 2\lambda xyz^3, 3\lambda xy^2z^2\rangle$.

Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so $2x = \lambda y^2z^3$ **(1)**, $\quad 1 = \lambda xz^3$ **(2)**, $\quad 2 = 3\lambda xy^2z$ **(3)**. Then **(2)** and **(3)** imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly **(1)** and **(3)** imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x,y,z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4})$, $(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z) = 2$ and $f(2,1,1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x,y,z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize $f(x,z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then

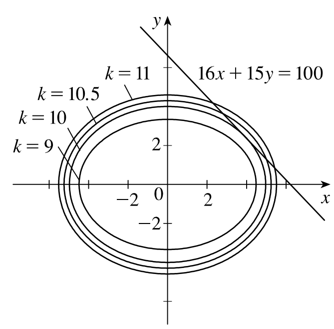
$f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and $f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2+4)\left(2+\frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

64. (a) The distance from a point (x,y) to the point $(-3,0)$ is $\sqrt{(x+3)^2 + y^2}$.

The distance from (x,y) to $(3,0)$ is $\sqrt{(x-3)^2 + y^2}$. Then the function that gives the sum of the distances is

$f(x,y) = \sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2}$.

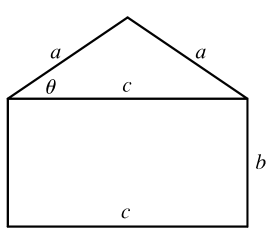
The graph shows several curves of $f(x,y) = k$ for $k = 9, 10, 10.5$, and 11. The smallest value of k that $g(x,y) = 16x + 15y$ intersects appears to be 10, so the point that minimizes the distance is approximately $(x,y) = (4, 2.4)$.



(b) $\nabla f = \left\langle \dfrac{x+3}{\sqrt{(x+3)^2+y^2}} + \dfrac{x-3}{\sqrt{(x-3)^2+y^2}}, \dfrac{y}{\sqrt{(x+3)^2+y^2}} + \dfrac{y}{\sqrt{(x-3)^2+y^2}} \right\rangle$. $\nabla g = \langle 16, 15\rangle$. Now

$\nabla f(4, 2.4) = \left\langle \frac{7}{7.4} + \frac{1}{2.6}, \frac{2.4}{7.4} + \frac{2.4}{2.6}\right\rangle = \left\langle \frac{640}{481}, \frac{600}{481}\right\rangle$ and $\nabla g(4, 2.4) = \langle 16, 15\rangle$. ∇f and ∇g are parallel if $\nabla f = \lambda\nabla g$ for some value of λ, which is true when $\lambda = \frac{40}{481}$. Thus, ∇f and ∇g are parallel.

65.



The area of the triangle is $\frac{1}{2}ca\sin\theta$ [Formula 6 in Appendix D] and the area of the rectangle is bc. Thus, the area of the whole object is $f(a,b,c) = \frac{1}{2}ca\sin\theta + bc$. The perimeter of the object is $g(a,b,c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \quad\Rightarrow$

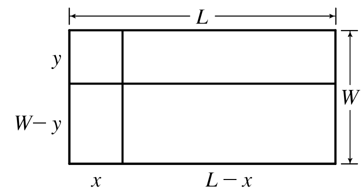
$\sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus, $f(a,b,c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: $ca(4a^2 - c^2)^{-1/2} = 2\lambda$ **(1)**, $c = 2\lambda$ **(2)**, $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$ **(3)**, and $2a + 2b + c = P$ **(4)**. From **(2)**, $\lambda = \frac{1}{2}c$ and so **(1)** produces $ca(4a^2 - c^2)^{-1/2} = c \;\Rightarrow\; (4a^2 - c^2)^{1/2} = a \;\Rightarrow$ $4a^2 - c^2 = a^2 \;\Rightarrow\; c = \sqrt{3}\,a$ **(5)**. Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, **(3)** gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from **(5)**, $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}\,a}{2} \;\Rightarrow\; -\dfrac{a}{2} - \dfrac{\sqrt{3}\,a}{2} = -b \;\Rightarrow\; b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$ **(6)**. Substituting **(5)** and **(6)** into **(4)** we get:

$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \dfrac{P}{3 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{3}P$ and thus

$b = \dfrac{\left(2\sqrt{3} - 3\right)\left(1 + \sqrt{3}\right)}{6}P = \dfrac{3 - \sqrt{3}}{6}P$ and $c = \left(2 - \sqrt{3}\right)P$.

PROBLEMS PLUS

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L-x)y$, $A_3 = (L-x)(W-y)$, $A_4 = x(W-y)$. For $0 \le x \le L$, $0 \le y \le W$, let



$$\begin{aligned} f(x,y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2y^2 + (L-x)^2y^2 + (L-x)^2(W-y)^2 + x^2(W-y)^2 \\ &= [x^2 + (L-x)^2][y^2 + (W-y)^2] \end{aligned}$$

Then we need to find the maximum and minimum values of $f(x,y)$. Here

$f_x(x,y) = [2x - 2(L-x)][y^2 + (W-y)^2] = 0 \quad\Rightarrow\quad 4x - 2L = 0$ or $x = \frac{1}{2}L$, and

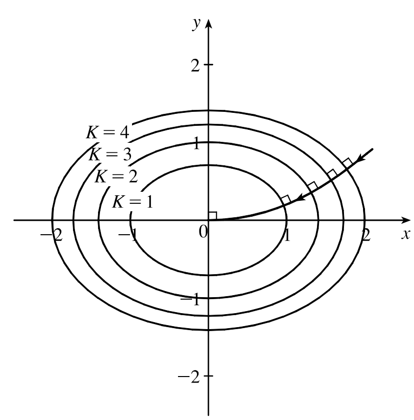
$f_y(x,y) = [x^2 + (L-x)^2][2y - 2(W-y)] = 0 \quad\Rightarrow\quad 4y - 2W = 0$ or $y = W/2$. Also

$f_{xx} = 4[y^2 + (W-y)^2]$, $f_{yy} = 4[x^2 + (L-x)^2]$, and $f_{xy} = (4x-2L)(4y-2W)$. Then

$D = 16[y^2 + (W-y)^2][x^2 + (L-x)^2] - (4x-2L)^2(4y-2W)^2$. Thus when $x = \frac{1}{2}L$ and $y = \frac{1}{2}W$, $D > 0$ and $f_{xx} = 2W^2 > 0$. Thus a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2W^2$.

There are no other critical points, so the maximum must occur on the boundary. Now along the width of the rectangle let $g(y) = f(0,y) = f(L,y) = L^2[y^2 + (W-y)^2]$, $0 \le y \le W$. Then $g'(y) = L^2[2y - 2(W-y)] = 0 \quad\Leftrightarrow\quad y = \frac{1}{2}W$. And $g\left(\frac{1}{2}\right) = \frac{1}{2}L^2W^2$. Checking the endpoints, we get $g(0) = g(W) = L^2W^2$. Along the length of the rectangle let $h(x) = f(x,0) = f(x,W) = W^2[x^2 + (L-x)^2]$, $0 \le x \le L$. By symmetry $h'(x) = 0 \quad\Leftrightarrow\quad x = \frac{1}{2}L$ and $h\left(\frac{1}{2}L\right) = \frac{1}{2}L^2W^2$. At the endpoints we have $h(0) = h(L) = L^2W^2$. Therefore L^2W^2 is the maximum value of f. This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

2. (a) The level curves of the function $C(x,y) = e^{-(x^2+2y^2)/10^4}$ are the curves $e^{-(x^2+2y^2)/10^4} = k$ (k is a positive constant). This equation is equivalent to $x^2 + 2y^2 = K \quad\Rightarrow\quad \dfrac{x^2}{\left(\sqrt{K}\right)^2} + \dfrac{y^2}{\left(\sqrt{K/2}\right)^2} = 1$, where $K = -10^4 \ln k$, a family of ellipses. We sketch level curves for $K = 1$, 2, 3, and 4. If the shark always swims in the direction of maximum increase of blood concentration, its direction at any point would coincide with the gradient vector. Then we know the shark's path is perpendicular to the level curves it intersects. We sketch one example of such a path.



(b) $\nabla C = -\frac{2}{10^4} e^{-(x^2+2y^2)/10^4} (x\,\mathbf{i} + 2y\,\mathbf{j})$. And ∇C points in the direction of most rapid increase in concentration; that is, ∇C is tangent to the most rapid increase curve. If $r(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a parametrization of the most rapid increase curve, then $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}$ is tangent to the curve, so $\frac{d\mathbf{r}}{dt} = \lambda \nabla C \quad \Rightarrow \quad \frac{dx}{dt} = \lambda\left[-\frac{2}{10^4}\, e^{-(x^2+2y^2)/10^4}\right] x$ and $\frac{dy}{dt} = \lambda\left[-\frac{2}{10^4} e^{-(x^2+2y^2)/10^4}\right](2y)$. Therefore $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2\,\frac{y}{x} \quad \Rightarrow \quad \frac{dy}{y} = 2\,\frac{dx}{x} \quad \Rightarrow \quad \ln|y| = 2\ln|x|$ so that $y = kx^2$ for some constant k. But $y(x_0) = y_0 \quad \Rightarrow \quad y_0 = kx_0^2 \quad \Rightarrow \quad k = y_0/x_0^2 \quad (x_0 = 0 \quad \Rightarrow \quad y_0 = 0 \quad \Rightarrow$ the shark is already at the origin, so we can assume $x_0 \neq 0$.) Therefore the path the shark will follow is along the parabola $y = y_0(x/x_0)^2$.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x\sin\theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x\cos\theta$. Therefore the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x,\theta) &= \tfrac{1}{2}x\sin\theta\,[(w-2x) + (w - 2x + 2x\cos\theta)] = (x\sin\theta)(w - 2x + x\cos\theta) \\ &= wx\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\ \ 0 < x \le \tfrac{1}{2}w,\ 0 < \theta \le \tfrac{\pi}{2} \end{aligned}$$

We look for the critical points of A: $\partial A/\partial x = w\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta$ and $\partial A/\partial\theta = wx\cos\theta - 2x^2\cos\theta + x^2(\cos^2\theta - \sin^2\theta)$, so $\partial A/\partial x = 0 \quad \Leftrightarrow \quad \sin\theta\,(w - 4x + 2x\cos\theta) = 0 \quad \Leftrightarrow$ $\cos\theta = \frac{4x - w}{2x} = 2 - \frac{w}{2x} \quad (0 < \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \sin\theta > 0)$. If, in addition, $\partial A/\partial\theta = 0$, then

$$\begin{aligned} 0 &= wx\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) \\ &= wx\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right] \\ &= 2wx - \tfrac{1}{2}w^2 - 4x^2 + wx + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$, $b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 14.7.6, we can argue from the physical nature of this problem that we have found a local maximum of A. Now checking the boundary of A, let $g(\theta) = A(w/2,\theta) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta$, $0 < \theta \le \frac{\pi}{2}$. Clearly g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also along the line $\theta = \frac{\pi}{2}$, let $h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2,\ 0 < x < \frac{1}{2}w \quad \Rightarrow$ $h'(x) = w - 4x = 0 \quad \Leftrightarrow \quad x = \frac{1}{4}w$, and $h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}\,w^2$, we conclude that the local maximum found earlier was an absolute maximum; that is, the base and the sides are of equal length.

(b) If the metal were bent into a semicircular gutter of radius r, we would have $w = \pi r$ and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{w}{\pi}\right)^2 = \frac{w^2}{2\pi}$.

Since $\dfrac{w^2}{2\pi} > \dfrac{\sqrt{3}\,w^2}{12}$, it *would* be better to bend the metal into a gutter with a semicircular cross-section.

4. Since $(x+y+z)^r/(x^2+y^2+z^2)$ is a rational function with domain $\{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$, f is continuous on $\mathbb{R}^3$ if and only if $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0) = 0$. Recall that $(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to $(x+y+z)^2$ gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4(x^2+y^2+z^2)$. Now for each r,

$$|(x+y+z)^r| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4(x^2+y^2+z^2)\right]^{r/2} = 2^r(x^2+y^2+z^2)^{r/2}$$

for $(x,y,z) \neq (0,0,0)$. Thus

$$|f(x,y,z) - 0| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{|(x+y+z)^r|}{x^2+y^2+z^2} \leq 2^r\frac{(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 2^r(x^2+y^2+z^2)^{(r/2)-1}$$

for $(x,y,z) \neq (0,0,0)$. Thus if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r(x^2+y^2+z^2)^{(r/2)-1} \to 0$ as $(x,y,z) \to (0,0,0)$ and so $\lim\limits_{(x,y,z)\to(0,0,0)} (x+y+z)^r/(x^2+y^2+z^2) = 0$. Hence for $r > 2$, f is continuous on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,0,0) = x^r/x^2 = x^{r-2}$ for $x \neq 0$. So when $r = 2$, $f(x,y,z) \to 1 \neq 0$ as $(x,y,z) \to (0,0,0)$ along the x-axis and when $r < 2$ the limit of $f(x,y,z)$ as $(x,y,z) \to (0,0,0)$ along the x-axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at $(0,0,0)$ and thus is not continuous on $\mathbb{R}^3$.

5. Let $g(x,y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x,y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$ and

$g_y(x,y) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$. Thus the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) \quad \Rightarrow$$

$\left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right]x + \left[f'\left(\frac{y_0}{x_0}\right)\right]y - z = 0$. But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Thus the origin is the common point of intersection.

6. (a) At $(x_1, y_1, 0)$ the equations of the tangent planes to $z = f(x,y)$ and $z = g(x,y)$ are

$$P_1\!: \; z - f(x_1,y_1) = f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1)$$

and

$$P_2\!: \; z - g(x_1,y_1) = g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1)$$

respectively. P_1 intersects the xy-plane in the line given by $f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1) = -f(x_1,y_1)$, $z = 0$; and P_2 intersects the xy-plane in the line given by $g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1) = -g(x_1,y_1)$, $z = 0$. The point $(x_2, y_2, 0)$ is the point of intersection of these two lines, since $(x_2, y_2, 0)$ is the point where the line of

intersection of the two tangent planes intersects the xy-plane. Thus (x_2, y_2) is the solution of the simultaneous equations

$$f_x(x_1, y_1)(x_2 - x_1) + f_y(x_1, y_1)(y_2 - y_1) = -f(x_1, y_1)$$

and $$g_x(x_1, y_1)(x_2 - x_1) + g_y(x_1, y_1)(y_2 - y_1) = -g(x_1, y_1)$$

For simplicity, rewrite $f_x(x_1, y_1)$ as f_x and similarly for f_y, g_x, g_y, f and g and solve the equations $(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f$ and $(g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$ simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \dfrac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or $y_2 = y_1 - \dfrac{gf_x - fg_x}{f_x g_y - g_x f_y}$ and $(f_x)(x_2 - x_1) + \dfrac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f$ so $x_2 - x_1 = \dfrac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \dfrac{fg_y - f_y g}{g_x f_y - f_x g_y}$. Hence $x_2 = x_1 - \dfrac{fg_y - f_y g}{f_x g_y - g_x f_y}$.

(b) Let $f(x, y) = x^x + y^y - 1000$ and $g(x, y) = x^y + y^x - 100$. Then we wish to solve the system of equations $f(x, y) = 0$, $g(x, y) = 0$. Recall $\dfrac{d}{dx}[x^x] = x^x(1 + \ln x)$ (differentiate logarithmically), so $f_x(x, y) = x^x(1 + \ln x)$, $f_y(x, y) = y^y(1 + \ln y)$, $g_x(x, y) = yx^{y-1} + y^x \ln y$, and $g_y(x, y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

$$x_2 = 2.5 - \frac{f(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5} \approx 2.447674117$$

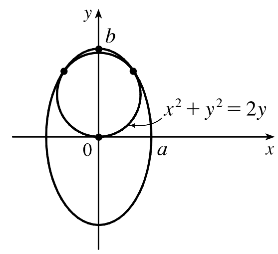
$$y_2 = 4.5 - \frac{f_x(2.5, 4.5)\, g(2.5, 4.5) - f(2.5, 4.5)\, g_x(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.

7. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \dfrac{a^2}{b^2}\left(b^2 - y^2\right)$. Substituting into the equation of the circle gives $\dfrac{a^2}{b^2}\left(b^2 - y^2\right) + y^2 - 2y = 0 \quad \Rightarrow \quad \left(\dfrac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$.



In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2\,\dfrac{b^2 - a^2}{b^2} = 0 \quad \Rightarrow$ $b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A(a, b) = \pi ab$, and we minimize this function subject to the constraint $g(a, b) = b^2 - a^2b^2 + a^4 = 0$.

Now $\nabla A = \lambda \nabla g \quad \Leftrightarrow \quad \pi b = \lambda(4a^3 - 2ab^2), \pi a = \lambda(2b - 2ba^2) \quad \Rightarrow \quad \lambda = \dfrac{\pi b}{2a(2a^2 - b^2)}$ **(1)**,

$\lambda = \dfrac{\pi a}{2b(1 - a^2)}$ **(2)**, $b^2 - a^2b^2 + a^4 = 0$ **(3)**. Comparing **(1)** and **(2)** gives $\dfrac{\pi b}{2a(2a^2 - b^2)} = \dfrac{\pi a}{2b(1 - a^2)} \quad \Rightarrow$

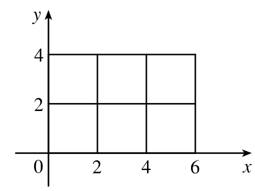
$2\pi b^2 = 4\pi a^4 \quad \Leftrightarrow \quad a^2 = \frac{1}{\sqrt{2}}\,b$. Substitute this into **(3)** to get $b = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad a = \sqrt{\frac{3}{2}}$.

8. Let $\mathbf{u} = \langle a, b, c\rangle$ and $\mathbf{v} = \langle x, y, 1\rangle$, so $|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$, $|\mathbf{v}| = \sqrt{x^2 + y^2 + 1}$, and $\mathbf{u} \cdot \mathbf{v} = ax + by + c$. Then by the Cauchy-Schwarz Inequality, $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}| \quad \Rightarrow \quad |ax + by + c| \le \sqrt{a^2 + b^2 + c^2}\,\sqrt{x^2 + y^2 + 1}$. Squaring both sides, we have $(ax + by + c)^2 \le \left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + 1\right) \quad \Rightarrow \quad \dfrac{(ax + by + c)^2}{x^2 + y^2 + 1} \le a^2 + b^2 + c^2$ (since $x^2 + y^2 + 1 > 0$). Thus $f(x, y) = \dfrac{(ax + by + c)^2}{x^2 + y^2 + 1} \le a^2 + b^2 + c^2$. We have equality if $(ax + by + c)^2 = \left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + 1\right)$ or equivalently $c^2\left[(a/c)x + (b/c)y + 1\right]^2 = c^2\left[(a/c)^2 + (b/c)^2 + 1\right]\left(x^2 + y^2 + 1\right)$ which is true when $x = a/c$ and $y = b/c$. Thus the maximum value of f is $f(a/c, b/c) = a^2 + b^2 + c^2$.

15 □ MULTIPLE INTEGRALS

15.1 Double Integrals over Rectangles

1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x,y) = xy$ and $\Delta A = 4$, so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

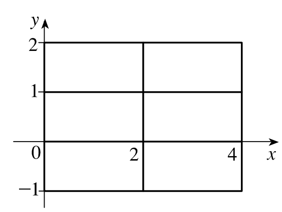
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $V \approx \sum\limits_{i=1}^{3} \sum\limits_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

2. (a) The subrectangles are shown in the figure.



Here $\Delta A = 2$ and we estimate

$$\iint_R \left(1 - xy^2\right) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$

$$= f(2,-1)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(4,-1)\,\Delta A + f(4,0)\,\Delta A + f(4,1)\,\Delta A$$

$$= (-1)(2) + 1(2) + (-1)(2) + (-3)(2) + 1(2) + (-3)(2) = -12$$

(b) $\iint_R \left(1 - xy^2\right) dA \approx \sum\limits_{i=1}^{2} \sum\limits_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

$$= f(0,0)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

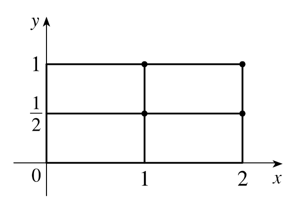
$$= 1(2) + 1(2) + 1(2) + 1(2) + (-1)(2) + (-7)(2) = -8$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = 1 \cdot \frac{1}{2} = \frac{1}{2}$, we estimate

$$\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$

$$= f\left(1, \tfrac{1}{2}\right)\Delta A + f(1,1)\,\Delta A + f\left(2, \tfrac{1}{2}\right)\Delta A + f(2,1)\,\Delta A$$

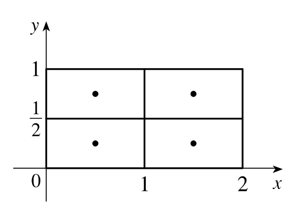
$$= e^{-1/2}\left(\tfrac{1}{2}\right) + e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-2}\left(\tfrac{1}{2}\right) \approx 0.990$$



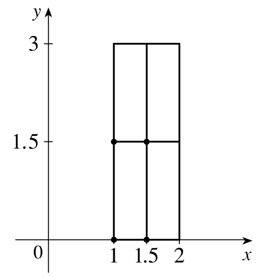
(b) $\iint_R xe^{-xy}\,dA \approx \sum\limits_{i=1}^{2} \sum\limits_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$

$$= f\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\Delta A + f\left(\tfrac{1}{2}, \tfrac{3}{4}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{1}{4}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{4}\right)\Delta A$$

$$= \tfrac{1}{2}e^{-1/8}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-9/8}\left(\tfrac{1}{2}\right) \approx 1.151$$



4. (a) The subrectangles are shown in the figure.



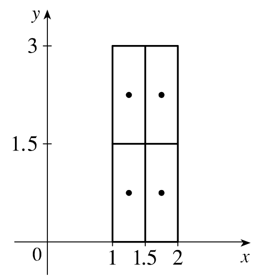
The surface is the graph of $f(x,y) = 1 + x^2 + 3y$ and $\Delta A = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, so we estimate

$$V = \iint_R (1 + x^2 + 3y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

$$= f(1,0)\,\Delta A + f\left(1, \tfrac{3}{2}\right) \Delta A + f\left(\tfrac{3}{2}, 0\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \Delta A$$

$$= 2\left(\tfrac{3}{4}\right) + \tfrac{13}{2}\left(\tfrac{3}{4}\right) + \tfrac{13}{4}\left(\tfrac{3}{4}\right) + \tfrac{31}{4}\left(\tfrac{3}{4}\right) = \tfrac{39}{2}\left(\tfrac{3}{4}\right) = \tfrac{117}{8} = 14.625$$

(b) $$V = \iint_R (1 + x^2 + 3y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$$

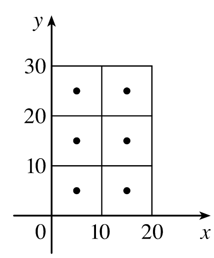


$$= f\left(\tfrac{5}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{5}{4}, \tfrac{9}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{9}{4}\right) \Delta A$$

$$= \tfrac{77}{16}\left(\tfrac{3}{4}\right) + \tfrac{149}{16}\left(\tfrac{3}{4}\right) + \tfrac{101}{16}\left(\tfrac{3}{4}\right) + \tfrac{173}{16}\left(\tfrac{3}{4}\right) = \tfrac{375}{16} = 23.4375$$

5. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$

$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\,\Delta A$$

$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$

$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

7. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

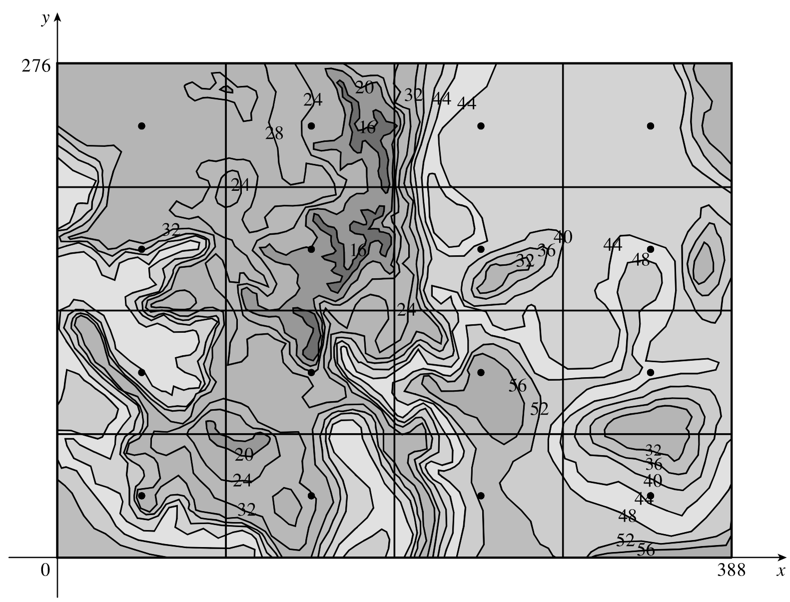
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{avg}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

8. As in Example 9, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y). The average temperature is given by

$$f_{\text{avg}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{388 \cdot 276}\iint_R f(x,y)\,dA$$

To use the Midpoint Rule with $m = n = 4$, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

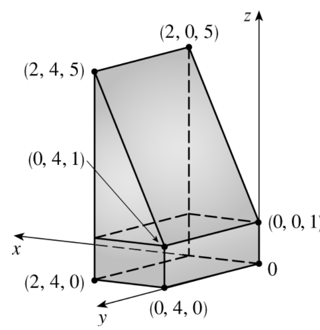
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A \\ &\approx \Delta A\,[31 + 28 + 52 + 43 + 43 + 25 + 57 + 46 + 36 + 20 + 42 + 45 + 30 + 23 + 43 + 41] \\ &= 6693(605)\end{aligned}$$

Therefore, $f_{\text{avg}} \approx \dfrac{6693 \cdot 605}{388 \cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on a day in February was approximately 37.8°F.

9. $z = \sqrt{2} > 0$, so we can interpret the double integral as the volume of the solid S that lies below the plane $z = \sqrt{2}$ and above the rectangle $[2, 6] \times [-1, 5]$. S is a rectangular solid, so $\iint_R \sqrt{2}\,dA = 4 \cdot 6 \cdot \sqrt{2} = 24\sqrt{2}$.

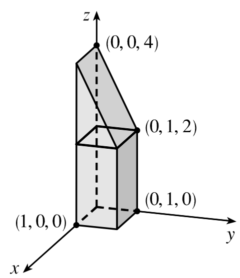
10. $z = 2x + 1 \geq 0$ for $0 \leq x \leq 2$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 2x + 1$ and above the rectangle $[0, 2] \times [0, 4]$. We can picture S as a rectangular solid (with height 1) surmounted by a triangular cylinder; thus

$$\iint_R (2x+1)\, dA = (2)(4)(1) + \tfrac{1}{2}(2)(4)(4) = 24$$

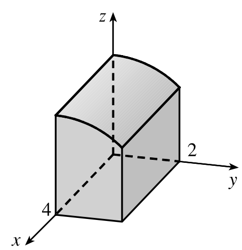


11. $z = 4 - 2y \geq 0$ for $0 \leq y \leq 1$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 4 - 2y$ and above the square $[0, 1] \times [0, 1]$. We can picture S as a rectangular solid (with height 2) surmounted by a triangular cylinder; thus

$$\iint_R (4-2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



12. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



13. $\int_0^2 (x + 3x^2y^2)\, dx = \left[\dfrac{x^2}{2} + 3\,\dfrac{x^3}{3}\,y^2\right]_{x=0}^{x=2} = \left[\frac{1}{2}x^2 + x^3y^2\right]_{x=0}^{x=2} = \left[\frac{1}{2}(2)^2 + (2)^3y^2\right] - \left[\frac{1}{2}(0)^2 + (0)^3y^2\right] = 2 + 8y^2,$

$\int_0^3 (x + 3x^2y^2)\, dy = \left[xy + 3x^2\,\dfrac{y^3}{3}\right]_{y=0}^{y=3} = \left[xy + x^2y^3\right]_{y=0}^{y=3} = \left[x(3) + x^2(3)^3\right] - \left[x(0) + x^2(0)^3\right] = 3x + 27x^2$

14. $\int_0^2 y\sqrt{x+2}\, dx = \left[y \cdot \frac{2}{3}(x+2)^{3/2}\right]_{x=0}^{x=2} = \frac{2}{3}y(4)^{3/2} - \frac{2}{3}y(2)^{3/2} = \frac{16}{3}y - \frac{4}{3}\sqrt{2}\,y = \frac{4}{3}(4 - \sqrt{2}\,)\,y,$

$\int_0^3 y\sqrt{x+2}\, dy = \left[\dfrac{y^2}{2}\,\sqrt{x+2}\right]_{y=0}^{y=3} = \frac{1}{2}(3)^2\sqrt{x+2} - \frac{1}{2}(0)^2\sqrt{x+2} = \frac{9}{2}\sqrt{x+2}$

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2} dx = \int_1^4 \left[(12x^2 - 4x) - (0 - 0)\right] dx$

$$= \int_1^4 (12x^2 - 4x)\, dx = \left[4x^3 - 2x^2\right]_1^4 = (256 - 32) - (4 - 2) = 222$$

16. $\int_0^1 \int_0^1 (x+y)^2\, dx\, dy = \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\, dx\, dy = \int_0^1 \left[\frac{1}{3}x^3 + x^2y + xy^2\right]_{x=0}^{x=1} dy$

$$= \int_0^1 (\tfrac{1}{3} + y + y^2)\, dy = \left[\tfrac{1}{3}y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{3} - 0 = \tfrac{7}{6}$$

17. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy = \int_0^1 \left[\frac{1}{2}x^2 + xe^{-y}\right]_{x=1}^{x=2} dy = \int_0^1 \left[(2 + 2e^{-y}) - \left(\frac{1}{2} + e^{-y}\right)\right] dy$

$$= \int_0^1 \left(\tfrac{3}{2} + e^{-y}\right) dy = \left[\tfrac{3}{2}y - e^{-y}\right]_0^1 = \left(\tfrac{3}{2} - e^{-1}\right) - (0 - 1) = \tfrac{5}{2} - e^{-1}$$

18. $\int_{-3}^1 \int_1^2 \left(x^2 + y^{-2}\right) dy\, dx = \int_{-3}^1 \left[x^2 y - y^{-1}\right]_{y=1}^{y=2} dx = \int_{-3}^1 \left[\left(2x^2 - \frac{1}{2}\right) - (x^2 - 1)\right] dx$

$$= \int_{-3}^1 \left(x^2 + \tfrac{1}{2}\right) dx = \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x\right]_{-3}^1 = \left(\tfrac{1}{3} + \tfrac{1}{2}\right) - \left(-\tfrac{27}{3} - \tfrac{3}{2}\right) = \tfrac{34}{3}$$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\, dx\, dy = \int_{-3}^3 \left[xy + y^2 \sin x\right]_{x=0}^{x=\pi/2} dy = \int_{-3}^3 \left(\frac{\pi}{2}y + y^2\right) dy$

$$= \left[\tfrac{\pi}{4}y^2 + \tfrac{1}{3}y^3\right]_{-3}^3 = \left[\left(\tfrac{9\pi}{4} + 9\right) - \left(\tfrac{9\pi}{4} - 9\right)\right] = 18$$

20. $\displaystyle\int_1^3 \int_1^5 \frac{\ln y}{xy}\, dy\, dx = \int_1^3 \frac{1}{x}\, dx \int_1^5 \frac{\ln y}{y}\, dy$ [by Equation 11]

$$= \Big[\ln|x|\Big]_1^3 \left[\tfrac{1}{2}(\ln y)^2\right]_1^5 \quad \text{[substitute } u = \ln y \;\Rightarrow\; du = (1/y)\, dy\text{]}$$

$$= (\ln 3 - 0) \cdot \tfrac{1}{2}[(\ln 5)^2 - 0] = \tfrac{1}{2}(\ln 3)(\ln 5)^2$$

21. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx = \int_1^4 \left[x \ln|y| + \frac{1}{x} \cdot \frac{1}{2}\, y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x}\right) dx = \left[\tfrac{1}{2}x^2 \ln 2 + \tfrac{3}{2}\ln|x|\right]_1^4$

$$= \left(8 \ln 2 + \tfrac{3}{2}\ln 4\right) - \left(\tfrac{1}{2}\ln 2 + 0\right) = \tfrac{15}{2}\ln 2 + \tfrac{3}{2}\ln 4 \text{ or } \tfrac{15}{2}\ln 2 + 3\ln(4^{1/2}) = \tfrac{21}{2}\ln 2$$

22. $\int_0^1 \int_0^2 ye^{x-y}\, dx\, dy = \int_0^1 \int_0^2 ye^x e^{-y}\, dx\, dy = \int_0^2 e^x\, dx \int_0^1 ye^{-y}\, dy$ [by Equation 11]

$$= [e^x]_0^2 \left[(-y - 1)e^{-y}\right]_0^1 \quad \text{[by integrating by parts]}$$

$$= \left(e^2 - e^0\right)\left[-2e^{-1} - \left(-e^0\right)\right] = (e^2 - 1)(1 - 2e^{-1}) \text{ or } e^2 - 2e + 2e^{-1} - 1$$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\, d\phi\, dt = \int_0^{\pi/2} \sin^3\phi\, d\phi \int_0^3 t^2\, dt$ [by Equation 11] $= \int_0^{\pi/2} (1 - \cos^2\phi) \sin\phi\, d\phi \int_0^3 t^2\, dt$

$$= \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/2} \left[\tfrac{1}{3}t^3\right]_0^3 = \left[(0 - 0) - \left(\tfrac{1}{3} - 1\right)\right] \cdot \tfrac{1}{3}(27 - 0) = \tfrac{2}{3}(9) = 6$$

24. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\, dy\, dx = \int_0^1 x\left[\frac{1}{3}(x^2 + y^2)^{3/2}\right]_{y=0}^{y=1} dx = \frac{1}{3}\int_0^1 x[(x^2 + 1)^{3/2} - x^3]\, dx = \frac{1}{3}\int_0^1 [x(x^2 + 1)^{3/2} - x^4] dx$

$$= \tfrac{1}{3}\left[\tfrac{1}{5}(x^2 + 1)^{5/2} - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{15}\left[(2^{5/2} - 1) - (1 - 0)\right] = \tfrac{2}{15}\left(2\sqrt{2} - 1\right)$$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\, du\, dv = \int_0^1 \left[\frac{1}{5}v(u + v^2)^5\right]_{u=0}^{u=1} dv = \frac{1}{5}\int_0^1 v\left[(1 + v^2)^5 - (0 + v^2)^5\right] dv$

$$= \tfrac{1}{5}\int_0^1 \left[v(1 + v^2)^5 - v^{11}\right] dv = \tfrac{1}{5}\left[\tfrac{1}{2} \cdot \tfrac{1}{6}(1 + v^2)^6 - \tfrac{1}{12}v^{12}\right]_0^1$$

[substitute $t = 1 + v^2 \;\Rightarrow\; dt = 2v\, dv$ in the first term]

$$= \tfrac{1}{60}\left[(2^6 - 1) - (1 - 0)\right] = \tfrac{1}{60}(63 - 1) = \tfrac{31}{30}$$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\, ds\, dt = \int_0^1 \left[\frac{2}{3}(s + t)^{3/2}\right]_{s=0}^{s=1} dt = \frac{2}{3}\int_0^1 [(1 + t)^{3/2} - t^{3/2}]\, dt = \frac{2}{3}\left[\frac{2}{5}(1 + t)^{5/2} - \frac{2}{5}t^{5/2}\right]_0^1$

$$= \tfrac{4}{15}[(2^{5/2} - 1) - (1 - 0)] = \tfrac{4}{15}(2^{5/2} - 2) \text{ or } \tfrac{8}{15}\left(2\sqrt{2} - 1\right)$$

27. $\iint_R x \sec^2 y\, dA = \int_0^2 \int_0^{\pi/4} x \sec^2 y\, dy\, dx = \int_0^2 x\, dx \int_0^{\pi/4} \sec^2 y\, dy = \left[\frac{1}{2}x^2\right]_0^2 \Big[\tan y\Big]_0^{\pi/4}$

$$= (2 - 0)\left(\tan\tfrac{\pi}{4} - \tan 0\right) = 2(1 - 0) = 2$$

28. $\iint_R (y + xy^{-2})\, dA = \int_1^2 \int_0^2 (y + xy^{-2})\, dx\, dy = \int_1^2 \left[xy + \frac{1}{2}x^2y^{-2}\right]_{x=0}^{x=2} dy = \int_1^2 \left(2y + 2y^{-2}\right) dy$

$= \left[y^2 - 2y^{-1}\right]_1^2 = (4 - 1) - (1 - 2) = 4$

29. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^1 \frac{x}{x^2+1}\, dx \int_{-3}^3 y^2\, dy = \left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 \left[\tfrac{1}{3}y^3\right]_{-3}^3$

$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \frac{1}{3}(27 + 27) = 9 \ln 2$

30. $\displaystyle\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA = \int_0^{1/2} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{1-t^2}}\, d\theta\, dt = \int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt \int_0^{\pi/3} \tan\theta\, d\theta = \left[\sin^{-1} t\right]_0^{1/2} \left[\ln|\sec\theta|\right]_0^{\pi/3}$

$= \left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right) = \left(\frac{\pi}{6} - 0\right)(\ln 2 - \ln 1) = \frac{\pi}{6}\ln 2$

31. $\iint_R x\sin(x+y)\, dA = \int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\, dy\, dx$

$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x + \frac{\pi}{3}\right)\right] dx$

$= x\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]

$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

32. $\displaystyle\iint_R \frac{x}{1+xy}\, dA = \int_0^1\int_0^1 \frac{x}{1+xy}\, dy\, dx = \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$

$= \int_0^1 \ln(1+x)\, dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]

$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

33. $\iint_R ye^{-xy}\, dA = \int_0^3\int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 \left[-e^{-xy}\right]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)\, dy = \left[\frac{1}{2}e^{-2y} + y\right]_0^3$

$= \frac{1}{2}e^{-6} + 3 - \left(\frac{1}{2} + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}$

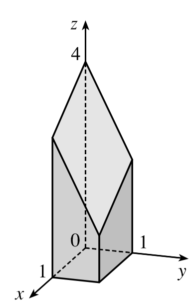
34. $\displaystyle\iint_R \frac{1}{1+x+y}\, dA = \int_1^3\int_1^2 \frac{1}{1+x+y}\, dy\, dx = \int_1^3 \left[\ln(1+x+y)\right]_{y=1}^{y=2} dx = \int_1^3 \left[\ln(x+3) - \ln(x+2)\right] dx$

$= \left[\left((x+3)\ln(x+3) - (x+3)\right) - \left((x+2)\ln(x+2) - (x+2)\right)\right]_1^3$

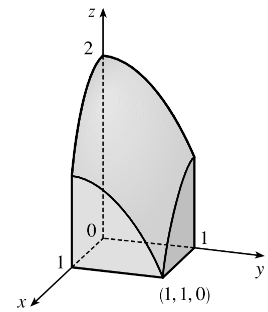
[by integrating by parts separately for each term]

$= (6\ln 6 - 6 - 5\ln 5 + 5) - (4\ln 4 - 4 - 3\ln 3 + 3) = 6\ln 6 - 5\ln 5 - 4\ln 4 + 3\ln 3$

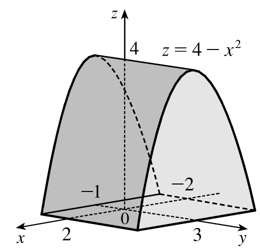
35. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.



36. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



37. $z = 4 - x^2 \geq 0$ for $-2 \leq x \leq 2$ and $-1 \leq y \leq 3$. So the solid is the region that lies below the parabolic cylinder $z = 4 - x^2$ and above $[-2, 2] \times [-1, 3]$.



38. (a) For any given value of $x = k$, $0 \leq k \leq 2$, we have the curve $z = k^2\sqrt{y}$, $x = k$, $1 \leq y \leq 4$.

The area below the curve and above xy-plane is given by $\displaystyle\int_1^4 k^2\sqrt{y}\,dy = \left[\frac{2k^2}{3}y^{3/2}\right]_{y=1}^{y=4} = \frac{2k^2}{3}(8-1) = \frac{14k^2}{3}$.

For $k = 1$, the area is $\dfrac{14(1^2)}{3} = \dfrac{14}{3}$. For $k = 2$, the area is $\dfrac{14(2^2)}{3} = \dfrac{56}{3}$.

(b) For any given value of $y = k$, $1 \leq k \leq 4$, we have the curve, $z = x^2\sqrt{k}$, $y = k$, $0 \leq x \leq 2$. The area below the curve and above xy-plane is given by $\displaystyle\int_0^2 x^2\sqrt{k}\,dx = \left[\frac{x^3}{3}\sqrt{k}\right]_{x=0}^{x=2} = \frac{8\sqrt{k}}{3}$. For $k = 1$, the area is $\dfrac{8\sqrt{1}}{3} = \dfrac{8}{3}$. For $k = 3$, the area is $\dfrac{8\sqrt{3}}{3}$.

(c) $$\int_1^4\int_0^2 x^2\sqrt{y}\,dx\,dy = \int_1^4\left[\frac{x^3}{3}\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 \frac{8}{3}\sqrt{y}\,dy = \left[\frac{16}{9}y^{3/2}\right]_1^4 = \frac{16}{9}(8-1) = \frac{112}{9}$$

39. (a) The volume under the surface $z = xy$ and over the square $R = [0, 2] \times [0, 2]$ is given by $\int_0^2\int_0^2 xy\,dx\,dy$.

(b) $$\int_0^2\int_0^2 xy\,dx\,dy = \int_0^2\left[\frac{x^2}{2}\right]_{x=0}^{x=2} y\,dy = \int_0^2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] y\,dy = \int_0^2 2y\,dy = \left[y^2\right]_0^2 = 4$$

40. (a) The volume under the surface $z = \cos x\cos y$ and over the square $R = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is given by $\int_{-\pi/4}^{\pi/4}\int_{-\pi/4}^{\pi/4} \cos x\cos y\,dx\,dy$.

(b) $$\begin{aligned}\int_{-\pi/4}^{\pi/4}\int_{-\pi/4}^{\pi/4} \cos x\cos y\,dx\,dy &= \int_{-\pi/4}^{\pi/4}\cos x\,dx\int_{-\pi/4}^{\pi/4}\cos y\,dy \qquad \text{[by Equation 11]}\\ &= \Big[\sin x\Big]_{x=-\pi/4}^{x=\pi/4}\Big[\sin y\Big]_{y=-\pi/4}^{y=\pi/4} = \left[\sin\tfrac{\pi}{4} - \sin\left(-\tfrac{\pi}{4}\right)\right]^2\\ &= \left[\tfrac{\sqrt{2}}{2} - \left(-\tfrac{\sqrt{2}}{2}\right)\right]^2 = \left(\sqrt{2}\right)^2 = 2\end{aligned}$$

41. (a) The volume under the surface $z = 1 + ye^{xy}$ and over the rectangle $R = [0,1] \times [1,2]$ is given by

$\int_1^2 \int_0^1 (1 + ye^{xy})\, dx\, dy.$

(b) $$\int_1^2 \int_0^1 (1 + ye^{xy})\, dx\, dy = \int_1^2 \left[x + y \cdot \frac{1}{y} e^{xy}\right]_{x=0}^{x=1} dy = \int_1^2 \left[(1 + e^y) - (0 + 1)\right] dy = \int_1^2 e^y\, dy = e^2 - e$$

42. (a) The volume under the surface $z = x^2 + y^2$ and over the square $R = [1,2] \times [1,3]$ is given by $\int_1^3 \int_1^2 (x^2 + y^2)\, dx\, dy$.

(b) $$\int_1^3 \int_1^2 (x^2 + y^2)\, dx\, dy = \int_1^3 \left[\frac{x^3}{3} + xy^2\right]_{x=1}^{x=2} dy = \int_1^3 \left[\left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right)\right] dy = \int_1^3 \left(\frac{7}{3} + y^2\right) dy$$

$$= \left[\frac{7}{3}y + \frac{y^3}{3}\right]_1^3 = (7 + 9) - \left(\frac{7}{3} + \frac{1}{3}\right) = \frac{40}{3}$$

43. The solid lies under the plane $4x + 6y - 2z + 15 = 0$ or $z = 2x + 3y + \frac{15}{2}$ so

$$V = \iint_R (2x + 3y + \tfrac{15}{2})\, dA = \int_{-1}^1 \int_{-1}^2 (2x + 3y + \tfrac{15}{2})\, dx\, dy = \int_{-1}^1 \left[x^2 + 3xy + \tfrac{15}{2}x\right]_{x=-1}^{x=2} dy$$

$$= \int_{-1}^1 \left[(19 + 6y) - (-\tfrac{13}{2} - 3y)\right] dy = \int_{-1}^1 (\tfrac{51}{2} + 9y)\, dy = \left[\tfrac{51}{2}y + \tfrac{9}{2}y^2\right]_{-1}^1 = 30 - (-21) = 51$$

44. $$V = \iint_R (3y^2 - x^2 + 2)\, dA = \int_{-1}^1 \int_1^2 (3y^2 - x^2 + 2)\, dy\, dx = \int_{-1}^1 \left[y^3 - x^2y + 2y\right]_{y=1}^{y=2} dx$$

$$= \int_{-1}^1 \left[(12 - 2x^2) - (3 - x^2)\right] dx = \int_{-1}^1 \left(9 - x^2 dx\right) = \left[9x - \tfrac{1}{3}x^3\right]_{-1}^1 = \tfrac{26}{3} + \tfrac{26}{3} = \tfrac{52}{3}$$

45. $$V = \int_{-2}^2 \int_{-1}^1 \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy = 4 \int_0^2 \int_0^1 \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy$$

$$= 4 \int_0^2 \left[x - \tfrac{1}{12}x^3 - \tfrac{1}{9}y^2x\right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\tfrac{11}{12} - \tfrac{1}{9}y^2\right) dy = 4\left[\tfrac{11}{12}y - \tfrac{1}{27}y^3\right]_0^2 = 4 \cdot \tfrac{83}{54} = \tfrac{166}{27}$$

46. The solid lies under the surface $z = x^2 + xy^2$ and above the rectangle $R = [0,5] \times [-2,2]$, so its volume is

$$V = \iint_R (x^2 + xy^2)\, dA = \int_0^5 \int_{-2}^2 (x^2 + xy^2)\, dy\, dx = \int_0^5 \left[x^2y + \tfrac{1}{3}xy^3\right]_{y=-2}^{y=2} dx$$

$$= \int_0^5 \left[\left(2x^2 + \tfrac{8}{3}x\right) - \left(-2x^2 - \tfrac{8}{3}x\right)\right] dx = \int_0^5 (4x^2 + \tfrac{16}{3}x)\, dx$$

$$= \left[\tfrac{4}{3}x^3 + \tfrac{8}{3}x^2\right]_0^5 = \tfrac{500}{3} + \tfrac{200}{3} - 0 = \tfrac{700}{3}$$

47. The solid lies under the surface $z = 1 + x^2ye^y$ and above the rectangle $R = [-1,1] \times [0,1]$, so its volume is

$$V = \iint_R (1 + x^2ye^y)\, dA = \int_0^1 \int_{-1}^1 (1 + x^2ye^y)\, dx\, dy = \int_0^1 \left[x + \tfrac{1}{3}x^3ye^y\right]_{x=-1}^{x=1} dy$$

$$= \int_0^1 (2 + \tfrac{2}{3}ye^y)\, dy = \left[2y + \tfrac{2}{3}(y - 1)\, e^y\right]_0^1 \qquad \text{[by integrating by parts in the second term]}$$

$$= (2 + 0) - \left(0 - \tfrac{2}{3}e^0\right) = 2 + \tfrac{2}{3} = \tfrac{8}{3}$$

48. The cylinder $z = 16 - x^2$ intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $R = [0,4] \times [0,5]$ in the xy-plane.

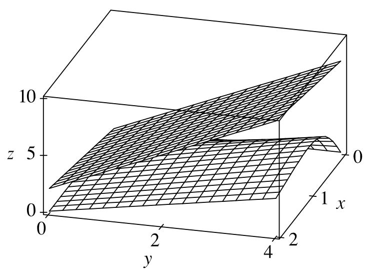
$$V = \int_0^5 \int_0^4 (16 - x^2)\, dx\, dy = \int_0^4 (16 - x^2)\, dx \int_0^5 dy$$

$$= \left[16x - \tfrac{1}{3}x^3\right]_0^4 \left[\,y\,\right]_0^5 = (64 - \tfrac{64}{3} - 0)(5 - 0) = \tfrac{640}{3}$$

49. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1, 1] \times [0, 4]$ and the solid that lies under $z = 1$ over R.

$$\begin{aligned} V &= \int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\,dx\,dy - \int_0^4 \int_{-1}^1 (1)\,dx\,dy \\ &= \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy \\ &= \int_0^4 \left[(2 + \tfrac{1}{3} + (y-2)^2) - (-2 - \tfrac{1}{3} - (y-2)^2)\right] dy - [x]_{-1}^1\,[y]_0^4 \\ &= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right] dy - [1 - (-1)][4 - 0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4) \\ &= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3} \end{aligned}$$

50.



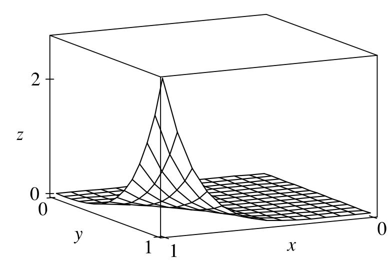
The solid lies below the plane $z = x + 2y$ and above the surface $z = \dfrac{2xy}{x^2+1}$ for $0 \le x \le 2$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = x + 2y$ over the rectangle $R = [0, 2] \times [0, 4]$ and the solid that lies under $z = \dfrac{2xy}{x^2+1}$ over R.

$$\begin{aligned} V &= \int_0^2 \int_0^4 (x + 2y)dy\,dx - \int_0^2 \int_0^4 \frac{2xy}{x^2+1}\,dy\,dx = \int_0^2 \left[xy + y^2\right]_{y=0}^{y=4} dx - \int_0^2 \frac{2x}{x^2+1}\,dx \int_0^4 y\,dy \\ &= \int_0^2 [(4x + 16) - (0 + 0)]\,dx - \left[\ln\left|x^2+1\right|\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^4 = \left[2x^2 + 16x\right]_0^2 - (\ln 5 - \ln 1)\,(8 - 0) \\ &= (8 + 32 - 0) - 8\ln 5 = 40 - 8\ln 5 \end{aligned}$$

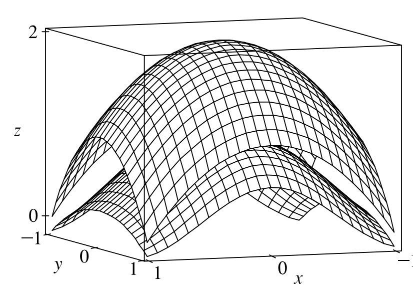
51. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command

`Integrate[f,{x,0,1},{y,0,1}]`

We find that $\iint_R x^5 y^3 e^{xy}\,dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



52. In Maple, we can calculate the integral by defining `f:=exp(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2}\cos(x^2 + y^2)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(Int(Int(g-f,x=-1..1),y=-1..1));`. Using `Int` rather than `int` forces Maple to use purely numerical techniques in evaluating the integral.



In Mathematica, we can use the command `NIntegrate[g-f,{x,-1,1},{y,-1,1}]`. We find that

$\iint_R \left[(2-x^2-y^2)-\left(e^{-x^2}\cos(x^2+y^2)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3D` (in Mathematica) to graph both functions on the same screen.

53. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R)=2\cdot 5=10$ and

$$f_{\text{avg}}=\frac{1}{A(R)}\iint_R f(x,y)\,dA=\tfrac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy=\tfrac{1}{10}\int_0^5\left[\tfrac{1}{3}x^3y\right]_{x=-1}^{x=1}dy=\tfrac{1}{10}\int_0^5\tfrac{2}{3}y\,dy=\tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5=\tfrac{5}{6}.$$

54. $A(R)=4\cdot 1=4$, so

$$\begin{aligned} f_{\text{avg}}=\frac{1}{A(R)}\iint_R f(x,y)\,dA &= \tfrac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dy\,dx=\tfrac{1}{4}\int_0^4\left[\tfrac{2}{3}(x+e^y)^{3/2}\right]_{y=0}^{y=1}dx \\ &= \tfrac{1}{4}\cdot\tfrac{2}{3}\int_0^4\left[(x+e)^{3/2}-(x+1)^{3/2}\right]dx=\tfrac{1}{6}\left[\tfrac{2}{5}(x+e)^{5/2}-\tfrac{2}{5}(x+1)^{5/2}\right]_0^4 \\ &= \tfrac{1}{6}\cdot\tfrac{2}{5}\left[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1\right]=\tfrac{1}{15}\left[(4+e)^{5/2}-e^{5/2}-5^{5/2}+1\right]\approx 3.327 \end{aligned}$$

55. $\displaystyle\iint_R \frac{xy}{1+x^4}\,dA=\int_{-1}^1\int_0^1\frac{xy}{1+x^4}\,dy\,dx=\int_{-1}^1\frac{x}{1+x^4}\,dx\int_0^1 y\,dy$ [by Equation 11] but $f(x)=\dfrac{x}{1+x^4}$ is an odd function so $\displaystyle\int_{-1}^1 f(x)\,dx=0$ (by Theorem 5.5.7). Thus $\displaystyle\iint_R\frac{xy}{1+x^4}\,dA=0\cdot\int_0^1 y\,dy=0$.

56.
$$\begin{aligned} \iint_R(1+x^2\sin y+y^2\sin x)\,dA &= \iint_R 1\,dA+\iint_R x^2\sin y\,dA+\iint_R y^2\sin x\,dA \\ &= A(R)+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}x^2\sin y\,dy\,dx+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}y^2\sin x\,dy\,dx \\ &= (2\pi)(2\pi)+\int_{-\pi}^{\pi}x^2dx\int_{-\pi}^{\pi}\sin y\,dy+\int_{-\pi}^{\pi}\sin x\,dx\int_{-\pi}^{\pi}y^2\,dy \end{aligned}$$

But $\sin x$ is an odd function, so $\int_{-\pi}^{\pi}\sin x\,dx=\int_{-\pi}^{\pi}\sin y\,dy=0$ (by Theorem 5.5.7) and $\iint_R(1+x^2\sin y+y^2\sin x)\,dA=4\pi^2+0+0=4\pi^2$.

57. Let $f(x,y)=\dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx=\frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy=-\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals that diverge at their lower limits of integration.

58. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1:

$g_x=\dfrac{d}{dx}g(x,y)=\dfrac{d}{dx}\displaystyle\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds=\int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again:

$g_{xy}=\dfrac{d}{dy}\displaystyle\int_c^y f(x,t)\,dt=f(x,y)$.

[continued]

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

15.2 Double Integrals over General Regions

1. $\int_1^5 \int_0^x (8x-2y)\,dy\,dx = \int_1^5 \left[8xy - y^2\right]_{y=0}^{y=x} dx = \int_1^5 [8x(x) - (x)^2 - 8x(0) + (0)^2]dx$

$= \int_1^5 7x^2\,dx = \left.\frac{7}{3}x^3\right]_1^5 = \frac{7}{3}(125-1) = \frac{868}{3}$

2. $\int_0^2 \int_0^{y^2} x^2 y\,dx\,dy = \int_0^2 \left[\frac{1}{3}x^3 y\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{1}{3}y\left[(y^2)^3 - (0)^3\right] dy$

$= \int_0^2 \frac{1}{3}y^7\,dy = \frac{1}{3}\left[\frac{1}{8}y^8\right]_0^2 = \frac{1}{3}(32-0) = \frac{32}{3}$

3. $\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 e^{y^3}\right]_{x=0}^{x=y} dy = \int_0^1 \frac{1}{2}e^{y^3}\left[(y)^2 - (0)^2\right] dy$

$= \frac{1}{2}\int_0^1 y^2 e^{y^3}\,dy = \frac{1}{2}\left[\frac{1}{3}e^{y^3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3}\left(e^1 - e^0\right) = \frac{1}{6}(e-1)$

4. $\int_0^{\pi/2} \int_0^x x\sin y\,dy\,dx = \int_0^{\pi/2} \left[x(-\cos y)\right]_{y=0}^{y=x} dx = \int_0^{\pi/2} (-x\cos x + x)\,dx = \int_0^{\pi/2} (x - x\cos x)\,dx$

$= \left[\frac{1}{2}x^2 - (x\sin x + \cos x)\right]_0^{\pi/2}$ (by integrating by parts in the second term)

$= \left(\frac{1}{2}\cdot\frac{\pi^2}{4} - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi^2}{8} - \frac{\pi}{2} + 1$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\,dt\,ds = \int_0^1 \left[t\cos(s^3)\right]_{t=0}^{t=s^2} ds = \int_0^1 s^2\cos(s^3)\,ds = \left.\frac{1}{3}\sin(s^3)\right]_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = \frac{1}{3}\sin 1$

6. $\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\,dw\,dv = \int_0^1 \left[w\sqrt{1+e^v}\right]_{w=0}^{w=e^v} dv = \int_0^1 e^v\sqrt{1+e^v}\,dv = \left.\frac{2}{3}(1+e^v)^{3/2}\right]_0^1$

$= \frac{2}{3}(1+e)^{3/2} - \frac{2}{3}(1+1)^{3/2} = \frac{2}{3}(1+e)^{3/2} - \frac{4}{3}\sqrt{2}$

7. (a) We express the iterated integral as a Type I: $\displaystyle\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx$. A Type II would require the sum of two integrals.

(b) $\displaystyle\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx = \int_0^2 \left[y^2\right]_{y=x}^{y=3x-x^2} dx = \int_0^2 \left[(3x-x^2)^2 - (x)^2\right] dx = \int_0^2 \left(8x^2 - 6x^3 + x^4\right) dx$

$= \left[\frac{8}{3}x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5\right]_0^2 = \frac{64}{3} - 24 + \frac{32}{5} = \frac{56}{15}$

8. (a) We express the iterated integral as a Type II: $\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy$. A Type I would require the sum of two integrals.

(b) $\displaystyle\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=y}^{x=2-y} dy = \int_0^1 \left[\left(\frac{(2-y)^2}{2} + (2-y)y\right) - \left(\frac{y^2}{2} + y^2\right)\right] dy$

$= \int_0^1 (2 - 2y^2)\,dy = \left[2y - \frac{2}{3}y^3\right]_0^1 = 2 - \frac{2}{3} = \frac{4}{3}$

9. (a) We express the iterated integral as a Type II. A Type I would require the sum of two integrals. The curves intersect when $\sqrt{x} = x - 2 \;\Rightarrow\; x = x^2 - 4x + 4 \;\Leftrightarrow\; 0 = x^2 - 5x + 4 \;\Leftrightarrow\; (x-4)(x-1) = 0 \;\Leftrightarrow\; x = 1$ or $x = 4$. The point for $x = 1$ is not in D. Thus, the point of intersection of the curves is $(4,2)$ and the integral is $\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy$.

(b) $\displaystyle\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^2 y\left[\frac{x^2}{2}\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^2 y\big[(y+2)^2 - (y^2)^2\big]\,dy = \frac{1}{2}\int_0^2 [y^3 + 4y^2 + 4y - y^5]\,dy$

$$= \tfrac{1}{2}\left[\tfrac{1}{4}y^4 + \tfrac{4}{3}y^3 + 2y^2 - \tfrac{1}{6}y^6\right]_0^2 = \tfrac{1}{2}\left(4 + \tfrac{32}{3} + 8 - \tfrac{32}{3}\right) = 6$$

10. (a) We express the iterated integral as a Type I. A Type II would require the sum of two integrals. The curves intersect when $6 - x = x^2 \;\Leftrightarrow\; x^2 + x - 6 = 0 \;\Leftrightarrow\; (x+3)(x-2) = 0 \;\Leftrightarrow\; x = -3$ or $x = 2$. The point for $x = -3$ is not in D. Thus, the point of intersection of the two curves is $(2, 4)$ and the integral is $\displaystyle\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx$.

(b) $\displaystyle\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx = \int_0^2 x\Big[y\Big]_{y=x^2}^{y=6-x} dx = \int_0^2 x[(6-x) - x^2]\,dx = \int_0^2 [6x - x^2 - x^3]\,dx$

$$= \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 12 - \frac{8}{3} - 4 = \frac{16}{3}$$

11. $\displaystyle\iint_D \frac{y}{x^2+1}\,dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \left[\frac{1}{x^2+1}\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{x^2+1}\,dx$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\ln\left|x^2+1\right|\right]_0^4 = \tfrac{1}{4}\left[\ln\left(x^2+1\right)\right]_0^4 = \tfrac{1}{4}(\ln 17 - \ln 1) = \tfrac{1}{4}\ln 17$$

12. $\iint_D (2x+y)\,dA = \int_1^2 \int_{y-1}^1 (2x+y)\,dx\,dy = \int_1^2 \left[x^2 + xy\right]_{x=y-1}^{x=1} dy = \int_1^2 \left[1 + y - (y-1)^2 - y(y-1)\right] dy$

$$= \int_1^2 (-2y^2 + 4y)\,dy = \left[-\tfrac{2}{3}y^3 + 2y^2\right]_1^2 = \left(-\tfrac{16}{3} + 8\right) - \left(-\tfrac{2}{3} + 2\right) = \tfrac{4}{3}$$

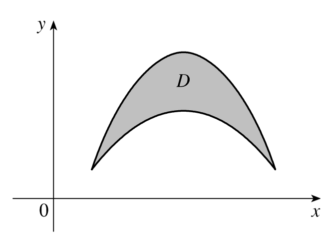
13. $\iint_D e^{-y^2}\,dA = \int_0^3 \int_0^y e^{-y^2}\,dx\,dy = \int_0^3 \left[xe^{-y^2}\right]_{x=0}^{x=y} dy = \int_0^3 \left(ye^{-y^2} - 0\right) dy = \int_0^3 ye^{-y^2}\,dy$

$$= -\tfrac{1}{2}e^{-y^2}\Big]_0^3 = -\tfrac{1}{2}\left(e^{-9} - e^0\right) = \tfrac{1}{2}\left(1 - e^{-9}\right)$$

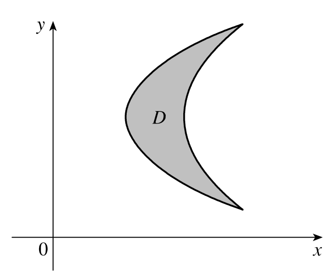
14. $\iint_D y\sqrt{x^2 - y^2}\,dA = \int_0^2 \int_0^x y\sqrt{x^2 - y^2}\,dy\,dx = \int_0^2 \left[-\tfrac{1}{3}(x^2 - y^2)^{3/2}\right]_{y=0}^{y=x} dx = \int_0^2 \left[0 + \tfrac{1}{3}(x^2)^{3/2}\right] dx$

$$= \int_0^2 \tfrac{1}{3}x^3\,dx = \tfrac{1}{3}\cdot\tfrac{1}{4}x^4\Big]_0^2 = \tfrac{1}{12}(16 - 0) = \tfrac{4}{3}$$

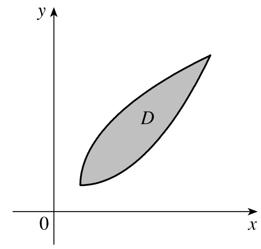
15. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



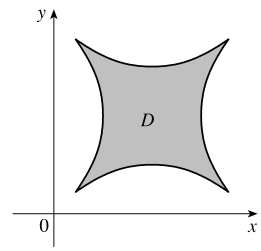
(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



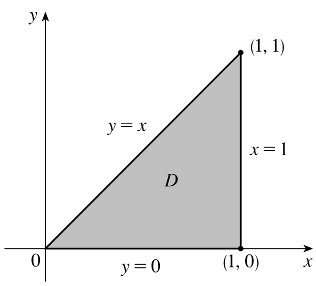
16. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) and also as lying between graphs of two continuous functions of y (a type II region). For additional examples see Figures 9–10 and 15–16 in the text.



(b) Now we sketch an example of a region D that can't be described as lying between the graphs of two continuous functions of x or between graphs of two continuous functions of y. The region shown in Figure 18 is another example.



17.

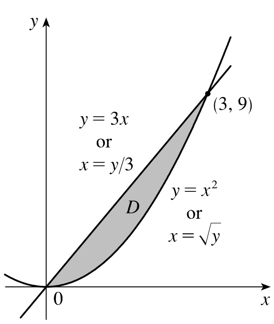


As a type I region, D lies between the lower boundary $y = 0$ and the upper boundary $y = x$ for $0 \le x \le 1$, so $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$. If we describe D as a type II region, D lies between the left boundary $x = y$ and the right boundary $x = 1$ for $0 \le y \le 1$, so $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$.

Thus $\iint_D x\,dA = \int_0^1 \int_0^x x\,dy\,dx = \int_0^1 \left[xy\right]_{y=0}^{y=x} dx = \int_0^1 x^2\,dx = \left.\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$ or

$\iint_D x\,dA = \int_0^1 \int_y^1 x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_{x=y}^{x=1} dy = \frac{1}{2}\int_0^1 (1-y^2)\,dy = \frac{1}{2}\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2}\left[\left(1-\frac{1}{3}\right) - 0\right] = \frac{1}{3}$.

18.



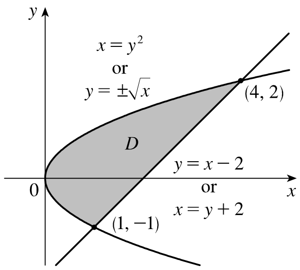
The curves $y = x^2$ and $y = 3x$ intersect at points $(0, 0)$, $(3, 9)$. As a type I region, D is enclosed by the lower boundary $y = x^2$ and the upper boundary $y = 3x$ for $0 \le x \le 3$, so $D = \{(x, y) \mid 0 \le x \le 3, x^2 \le y \le 3x\}$. If we describe D as a type II region, D is enclosed by the left boundary $x = y/3$ and the right boundary $x = \sqrt{y}$ for $0 \le y \le 9$, so $D = \{(x, y) \mid 0 \le y \le 9, y/3 \le x \le \sqrt{y}\}$. Thus

$$\begin{aligned}\iint_D xy\,dA &= \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \left[x \cdot \tfrac{1}{2}y^2\right]_{y=x^2}^{y=3x} dx = \tfrac{1}{2}\int_0^3 x(9x^2 - x^4)\,dx = \tfrac{1}{2}\int_0^3 (9x^3 - x^5)\,dx \\ &= \tfrac{1}{2}\left[9 \cdot \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^3 = \tfrac{1}{2}\left[\left(\tfrac{9}{4} \cdot 81 - \tfrac{1}{6} \cdot 729\right) - 0\right] = \tfrac{243}{8}\end{aligned}$$

or

$$\begin{aligned}\iint_D xy\,dA &= \int_0^9 \int_{y/3}^{\sqrt{y}} xy\,dx\,dy = \int_0^9 \left[\tfrac{1}{2}x^2 y\right]_{x=y/3}^{x=\sqrt{y}} dy = \tfrac{1}{2}\int_0^9 \left(y - \tfrac{1}{9}y^2\right)y\,dy = \tfrac{1}{2}\int_0^9 \left(y^2 - \tfrac{1}{9}y^3\right) dy \\ &= \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{9} \cdot \tfrac{1}{4}y^4\right]_0^9 = \tfrac{1}{2}\left[\left(\tfrac{1}{3} \cdot 729 - \tfrac{1}{36} \cdot 6561\right) - 0\right] = \tfrac{243}{8}\end{aligned}$$

19.

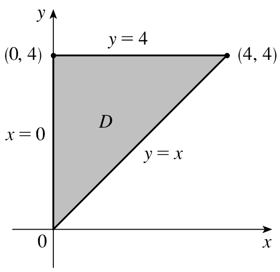


The curves $y = x - 2$ or $x = y + 2$ and $x = y^2$ intersect when $y + 2 = y^2 \Leftrightarrow y^2 - y - 2 = 0 \Leftrightarrow (y-2)(y+1) = 0 \Leftrightarrow y = -1, y = 2$, so the points of intersection are $(1, -1)$ and $(4, 2)$. If we describe D as a type I region, the upper boundary curve is $y = \sqrt{x}$ but the lower boundary curve consists of two parts, $y = -\sqrt{x}$ for $0 \le x \le 1$ and $y = x - 2$ for $1 \le x \le 4$.

Thus $D = \{(x,y) \mid 0 \le x \le 1, -\sqrt{x} \le y \le \sqrt{x}\} \cup \{(x,y) \mid 1 \le x \le 4, x - 2 \le y \le \sqrt{x}\}$ and $\iint_D y\,dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx$. If we describe D as a type II region, D is enclosed by the left boundary $x = y^2$ and the right boundary $x = y + 2$ for $-1 \le y \le 2$, so $D = \{(x,y) \mid -1 \le y \le 2, y^2 \le x \le y + 2\}$ and $\iint_D y\,dA = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy$. In either case, the resulting iterated integrals are not difficult to evaluate but the region D is more simply described as a type II region, giving one iterated integral rather than a sum of two, so we evaluate the latter integral:

$$\iint_D y\,dA = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 \left[xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^2 (y + 2 - y^2)y\,dy = \int_{-1}^2 (y^2 + 2y - y^3)\,dy$$
$$= \left[\tfrac{1}{3}y^3 + y^2 - \tfrac{1}{4}y^4\right]_{-1}^2 = \left(\tfrac{8}{3} + 4 - 4\right) - \left(-\tfrac{1}{3} + 1 - \tfrac{1}{4}\right) = \tfrac{9}{4}$$

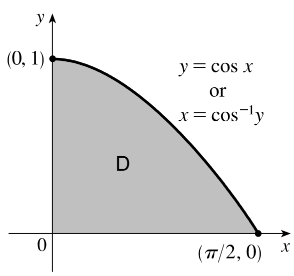
20.



As a type I region, $D = \{(x,y) \mid 0 \le x \le 4, x \le y \le 4\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_x^4 y^2 e^{xy}\,dy\,dx$. As a type II region, $D = \{(x,y) \mid 0 \le y \le 4, 0 \le x \le y\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy$. Evaluating $\int y^2 e^{xy}\,dy$ requires integration by parts whereas $\int y^2 e^{xy}\,dx$ does not, so the iterated integral corresponding to D as a type II region appears easier to evaluate.

$$\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4 \left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4 \left(ye^{y^2} - y\right) dy$$
$$= \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}y^2\right]_0^4 = \left(\tfrac{1}{2}e^{16} - 8\right) - \left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}e^{16} - \tfrac{17}{2}$$

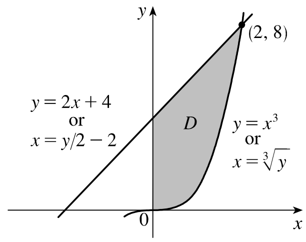
21.



If we describe D as a type I region, $D = \{(x,y) \mid 0 \le x \le \pi/2, 0 \le y \le \cos x\}$ and $\iint_D \sin^2 x\,dA = \int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\,dy\,dx$. As a type II region, $D = \{(x,y) \mid 0 \le x \le \cos^{-1} y, 0 \le y \le 1\}$ and $\iint_D \sin^2 x\,dA = \int_0^1 \int_0^{\cos^{-1} y} \sin^2 x\,dx\,dy$. Evaluating $\int_0^{\cos^{-1} y} \sin^2 x\,dx$ will result in a very difficult integral. Therefore, we evaluate the iterated integral that describes D as a type I region because integrating $\sin^2 x$ with respect to y is easy.

$$\int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\,dy\,dx = \int_0^{\pi/2} \sin^2 x \left[y\right]_{y=0}^{y=\cos x} dx = \int_0^{\pi/2} \cos x \sin^2 x\,dx$$
$$= \int_0^1 u^2\,du \quad \begin{bmatrix} u = \sin x, \\ du = \cos x\,dx \end{bmatrix} \quad = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

22.



By inspection, the curves $y = 2x + 4$ and $y = x^3$ intersect when $x^3 = 2x + 4 \quad \Leftrightarrow$ $x = 2$, so the point of intersection is $(2, 8)$. If we describe D as a type 1 region, $D = \{(x, y) \mid 0 \le x \le 2, x^3 \le y \le 2x + 4\}$ and the integral is

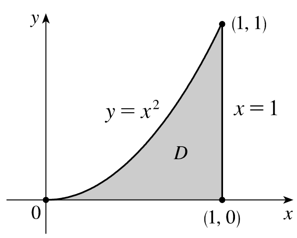
$$\iint_D 6x^2\,dA = \int_0^2 \int_{x^3}^{2x+4} 6x^2\,dy\,dx.$$

If we describe D as a type II region, the right boundary curve is $x = \sqrt[3]{y}$, but the left boundary curve consists of two parts, $x = 0$ for $0 \le y \le 4$ and $x = y/2 - 2$ for $4 \le y \le 8$.

In either case, the resulting iterated integrals are not difficult to evaluate, but the region D is more simply described as a type I region, giving one iterated integral rather than a sum of two, so we evaluate that integral:

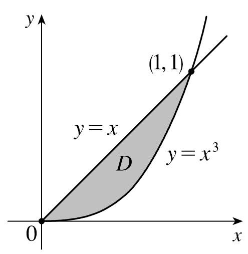
$$\int_0^2 \int_{x^3}^{2x+4} 6x^2\,dy\,dx = \int_0^2 \Big[6x^2 y\Big]_{y=x^3}^{y=2x+4} dx = \int_0^2 [6x^2(2x + 4 - x^3)]\,dx = \int_0^2 (12x^3 + 24x^2 - 6x^5)\,dx$$
$$= \big[3x^4 + 8x^3 - x^6\big]_0^2 = 48 + 64 - 64 = 48$$

23.



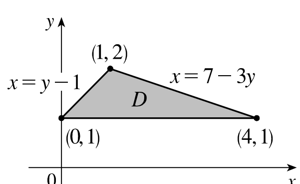
$$\int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \big[x\sin y\big]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx$$
$$= -\tfrac{1}{2}\cos x^2\Big]_0^1 = -\tfrac{1}{2}(\cos 1 - \cos 0) = \tfrac{1}{2}(1 - \cos 1)$$

24.



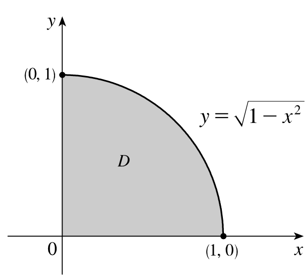
$$\iint_D (x^2 + 2y)\,dA = \int_0^1 \int_{x^3}^{x} (x^2 + 2y)\,dy\,dx = \int_0^1 \big[x^2 y + y^2\big]_{y=x^3}^{y=x} dx$$
$$= \int_0^1 (x^3 + x^2 - x^5 - x^6)\,dx = \big[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7\big]_0^1$$
$$= \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{7} = \tfrac{23}{84}$$

25.



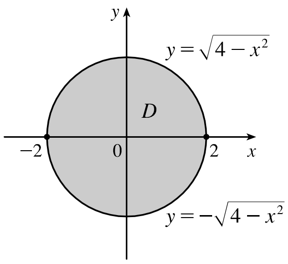
$$\iint_D y^2\,dA = \int_1^2 \int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \big[xy^2\big]_{x=y-1}^{x=7-3y} dy$$
$$= \int_1^2 [(7 - 3y) - (y - 1)]\,y^2\,dy = \int_1^2 (8y^2 - 4y^3)\,dy$$
$$= \big[\tfrac{8}{3}y^3 - y^4\big]_1^2 = \tfrac{64}{3} - 16 - \tfrac{8}{3} + 1 = \tfrac{11}{3}$$

26.



$$\iint_D xy\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$
$$= \int_0^1 \big[\tfrac{1}{2}xy^2\big]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1 - x^2)\,dx$$
$$= \tfrac{1}{2}\int_0^1 (x - x^3)\,dx = \tfrac{1}{2}\big[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\big]_0^1$$
$$= \tfrac{1}{2}\big(\tfrac{1}{2} - \tfrac{1}{4} - 0\big) = \tfrac{1}{8}$$

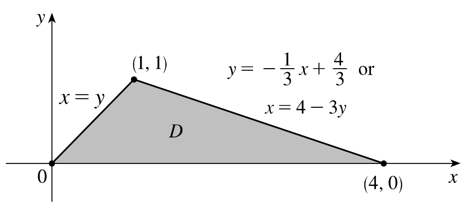
27.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx$$
$$= \int_{-2}^{2}\left[2xy-\tfrac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$
$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\tfrac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$
$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

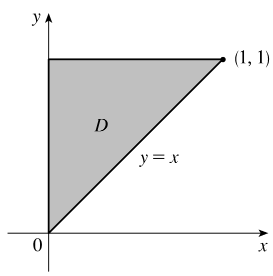
[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx=0$.]

28.



$$\iint_D y\,dA = \int_0^1\int_y^{4-3y} y\,dx\,dy$$
$$= \int_0^1 [xy]_{x=y}^{x=4-3y}\,dy = \int_0^1(4y-3y^2-y^2)\,dy$$
$$= \int_0^1(4y-4y^2)\,dy = \left[2y^2-\tfrac{4}{3}y^3\right]_0^1 = 2-\tfrac{4}{3}-0=\tfrac{2}{3}$$

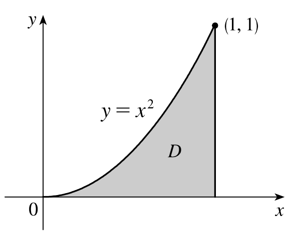
29. (a)



As a Type I region, $D=\{(x,y)\mid 0\le x\le 1,\,x\le y\le 1\}$ and the volume V of the solid that lies under the surface and above D is given by $V=\iint_D(1+xy)\,dA=\int_0^1\int_x^1(1+xy)\,dy\,dx$. As a Type II region, $D=\{(x,y)\mid 0\le y\le 1,\,0\le x\le y\}$ and $V=\int_0^1\int_0^y(1+xy)\,dx\,dy$. Evaluate either integral in part (b).

(b) $$\int_0^1\int_0^y(1+xy)\,dx\,dy=\int_0^1\left[x+y\,\frac{x^2}{2}\right]_{x=0}^{x=y}dy=\int_0^1\left[\left(y+\frac{y^3}{2}\right)-0\right]dy=\left[\frac{y^2}{2}+\frac{y^4}{8}\right]_0^1=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$$

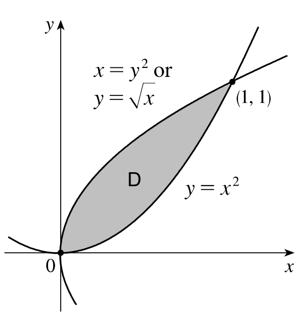
30. (a)



As a Type I region, $D=\{(x,y)\mid 0\le x\le 1,\,0\le y\le x^2\}$ and the volume V of the solid that lies under the surface and above D is given by $V=\iint_D(x^2+y^2)\,dA=\int_0^1\int_0^{x^2}(x^2+y^2)\,dy\,dx$. As a Type II region, $D=\{(x,y)\mid 0\le y\le 1,\,\sqrt{y}\le x\le 1\}$ and $V=\int_0^1\int_{\sqrt{y}}^1(x^2+y^2)\,dx\,dy$.

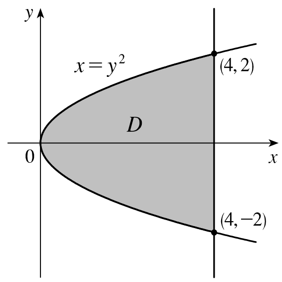
(b) $$\int_0^1\int_0^{x^2}(x^2+y^2)\,dy\,dx=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=x^2}dx=\int_0^1\left[x^4+\frac{x^6}{3}\right]dx=\left[\frac{x^5}{5}+\frac{x^7}{21}\right]_0^1=\frac{1}{5}+\frac{1}{21}=\frac{26}{105}$$

31.



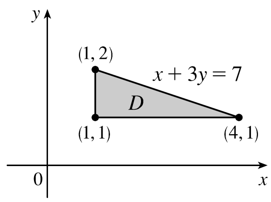
$$V=\int_0^1\int_{x^2}^{\sqrt{x}}(3x+2y)\,dy\,dx=\int_0^1\left[3xy+y^2\right]_{y=x^2}^{y=\sqrt{x}}dx$$
$$=\int_0^1\left[(3x\sqrt{x}+x)-(3x^3+x^4)\right]dx=\int_0^1(3x^{3/2}+x-3x^3-x^4)\,dx$$
$$=\left[3\cdot\tfrac{2}{5}x^{5/2}+\tfrac{1}{2}x^2-\tfrac{3}{4}x^4-\tfrac{1}{5}x^5\right]_0^1=\tfrac{6}{5}+\tfrac{1}{2}-\tfrac{3}{4}-\tfrac{1}{5}-0=\tfrac{3}{4}$$

32.



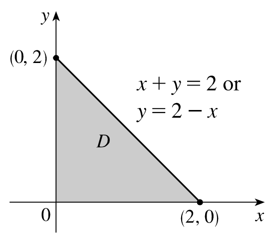
$$V = \int_{-2}^{2}\int_{y^2}^{4}(1+x^2y^2)\,dx\,dy$$
$$= \int_{-2}^{2}\left[x+\tfrac{1}{3}x^3y^2\right]_{x=y^2}^{x=4}\,dy = \int_{-2}^{2}(4+\tfrac{61}{3}y^2-\tfrac{1}{3}y^8)\,dy$$
$$= \left[4y+\tfrac{61}{9}y^3-\tfrac{1}{27}y^9\right]_{-2}^{2} = 8+\tfrac{488}{9}-\tfrac{512}{27}+8+\tfrac{488}{9}-\tfrac{512}{27} = \tfrac{2336}{27}$$

33.



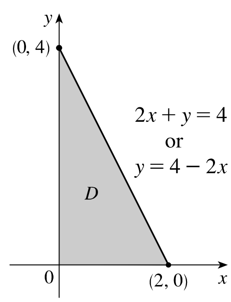
$$V = \int_1^2\int_1^{7-3y} xy\,dx\,dy = \int_1^2\left[\tfrac{1}{2}x^2y\right]_{x=1}^{x=7-3y}\,dy$$
$$= \tfrac{1}{2}\int_1^2 y\left[(7-3y)^2-1\right]dy = \tfrac{1}{2}\int_1^2(48y-42y^2+9y^3)\,dy$$
$$= \tfrac{1}{2}\left[24y^2-14y^3+\tfrac{9}{4}y^4\right]_1^2 = \tfrac{31}{8}$$

34.



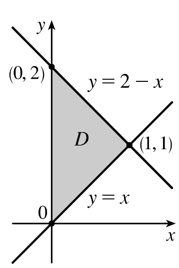
$$V = \int_0^2\int_0^{2-x}(x^2+y^2+1)\,dy\,dx = \int_0^2\left[x^2y+\tfrac{1}{3}y^3+y\right]_{y=0}^{y=2-x}\,dx$$
$$= \int_0^2\left[x^2(2-x)+\tfrac{1}{3}(2-x)^3+(2-x)-0\right]dx$$
$$= \int_0^2\left(-\tfrac{4}{3}x^3+4x^2-5x+\tfrac{14}{3}\right)dx = \left[-\tfrac{1}{3}x^4+\tfrac{4}{3}x^3-\tfrac{5}{2}x^2+\tfrac{14}{3}x\right]_0^2$$
$$= -\tfrac{16}{3}+\tfrac{32}{3}-10+\tfrac{28}{3}-0 = \tfrac{14}{3}$$

35.



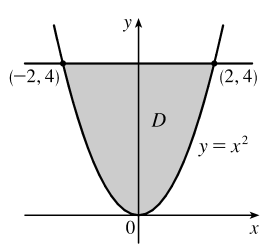
$$V = \int_0^2\int_0^{4-2x}(4-2x-y)\,dy\,dx = \int_0^2\left[4y-2xy-\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x}\,dx$$
$$= \int_0^2\left[4(4-2x)-2x(4-2x)-\tfrac{1}{2}(4-2x)^2-0\right]dx$$
$$= \int_0^2\left(2x^2-8x+8\right)dx = \left[\tfrac{2}{3}x^3-4x^2+8x\right]_0^2 = \tfrac{16}{3}-16+16-0 = \tfrac{16}{3}$$

36.



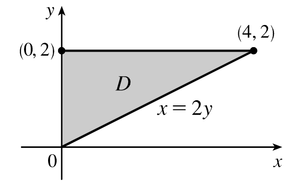
$$V = \int_0^1\int_x^{2-x} x\,dy\,dx$$
$$= \int_0^1\left[xy\right]_{y=x}^{y=2-x}\,dx = \int_0^1(2x-2x^2)\,dx$$
$$= \left[x^2-\tfrac{2}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

37.



$$V = \int_{-2}^{2}\int_{x^2}^{4} x^2\,dy\,dx$$
$$= \int_{-2}^{2}\left[x^2y\right]_{y=x^2}^{y=4}\,dx = \int_{-2}^{2}(4x^2-x^4)\,dx$$
$$= \left[\tfrac{4}{3}x^3-\tfrac{1}{5}x^5\right]_{-2}^{2} = \tfrac{32}{3}-\tfrac{32}{5}+\tfrac{32}{3}-\tfrac{32}{5} = \tfrac{128}{15}$$

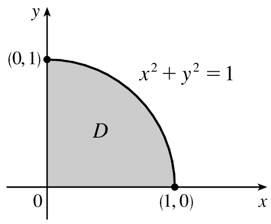
38.



$$V = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dx\,dy = \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\,dy = \left[-\tfrac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

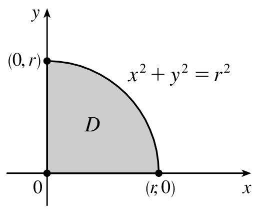
39.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

40.



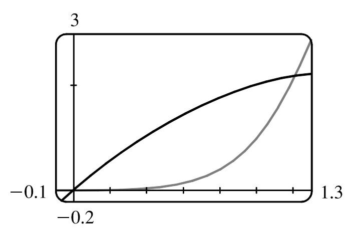
By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r \left(r^2-y^2\right) dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

41.



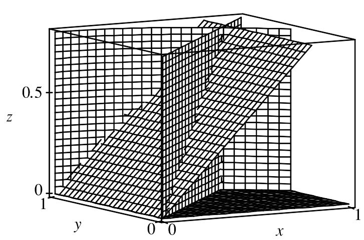
From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

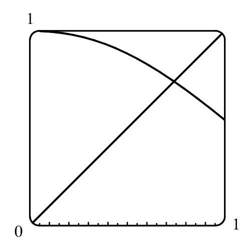
$$\iint_D x\,dA \approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\left[xy\right]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213}\left(3x^2 - x^3 - x^5\right) dx = \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213}$$

$$\approx 0.713$$

42.





The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391}\int_x^{\cos x} x\,dy\,dx = \int_0^{0.7391}\left[xy\right]_{y=x}^{y=\cos x} dx$$

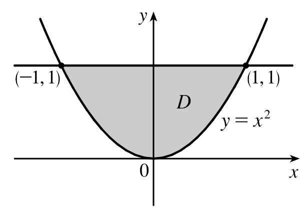
$$= \int_0^{0.7391}\left(x\cos x - x^2\right) dx = \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391}\int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2}\int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

43. The region of integration is bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$ which intersect at $(\pm 1, 0)$ with $1 - x^2 \ge x^2 - 1$ on $[-1, 1]$. Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

$$\begin{aligned}
V &= \int_{-1}^{1}\int_{x^2-1}^{1-x^2}(2x+2y+10)\,dy\,dx - \int_{-1}^{1}\int_{x^2-1}^{1-x^2}(2-x-y)\,dy\,dx \\
&= \int_{-1}^{1}\int_{x^2-1}^{1-x^2}(2x+2y+10-(2-x-y))\,dy\,dx \\
&= \int_{-1}^{1}\int_{x^2-1}^{1-x^2}(3x+3y+8)\,dy\,dx = \int_{-1}^{1}\Big[3xy + \tfrac{3}{2}y^2 + 8y\Big]_{y=x^2-1}^{y=1-x^2}\,dx \\
&= \int_{-1}^{1}\big[3x(1-x^2) + \tfrac{3}{2}(1-x^2)^2 + 8(1-x^2) - 3x(x^2-1) - \tfrac{3}{2}(x^2-1)^2 - 8(x^2-1)\big]\,dx \\
&= \int_{-1}^{1}(-6x^3 - 16x^2 + 6x + 16)\,dx = \big[-\tfrac{3}{2}x^4 - \tfrac{16}{3}x^3 + 3x^2 + 16x\big]_{-1}^{1} \\
&= -\tfrac{3}{2} - \tfrac{16}{3} + 3 + 16 + \tfrac{3}{2} - \tfrac{16}{3} - 3 + 16 = \tfrac{64}{3}
\end{aligned}$$

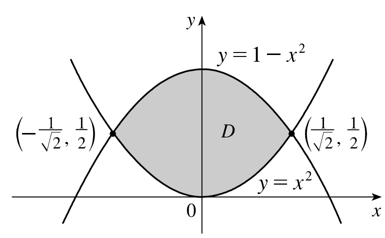
44. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \ge 3y$ for $0 \le y \le 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$\begin{aligned}
V &= \int_{-1}^{1}\int_{x^2}^{1}(2+y)\,dy\,dx - \int_{-1}^{1}\int_{x^2}^{1}(3y)\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}(2+y-3y)\,dy\,dx \\
&= \int_{-1}^{1}\int_{x^2}^{1}(2-2y)\,dy\,dx = \int_{-1}^{1}\Big[2y - y^2\Big]_{y=x^2}^{y=1}\,dx \\
&= \int_{-1}^{1}(1 - 2x^2 + x^4)\,dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\big]_{-1}^{1} = \tfrac{16}{15}
\end{aligned}$$

45. The region of integration is bounded by the curves $y = x^2$ and $y = 1 - x^2$ which intersect at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

The solid lies under the graph of $z = 3$ and above the graph of $z = y$, so its volume is



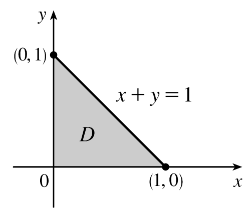
$$\begin{aligned}
V &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x^2}^{1-x^2}3\,dy\,dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x^2}^{1-x^2}y\,dy\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x^2}^{1-x^2}(3-y)\,dy\,dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\Big[3y - \tfrac{1}{2}y^2\Big]_{y=x^2}^{y=1-x^2}\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\big[\big(3(1-x^2) - \tfrac{1}{2}(1-x^2)^2\big) - \big(3x^2 - \tfrac{1}{2}(x^2)^2\big)\big]\,dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\tfrac{5}{2} - 5x^2\right)dx = \left[\tfrac{5}{2}x - \tfrac{5}{3}x^3\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = \left(\tfrac{5}{2\sqrt{2}} - \tfrac{5}{6\sqrt{2}}\right) - \left(-\tfrac{5}{2\sqrt{2}} + \tfrac{5}{6\sqrt{2}}\right) \\
&= \tfrac{10}{3\sqrt{2}} \text{ or } \tfrac{5\sqrt{2}}{3}
\end{aligned}$$

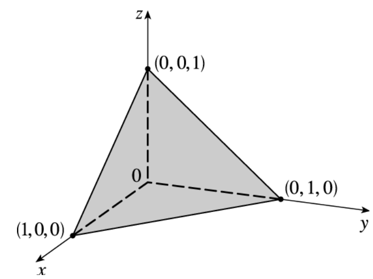
46. The region of integration is the portion of the first quadrant bounded by the axes and the curve $y = \sqrt{4-x^2}$. The solid lies under the graph of $z = x + y$ and above the graph of $z = xy$, so its volume is

$$\begin{aligned} V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx - \int_0^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y-xy)\,dy\,dx \\ &= \int_0^2 \left[xy + \tfrac{1}{2}y^2 - \tfrac{1}{2}xy^2\right]_{y=0}^{y=\sqrt{4-x^2}} dx = \int_0^2 \left[x\sqrt{4-x^2} + \tfrac{1}{2}(4-x^2) - \tfrac{1}{2}x(4-x^2) - 0\right] dx \\ &= \int_0^2 \left(x\sqrt{4-x^2} + 2 - \tfrac{1}{2}x^2 - 2x + \tfrac{1}{2}x^3\right) dx = \left[-\tfrac{1}{3}(4-x^2)^{3/2} + 2x - \tfrac{1}{6}x^3 - x^2 + \tfrac{1}{8}x^4\right]_0^2 \\ &= \left(4 - \tfrac{4}{3} - 4 + 2\right) - \left(-\tfrac{1}{3}\cdot 4^{3/2}\right) = \tfrac{2}{3} + \tfrac{8}{3} = \tfrac{10}{3} \end{aligned}$$

47. $\int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$.

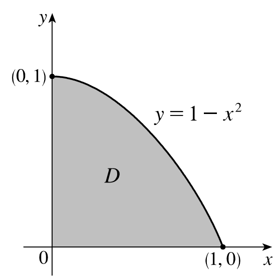
The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.

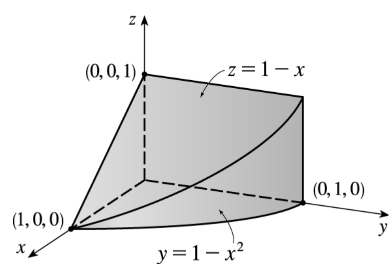




48. $\displaystyle\int_0^1 \int_0^{1-x^2} (1-x)\,dy\,dx$.

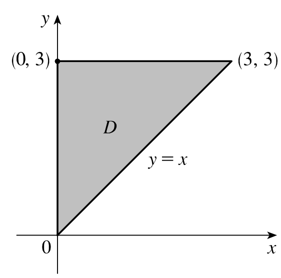
The solid lies below the plane $z = 1 - x$ and above the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x^2\}$ in the xy-plane.

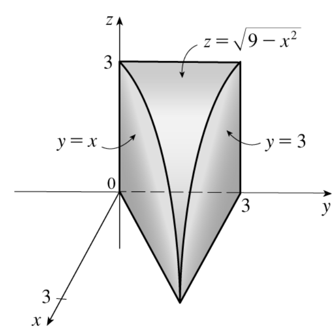




49. $\int_0^3 \int_0^y \sqrt{9-x^2}\,dx\,dy$.

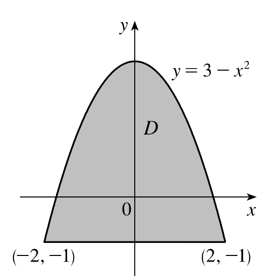
The solid lies under the top half of the cylinder $x^2 + z^2 = 9$; that is, $z = \sqrt{9-x^2}$, and above the region $D = \{(x,y) \mid 0 \le x \le y, 0 \le y \le 3\}$.

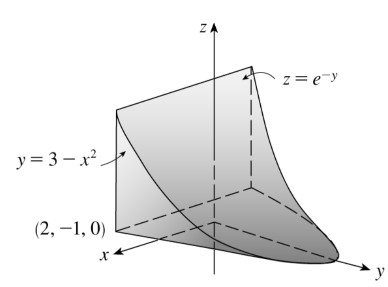




50. $\displaystyle\int_{-2}^2 \int_{-1}^{3-x^2} e^{-y}\,dy\,dx$.

The solid lies below the surface $z = e^{-y}$ and above the region $D = \{(x,y) \mid -2 \le x \le 2, -1 \le y \le 3 - x^2\}$.





51. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume of the solid is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

52. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. So the volume is given by

$$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2)\right] dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

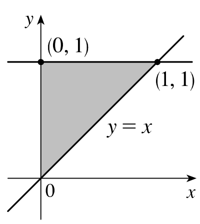
53. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\displaystyle\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

54. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1 - x^2} \le y \le 1 + \sqrt{1 - x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)]\, dy\, dx = \frac{\pi}{2}$$

55.

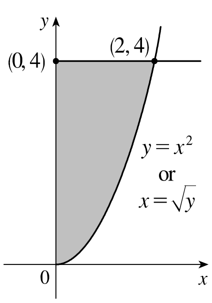


Because the region of integration is

$D = \{(x, y) \mid 0 \le x \le y, 0 \le y \le 1\} = \{(x, y) \mid x \le y \le 1, 0 \le x \le 1\}$

we have $\int_0^1 \int_0^y f(x, y)\, dx\, dy = \iint_D f(x, y)\, dA = \int_0^1 \int_x^1 f(x, y)\, dy\, dx$.

56.

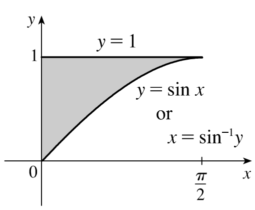


Because the region of integration is

$$D = \{(x, y) \mid x^2 \le y \le 4, 0 \le x \le 2\}$$
$$= \{(x, y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$$

we have $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^4 \int_0^{\sqrt{y}} f(x, y)\, dx\, dy$.

57.



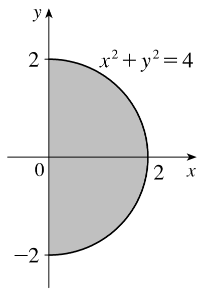
Because the region of integration is

$$D = \{(x, y) \mid 0 \le x \le \pi/2,\ \sin x \le y \le 1\}$$
$$= \{(x, y) \mid 0 \le x \le \sin^{-1} y, 0 \le y \le 1\}$$

we have

$\int_0^{\pi/2} \int_{\sin x}^1 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^1 \int_0^{\sin^{-1} y} f(x, y)\, dx\, dy$

58.



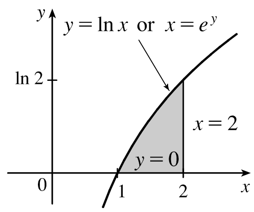
Because the region of integration is

$$D = \left\{(x, y) \mid 0 \le x \le \sqrt{4-y^2}, -2 \le y \le 2\right\}$$
$$= \left\{(x, y) \mid -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, 0 \le x \le 2\right\}$$

we have

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x, y)\,dx\,dy = \iint_D f(x, y)\,dA = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy\,dx.$$

59.



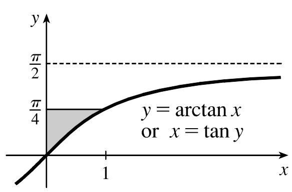
Because the region of integration is

$$D = \{(x, y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x, y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x, y)\,dy\,dx = \iint_D f(x, y)\,dA = \int_0^{\ln 2} \int_{e^y}^{2} f(x, y)\,dx\,dy$$

60.



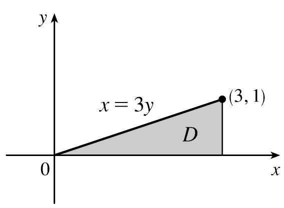
Because the region of integration is

$$D = \left\{(x, y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\right\}$$
$$= \left\{(x, y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\right\}$$

we have

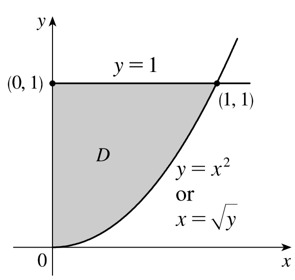
$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\,dy\,dx = \iint_D f(x, y)\,dA = \int_0^{\pi/4} \int_0^{\tan y} f(x, y)\,dx\,dy$$

61.



$$\int_0^1 \int_{3y}^{3} e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx$$
$$= \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\,dx = \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6}$$

62.

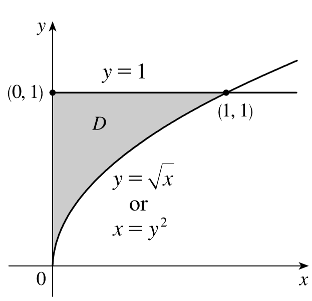


$$\int_0^1 \int_{x^2}^{1} \sqrt{y}\,\sin y\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,\sin y\,dx\,dy = \int_0^1 \sqrt{y}\,\sin y\,[x]_{x=0}^{x=\sqrt{y}}\,dy$$
$$= \int_0^1 \left(\sqrt{y}\,\sin y\right)\left(\sqrt{y} - 0\right) dy = \int_0^1 y\,\sin y\,dy$$
$$= -y\cos y\big]_0^1 + \int_0^1 \cos y\,dy$$

[by integrating by parts with $u = y$, $dv = \sin y\,dy$]

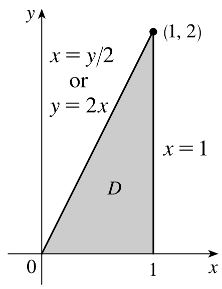
$$= [-y\cos y + \sin y]_0^1 = -\cos 1 + \sin 1 - 0 = \sin 1 - \cos 1$$

63.



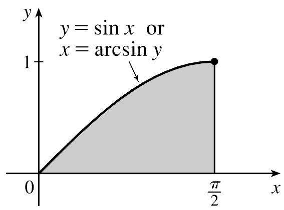
$$\int_0^1 \int_{\sqrt{x}}^{1} \sqrt{y^3+1}\,dy\,dx = \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \sqrt{y^3+1}\,[x]_{x=0}^{x=y^2}\,dy$$
$$= \int_0^1 y^2 \sqrt{y^3+1}\,dy = \tfrac{2}{9}\left(y^3+1\right)^{3/2}\Big]_0^1$$
$$= \tfrac{2}{9}\left(2^{3/2} - 1^{3/2}\right) = \tfrac{2}{9}\left(2\sqrt{2} - 1\right)$$

64.



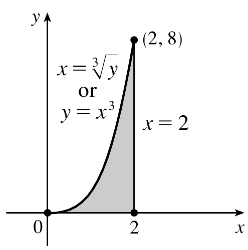
$$\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$
$$= \int_0^1 \cos(x^3-1)\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x} dx$$
$$= \int_0^1 2x^2\cos(x^3-1)\,dx = \tfrac{2}{3}\sin(x^3-1)\Big]_0^1$$
$$= \tfrac{2}{3}\left[0-\sin(-1)\right] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1$$

65.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,\big[\,y\,\big]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \left[\begin{array}{c}\text{Let } u=\cos x,\, du=-\sin x\,dx,\\ dx=du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

66.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\Big[\,y\,\Big]_{y=0}^{y=x^3} dx = \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

67. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0,\, x+1 \le y \le 1\}$

$\cup\,\{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \qquad [\text{by symmetry of the regions and because } f(x,y)=x^2 \ge 0]$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

68. $D = \left\{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y-y^3\right\} \cup \left\{(x,y) \mid 0 \le y \le 1,\, \sqrt{y}-1 \le x \le y-y^3\right\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0\int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1\int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 \left[xy\right]_{x=-1}^{x=y-y^3} dy + \int_0^1 \left[xy\right]_{x=\sqrt{y}-1}^{x=y-y^3} dy$$
$$= \int_{-1}^0 (y^2-y^4+y)\,dy + \int_0^1 (y^2-y^4-y^{3/2}+y)\,dy$$
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$
$$= \left(0-\tfrac{11}{30}\right) + \left(\tfrac{7}{30}-0\right) = -\tfrac{2}{15}$$

69. Since $x^2 + y^2 \le 1$ on S, we must have $0 \le x^2 \le 1$ and $0 \le y^2 \le 1$, so $0 \le x^2y^2 \le 1 \quad \Rightarrow$

$3 \le 4 - x^2y^2 \le 4 \quad \Rightarrow \quad \sqrt{3} \le \sqrt{4 - x^2y^2} \le 2$. Here we have $A(S) = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$, so by Property 10,

$\sqrt{3}A(S) \le \iint_S \sqrt{4 - x^2y^2}\, dA \le 2A(S) \quad \Rightarrow \quad \frac{\sqrt{3}}{2}\pi \le \iint_S \sqrt{4 - x^2y^2}\, dA \le \pi$ or we can say

$2.720 < \iint_S \sqrt{4 - x^2y^2}\, dA < 3.142$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

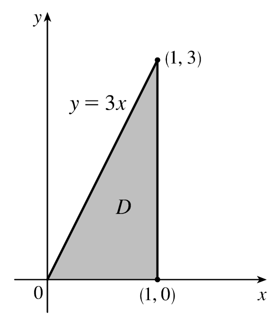
70. T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ so $A(T) = \frac{1}{2}(1)(2) = 1$. We have $0 \le \sin^4(x+y) \le 1$ for all x, y, and Property 10 gives $0 \cdot A(T) \le \iint_T \sin^4(x+y)dA \le 1 \cdot A(T) \quad \Rightarrow \quad 0 \le \iint_T \sin^4(x+y)dA \le 1$.

71. The average value of a function f of two variables defined on a rectangle R was defined in Section 15.1 as $f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x,y)dA$.

Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x,y)dA = \frac{1}{3/2} \int_0^1 \int_0^{3x} xy\, dy\, dx$$

$$= \frac{2}{3} \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \frac{1}{3} \int_0^1 9x^3\, dx = \left.\tfrac{3}{4}x^4\right]_0^1 = \frac{3}{4}$$



72. Here $D = \left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\right\}$, so

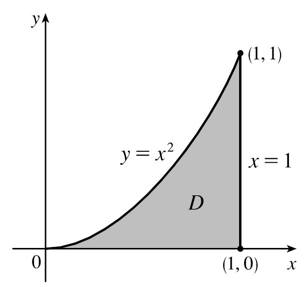
$A(D) = \int_0^1 x^2\, dx = \left.\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$ and

$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x,y)dA = \frac{1}{1/3} \int_0^1 \int_0^{x^2} x \sin y\, dy\, dx$$

$$= 3 \int_0^1 \left[-x \cos y\right]_{y=0}^{y=x^2} dx$$

$$= 3 \int_0^1 \left[x - x\cos(x^2)\right] dx = 3\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}\sin(x^2)\right]_0^1$$

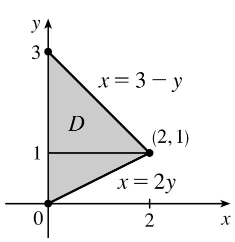
$$= 3\left(\tfrac{1}{2} - \tfrac{1}{2}\sin 1 - 0\right) = \tfrac{3}{2}(1 - \sin 1)$$



73. Since $m \le f(x,y) \le M$, $\iint_D m\, dA \le \iint_D f(x,y)\, dA \le \iint_D M\, dA$ by (7) $\Rightarrow$

$m \iint_D 1\, dA \le \iint_D f(x,y)\, dA \le M \iint_D 1\, dA$ by (6) $\Rightarrow \quad m \cdot A(D) \le \iint_D f(x,y)\, dA \le M \cdot A(D)$ by (9).

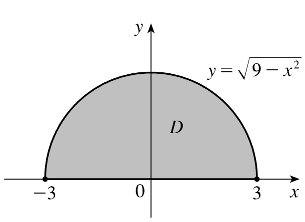
74.



$$\iint_D f(x,y)\, dA = \int_0^1 \int_0^{2y} f(x,y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x,y)\, dx\, dy$$

$$= \int_0^2 \int_{x/2}^{3-x} f(x,y)\, dy\, dx$$

75.

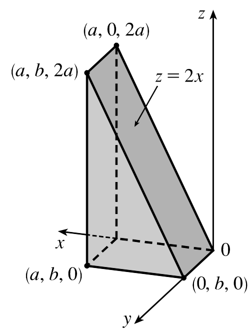


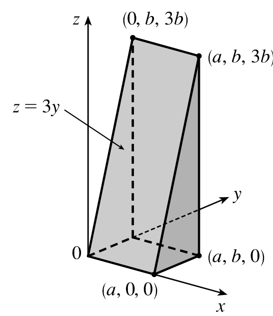
First we can write $\iint_D (x+2)\, dA = \iint_D x\, dA + \iint_D 2\, dA$. But $f(x,y) = x$ is an odd function with respect to x [that is, $f(-x,y) = -f(x,y)$] and D is symmetric with respect to x. Consequently, the volume above D and below the graph of f is the same as the volume below D and above the graph of f, so

$\iint_D x\,dA = 0$. Also, $\iint_D 2\,dA = 2 \cdot A(D) = 2 \cdot \frac{1}{2}\pi(3)^2 = 9\pi$ since D is a half disk of radius 3. Thus $\iint_D (x+2)\,dA = 0 + 9\pi = 9\pi$.

76. The graph of $f(x,y) = \sqrt{R^2 - x^2 - y^2}$ is the top half of the sphere $x^2 + y^2 + z^2 = R^2$, centered at the origin with radius R, and D is the disk in the xy-plane also centered at the origin with radius R. Thus $\iint_D \sqrt{R^2 - x^2 - y^2}\,dA$ represents the volume of a half ball of radius R which is $\frac{1}{2} \cdot \frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$.

77. We can write $\iint_D (2x+3y)\,dA = \iint_D 2x\,dA + \iint_D 3y\,dA$. $\iint_D 2x\,dA$ represents the volume of the solid lying under the plane $z = 2x$ and above the rectangle D. This solid region is a triangular cylinder with length b and whose cross-section is a triangle with width a and height $2a$. (See the first figure.)

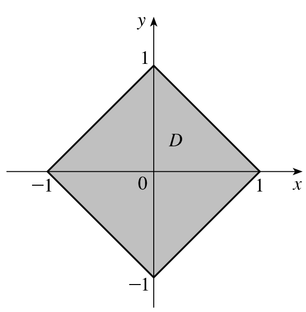




Thus its volume is $\frac{1}{2} \cdot a \cdot 2a \cdot b = a^2 b$. Similarly, $\iint_D 3y\,dA$ represents the volume of a triangular cylinder with length a, triangular cross-section with width b and height $3b$, and volume $\frac{1}{2} \cdot b \cdot 3b \cdot a = \frac{3}{2}ab^2$. (See the second figure.) Thus

$$\iint_D (2x+3y)\,dA = a^2 b + \tfrac{3}{2}ab^2$$

78.



In the first quadrant, x and y are positive and the boundary of D is $x + y = 1$. But D is symmetric with respect to both axes because of the absolute values, so the region of integration is the square shown at the left. To evaluate the double integral, we first write $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = \iint_D 2\,dA + \iint_D x^2y^3\,dA - \iint_D y^2 \sin x\,dA$. Now $f(x,y) = x^2y^3$ is odd with respect to y [that is, $f(x,-y) = -f(x,y)$] and D is symmetric with respect to y, so $\iint_D x^2y^3\,dA = 0$.

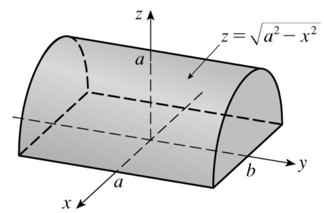
Similarly, $g(x,y) = y^2 \sin x$ is odd with respect to x [since $g(-x,y) = -g(x,y)$] and D is symmetric with respect to x, so $\iint_D y^2 \sin x\,dA = 0$. D is a square with side length $\sqrt{2}$, so $\iint_D 2\,dA = 2 \cdot A(D) = 2\left(\sqrt{2}\right)^2 = 4$, and $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = 4 + 0 + 0 = 4$.

79. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = \iint_D ax^3\,dA + \iint_D by^3\,dA + \iint_D \sqrt{a^2 - x^2}\,dA$. Now ax^3 is odd with respect to x and by^3 is odd with respect to y, and the region of integration is symmetric with respect to both x and y, so $\iint_D ax^3\,dA = \iint_D by^3\,dA = 0$.

[continued]

$\iint_D \sqrt{a^2 - x^2}\, dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D, namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is $\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2}\pi a^2(2b) = \pi a^2 b$. Thus

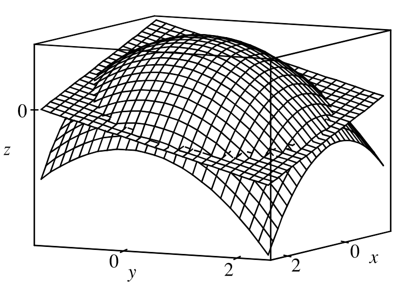
$\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = 0 + 0 + \pi a^2 b = \pi a^2 b.$



80. By the Extreme Value Theorem (14.7.8), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 15.2.10, $mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D)$. Dividing through by the positive number $A(D)$, we get $m \leq \dfrac{1}{A(D)} \displaystyle\iint_D f(x,y)\, dA \leq M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \dfrac{1}{A(D)} \displaystyle\iint_D f(x,y)\, dA$ or equivalently $\iint_D f(x,y)\, dA = f(x_0, y_0)\, A(D)$.

81. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for double integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2} \displaystyle\iint_{D_r} f(x,y)\, dA$. But $\lim\limits_{r \to 0^+} (x_r, y_r) = (a, b)$, so $\lim\limits_{r \to 0^+} \dfrac{1}{\pi r^2} \displaystyle\iint_{D_r} f(x,y)\, dA = \lim\limits_{r \to 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f.

82. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4 - x^2 - y^2) - (1 - x - y)\right] dy\, dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

1. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 4,\ 0 \leq \theta \leq 3\pi/2\}$. Thus, $\iint_R f(x,y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -1 \leq x \leq 1,\ -x \leq y \leq 1\}$. Thus, $\iint_R f(x,y)\, dA = \int_{-1}^{1} \int_{-x}^{1} f(x,y)\, dy\, dx$.

3. The region R is more easily described with polar coordinates: $R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}$.

Thus, $\iint_R f(x,y)\,dA = \int_0^\pi \int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 3, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$.

Thus, $\iint_R f(x,y)\,dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

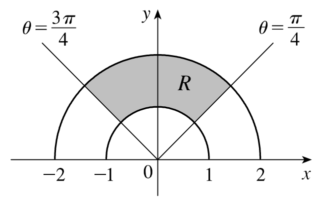
5. The region R is more easily described with rectangular coordinates: $R = \{(x,y) \mid 2y-2 \le x \le -2y+2, 0 \le y \le 1\}$.

Thus, $\iint_R f(x,y)\,dA = \int_0^1 \int_{2y-2}^{-2y+2} f(x,y)\,dx\,dy$.

6. The region R is more easily described with polar coordinates: $R = \{(r,\theta) \mid 8 \le r \le 10, 0 \le \theta \le 2\pi\}$.

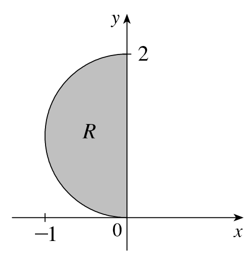
Thus, $\iint_R f(x,y)\,dA = \int_0^{2\pi} \int_8^{10} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$= \left[\,\theta\,\right]_{\pi/4}^{3\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) \cdot \tfrac{1}{2}(4-1) = \tfrac{\pi}{2} \cdot \tfrac{3}{2} = \tfrac{3\pi}{4}$$

8. The integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 2\sin\theta, \pi/2 \le \theta \le \pi\}$. Since $r = 2\sin\theta \;\Rightarrow\; r^2 = 2r\sin\theta \;\Leftrightarrow\; x^2+y^2 = 2y \;\Leftrightarrow\; x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0,1)$.



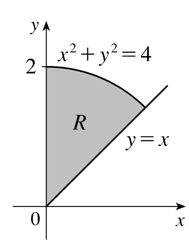
$$\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$
$$= \int_{\pi/2}^{\pi} 2 \cdot \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$
$$= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}$$

9. The half-disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y\,dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 (r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^\pi \cos^2\theta \sin\theta\,d\theta\right)\left(\int_0^5 r^4\,dr\right)$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^\pi \left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1-1)\cdot 625 = \tfrac{1250}{3}$$

10. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le r \le 2, \pi/4 \le \theta \le \pi/2\}$. Thus

$$\iint_R (2x-y)\,dA = \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\,d\theta \int_0^2 r^2\,dr$$
$$= \left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= \left(2 + 0 - \sqrt{2} - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right) = \tfrac{16}{3} - 4\sqrt{2}$$



11. $\iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_1^3 r\sin(r^2)\,dr = \big[\theta\big]_0^{\pi/2}\left[-\frac{1}{2}\cos(r^2)\right]_1^3$

$= \left(\frac{\pi}{2}\right)\left[-\frac{1}{2}(\cos 9 - \cos 1)\right] = \frac{\pi}{4}(\cos 1 - \cos 9)$

12. $\displaystyle\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_a^b r\,dr = \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\int_a^b r\,dr$

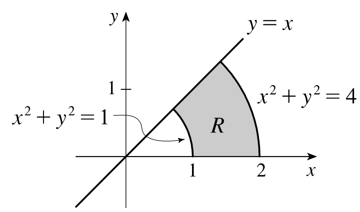
$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{2}r^2\right]_a^b = \frac{1}{2}(2\pi - 0 - 0)\cdot\frac{1}{2}\left(b^2-a^2\right) = \frac{\pi}{2}(b^2-a^2)$

13. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta\int_0^2 re^{-r^2}\,dr$

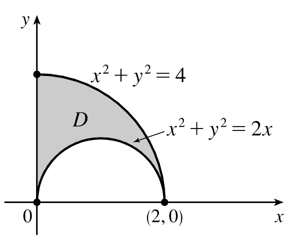
$= \big[\,\theta\,\big]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\frac{1}{2}\right)(e^{-4}-e^0) = \frac{\pi}{2}(1-e^{-4})$

14. $\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos\sqrt{r^2}\;r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = \big[\,\theta\,\big]_0^{2\pi}\,[r\sin r + \cos r]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

15. R is the region shown in the figure, and can be described by $R = \{(r,\theta)\mid 0\le\theta\le\pi/4, 1\le r\le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0\le\theta\le\pi/4$, so the integral becomes $\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2$.



16.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA \;-\; \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3}[\sin\theta]_0^{\pi/2} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

17. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1-\cos\theta$ (see the figure). Here $D = \{(r,\theta)\mid 0\le r\le 1-\cos\theta, 0\le\theta\le\pi/2\}$, so the total area is

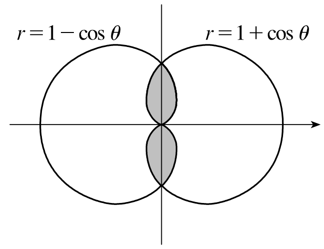
$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta}d\theta$

$= 2\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$

$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right]d\theta$

$= 2\left[\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right) = \frac{3\pi}{2} - 4$

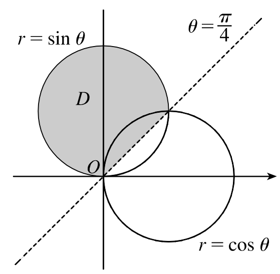


18. The region D is described by $D = \{(r,\theta) \mid 0 \le r \le \sqrt{\theta}, 0 \le \theta \le 2\pi\}$, so the area is

$$A(D) = \int_0^{2\pi} \int_0^{\sqrt{\theta}} r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\theta}} d\theta = \int_0^{2\pi} \frac{\theta}{2}\,d\theta = \left[\frac{\theta^2}{4}\right]_0^{2\pi} = \pi^2.$$

19. By symmetry, the total area is twice the area defined by

$D = \{(r,\theta) \mid 0 \le r \le \sin\theta, \pi/4 \le \theta \le \pi\}$ (see the figure).



The total area is

$$2A(D) = 2\int_{\pi/4}^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta = 2\cdot\tfrac{1}{2}\int_{\pi/4}^{\pi}\left[r^2\right]_{r=0}^{r=\sin\theta} d\theta = \int_{\pi/4}^{\pi}\sin^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi}$$

$$= \tfrac{1}{2}(\pi - 0) - \tfrac{1}{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) = \tfrac{3\pi}{8} + \tfrac{1}{4}$$

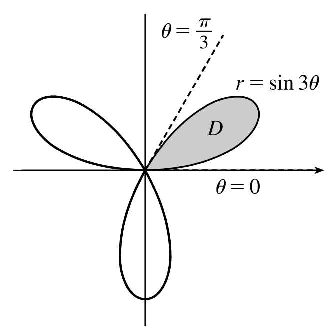
20. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant between the circle $r = 1/\sqrt{2}$ and the curve $r^2 = \cos 2\theta \;\Rightarrow\; r = \sqrt{\cos 2\theta}$. The curves intersect in the first quadrant when $\cos 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \;\Rightarrow$ $\cos 2\theta = \frac{1}{2} \;\Rightarrow\; 2\theta = \frac{\pi}{3} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Thus, $D = \{(r,\theta) \mid 1/\sqrt{2} \le r \le \sqrt{\cos 2\theta}, 0 \le \theta \le \pi/6\}$, so the total area is

$$4A(D) = 4\int_0^{\pi/6}\int_{1/\sqrt{2}}^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\cdot\frac{1}{2}\int_0^{\pi/6}\left[r^2\right]_{r=1/\sqrt{2}}^{r=\sqrt{\cos 2\theta}} d\theta = 2\int_0^{\pi/6}\left[\cos 2\theta - \frac{1}{2}\right] d\theta$$

$$= 2\left[\tfrac{1}{2}\sin 2\theta - \tfrac{\theta}{2}\right]_0^{\pi/6} = \tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}$$

21. One loop is given by the region

$D = \{(r,\theta) \mid 0 \le r \le \sin 3\theta, 0 \le \theta \le \pi/3\}$, so the area is

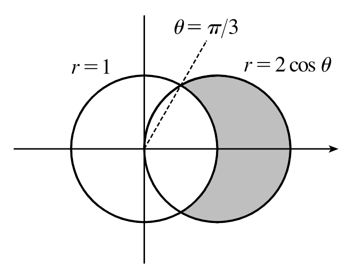


$$\iint_D dA = \int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/3}\left[r^2\right]_{r=0}^{r=\sin 3\theta} d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/3}\tfrac{1}{2}(1-\cos 6\theta)\,d\theta$$

$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \tfrac{\pi}{12}$$

22. In polar coordinates the circle $(x-1)^2 + y^2 = 1 \;\Leftrightarrow\; x^2 + y^2 = 2x$ is $r^2 = 2r\cos\theta \;\Rightarrow\; r = 2\cos\theta$, and the circle $x^2 + y^2 = 1$ is $r = 1$. The curves intersect in the first quadrant when $2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pi/3$, so the portion of the region in the first quadrant is given by $D = \{(r,\theta) \mid 1 \le r \le 2\cos\theta, 0 \le \theta \le \pi/3\}$. By symmetry, the total area is twice the area of D:



$$2A(D) = 2\iint_D dA = 2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\pi/3}\left(4\cos^2\theta - 1\right)d\theta = \int_0^{\pi/3}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 1\right]d\theta$$

$$= \int_0^{\pi/3}(1 + 2\cos 2\theta)\,d\theta = \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

23. (a) $V = \iint_D (1+xy)\,dA$, where D is the portion of the circle $x^2+y^2=4$ in the first quadrant. Thus,

$V = \iint_D (1+xy)\,dA = \int_0^{\pi/2}\int_0^2 (1+r^2\cos\theta\sin\theta)r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2}\int_0^2 (1+r^2\cos\theta\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \left(r+r^3\cos\theta\sin\theta\right)dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}+\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi/2}(2+4\cos\theta\sin\theta)\,d\theta = \left[2\theta + 2\sin^2\theta\right]_0^{\pi/2} \qquad [u=\sin\theta,\ du=\cos\theta\,d\theta]$$

$$= \pi + 2$$

24. (a) $V = \iint_D (x^2+y^2)\,dA$, where D is the region on or between the circles $x^2+y^2=1$ and $x^2+y^2=4$. Thus,

$V = \iint_{1\le x^2+y^2\le 4}(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 r^2 r\,dr\,d\theta$.

(b) $$\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta = \int_0^{2\pi}d\theta\int_1^2 r^3\,dr = \Big[\theta\Big]_{\theta=0}^{\theta=2\pi}\cdot\frac{1}{4}\left[r^4\right]_{r=1}^{r=2} = 2\pi\cdot\frac{1}{4}(16-1) = \frac{15\pi}{2}$$

25. (a) $V = \iint_D y\,dA$, where D is the portion of the circle $x^2+y^2=9$ in quadrants I–III. Thus,

$V = \iint_D y\,dA = \int_0^{3\pi/2}\int_0^3 (r\sin\theta)\,r\,dr\,d\theta$.

(b) $$\int_0^{3\pi/2}\int_0^3 r^2\sin\theta\,dr\,d\theta = \int_0^{3\pi/2}\sin\theta\,d\theta\int_0^3 r^2\,dr = \Big[-\cos\theta\Big]_{\theta=0}^{\theta=3\pi/2}\left[\frac{r^3}{3}\right]_{r=0}^{r=3} = [0-(-1)](9-0) = 9$$

26. (a) $V = \iint_D xy^2\,dA$, where D is the region on or between the circles $r=2$ and $r=3$ in the first quadrant. Thus,

$V = \iint_D xy^2\,dA = \int_0^{\pi/2}\int_2^3 (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2}\int_2^3 r^4\cos\theta\sin^2\theta\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\sin^2\theta\,d\theta\int_2^3 r^4\,dr = \left[\frac{\sin^3\theta}{3}\right]_{\theta=0}^{\theta=\pi/2}\left[\frac{r^5}{5}\right]_{r=2}^{r=3}$$

$$= \left(\tfrac{1}{3}-0\right)\left(\tfrac{243}{5}-\tfrac{32}{5}\right) = \tfrac{211}{15}$$

27. (a) The region is described by $D = \{(r,\theta)\mid 0\le r\le \sin\theta,\ 0\le\theta\le\pi/2\}$, so $\iint_D x\,dA = \int_0^{\pi/2}\int_0^{\sin\theta}(r\cos\theta)\,r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2}\int_0^{\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_{r=0}^{r=\sin\theta}d\theta = \frac{1}{3}\int_0^{\pi/2}\cos\theta\sin^3\theta\,d\theta$$

$$= \tfrac{1}{3}\int_0^1 u^3\,du \qquad [u=\sin\theta,\ du=\cos\theta\,d\theta]$$

$$= \tfrac{1}{3}\left[\tfrac{1}{4}u^4\right]_0^1 = \tfrac{1}{12}$$

28. (a) The region is described by $D = \{(r,\theta)\mid 0\le r\le 1+\cos\theta,\ 0\le\theta\le\pi\}$, so $\iint_D dA = \int_0^{\pi}\int_0^{1+\cos\theta} 1\cdot r\,dr\,d\theta$.

(b) $$\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=1+\cos\theta}d\theta = \frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi}\left[1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta = \tfrac{1}{2}\int_0^{\pi}\left[\tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos 2\theta\right]d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3\theta}{2}+2\sin\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{\pi} = \tfrac{3\pi}{4}$$

29. $V = \iint_{x^2+y^2\le 25}\left(x^2+y^2\right)dA = \int_0^{2\pi}\int_0^5 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^5 r^3\,dr = \left[\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^5 = 2\pi\left(\tfrac{625}{4}\right) = \tfrac{625}{2}\pi$

30. $V = \iint_{1\,\leq\,x^2\,+\,y^2\,\leq 4} \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{r^2}\;r\,dr\,d\theta = \int_0^{2\pi} d\theta\,\int_1^2 r^2\,dr = \big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{3}r^3\big]_1^2 = 2\pi\big(\tfrac{8}{3}-\tfrac{1}{3}\big) = \tfrac{14}{3}\pi$

31. $2x+y+z=4 \quad\Leftrightarrow\quad z=4-2x-y$, so the volume of the solid is

$$\begin{aligned} V &= \iint_{x^2\,+\,y^2\,\leq 1}(4-2x-y)\,dA = \int_0^{2\pi}\int_0^1 (4-2r\cos\theta-r\sin\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \big[4r-r^2\,(2\cos\theta+\sin\theta)\big]\,dr\,d\theta = \int_0^{2\pi}\big[2r^2-\tfrac{1}{3}r^3\,(2\cos\theta+\sin\theta)\big]_{r=0}^{r=1}\,d\theta \\ &= \int_0^{2\pi}\big[2-\tfrac{1}{3}\,(2\cos\theta+\sin\theta)\big]\,d\theta = \big[2\theta-\tfrac{1}{3}\,(2\sin\theta-\cos\theta)\big]_0^{2\pi} = 4\pi+\tfrac{1}{3}-0-\tfrac{1}{3}=4\pi \end{aligned}$$

32. The sphere $x^2+y^2+z^2=16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$\begin{aligned} V &= 2\iint\limits_{4\leq x^2+y^2\leq 16}\sqrt{16-x^2-y^2}\,dA \quad \text{[by symmetry]} \quad = 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\;r\,dr\,d\theta \\ &= 2\int_0^{2\pi} d\theta\int_2^4 r(16-r^2)^{1/2}dr = 2\big[\,\theta\,\big]_0^{2\pi}\Big[-\tfrac{1}{3}(16-r^2)^{3/2}\Big]_2^4 \\ &= -\tfrac{2}{3}(2\pi)(0-12^{3/2}) = \tfrac{4\pi}{3}\big(12\sqrt{12}\big) = 32\sqrt{3}\,\pi \end{aligned}$$

33. By symmetry,

$$\begin{aligned} V &= 2\iint\limits_{x^2\,+\,y^2\,\leq\,a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr \\ &= 2\big[\,\theta\,\big]_0^{2\pi}\Big[-\tfrac{1}{3}(a^2-r^2)^{3/2}\Big]_0^a = 2(2\pi)\big(0+\tfrac{1}{3}a^3\big) = \tfrac{4}{3}\pi a^3 \end{aligned}$$

34. The paraboloid $z=1+2x^2+2y^2$ intersects the plane $z=7$ when $7=1+2x^2+2y^2$ or $x^2+y^2=3$ and we are restricted to the first octant, so

$$\begin{aligned} V &= \iint\limits_{\substack{x^2+y^2\,\leq\,3,\\ x\geq 0,\,y\geq 0}}\big[7-\big(1+2x^2+2y^2\big)\big]\,dA = \int_0^{\pi/2}\int_0^{\sqrt{3}}\big[7-(1+2r^2)\big]\,r\,dr\,d\theta \\ &= \int_0^{\pi/2}d\theta\int_0^{\sqrt{3}}\big(6r-2r^3\big)\,dr = \big[\,\theta\,\big]_0^{\pi/2}\big[3r^2-\tfrac{1}{2}r^4\big]_0^{\sqrt{3}} = \tfrac{\pi}{2}\cdot\tfrac{9}{2} = \tfrac{9}{4}\pi \end{aligned}$$

35. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\Big(\sqrt{x^2+y^2}\Big)^2=1$ or $x^2+y^2=\frac{1}{2}$. So

$$\begin{aligned} V &= \iint\limits_{x^2\,+\,y^2\,\leq\,1/2}\Big(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\Big)\,dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\Big(\sqrt{1-r^2}-r\Big)r\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^{1/\sqrt{2}}\Big(r\sqrt{1-r^2}-r^2\Big)\,dr = \big[\,\theta\,\big]_0^{2\pi}\Big[-\tfrac{1}{3}(1-r^2)^{3/2}-\tfrac{1}{3}r^3\Big]_0^{1/\sqrt{2}} = 2\pi\big(-\tfrac{1}{3}\big)\Big(\tfrac{1}{\sqrt{2}}-1\Big) = \tfrac{\pi}{3}\big(2-\sqrt{2}\big) \end{aligned}$$

36. The two paraboloids intersect when $6-x^2-y^2=2x^2+2y^2$ or $x^2+y^2=2$. For $x^2+y^2\leq 2$, the paraboloid $z=6-x^2-y^2$ is above $z=2x^2+2y^2$ so

$$\begin{aligned} V &= \iint\limits_{x^2\,+\,y^2\,\leq\,2}\big[(6-x^2-y^2)-(2x^2+2y^2)\big]\,dA = \iint\limits_{x^2\,+\,y^2\,\leq\,2}\big[6-3(x^2+y^2)\big]\,dA = \int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r-3r^3)\,dr = \big[\,\theta\,\big]_0^{2\pi}\big[3r^2-\tfrac{3}{4}r^4\big]_0^{\sqrt{2}} = 2\pi\,(6-3) = 6\pi \end{aligned}$$

37. The given solid is the region inside the cylinder $x^2+y^2=4$ between the surfaces $z=\sqrt{64-4x^2-4y^2}$ and $z=-\sqrt{64-4x^2-4y^2}$. So

$$V=\iint\limits_{x^2+y^2\le 4}\left[\sqrt{64-4x^2-4y^2}-\left(-\sqrt{64-4x^2-4y^2}\right)\right]dA=\iint\limits_{x^2+y^2\le 4}2\cdot 2\sqrt{16-x^2-y^2}\,dA$$

$$=4\int_0^{2\pi}\int_0^2\sqrt{16-r^2}\;r\,dr\,d\theta=4\int_0^{2\pi}d\theta\int_0^2 r\sqrt{16-r^2}\,dr=4\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_0^2$$

$$=8\pi\left(-\tfrac{1}{3}\right)(12^{3/2}-16^{2/3})=\tfrac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

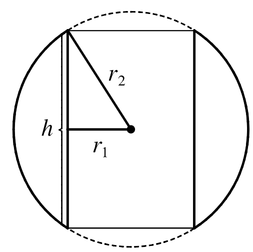
38. (a) Here the region in the xy-plane is the annular region $r_1^2\le x^2+y^2\le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V=2\iint\limits_{r_1^2\le x^2+y^2\le r_2^2}\sqrt{r_2^2-x^2-y^2}\,dA=2\int_0^{2\pi}\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\;r\,dr\,d\theta=2\int_0^{2\pi}d\theta\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\;r\,dr$$

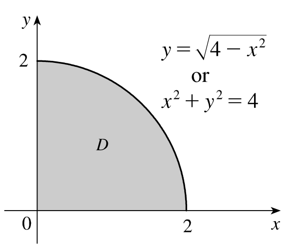
$$=2\,(2\pi)\left[-\tfrac{1}{3}(r_2^2-r^2)^{3/2}\right]_{r_1}^{r_2}=\tfrac{4\pi}{3}(r_2^2-r_1^2)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2=\left(\frac{1}{2}h\right)^2+r_1^2$ or $\frac{1}{4}h^2=r_2^2-r_1^2$.

Thus the volume in terms of h is $V=\frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2}=\frac{\pi}{6}h^3$.



39.

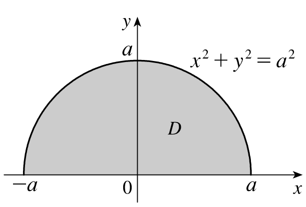


$$\int_0^2\int_0^{\sqrt{4-x^2}}e^{-x^2-y^2}\,dy\,dx=\int_0^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}d\theta\int_0^2 re^{-r^2}\,dr=\left[\,\theta\,\right]_0^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2$$

$$=\tfrac{\pi}{2}\left[-\tfrac{1}{2}\left(e^{-4}-1\right)\right]=\tfrac{\pi}{4}(1-e^{-4})$$

40.



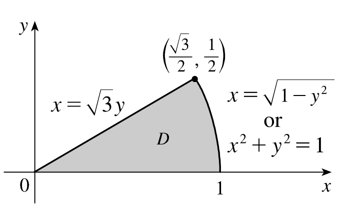
$$\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}(2x+y)\,dx\,dy=\int_0^{\pi}\int_0^a(2r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{\pi}(2\cos\theta+\sin\theta)\,d\theta\int_0^a r^2\,dr$$

$$=\left[2\sin\theta-\cos\theta\right]_0^{\pi}\left[\tfrac{1}{3}r^3\right]_0^a$$

$$=\left[(0+1)-(0-1)\right]\cdot\tfrac{1}{3}(a^3-0)=\tfrac{2}{3}a^3$$

41. The region D of integration is shown in the figure. In polar coordinates the line $x=\sqrt{3}\,y$ is $\theta=\pi/6$, so



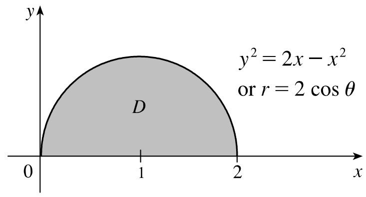
$$\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}}xy^2\,dx\,dy=\int_0^{\pi/6}\int_0^1(r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$$

$$=\int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta\int_0^1 r^4\,dr$$

$$=\left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/6}\left[\tfrac{1}{5}r^5\right]_0^1$$

$$=\left[\tfrac{1}{3}\left(\tfrac{1}{2}\right)^3-0\right]\left[\tfrac{1}{5}-0\right]=\tfrac{1}{120}$$

42.



$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/2} \left(\tfrac{8}{3}\cos^3\theta\right) d\theta$$
$$= \tfrac{8}{3}\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\,d\theta$$
$$= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

43. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, so

$\iint_D e^{(x^2+y^2)^2}\,dA = \int_0^{2\pi}\int_0^1 e^{(r^2)^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 re^{r^4}\,dr = 2\pi\int_0^1 re^{r^4}\,dr$. Using a calculator, we estimate $2\pi\int_0^1 re^{r^4}\,dr \approx 4.5951$.

44. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le \pi/2\}$, so

$$\iint_D xy\sqrt{1+x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\;r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \int_0^1 r^3\sqrt{1+r^2}\,dr = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \int_0^1 r^3\sqrt{1+r^2}\,dr$$
$$= \tfrac{1}{2}\int_0^1 r^3\sqrt{1+r^2}\,dr \approx 0.1609$$

45. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x,y) \mid x^2+y^2 \le 400\right\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

46. (a) If $R \le 100$, the total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi[-Re^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - Re^{-R} - e^{-R})\text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is

$\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function following Example 15.1.8.

47. As in Exercise 15.2.71, $f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x,y)\, dA$. Here $D = \{(r,\theta) \mid a \le r \le b, 0 \le \theta \le 2\pi\}$,

so $A(D) = \pi b^2 - \pi a^2 = \pi(b^2 - a^2)$ and

$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D \frac{1}{\sqrt{x^2+y^2}}\, dA = \frac{1}{\pi(b^2-a^2)} \int_0^{2\pi} \int_a^b \frac{1}{\sqrt{r^2}}\, r\, dr\, d\theta = \frac{1}{\pi(b^2-a^2)} \int_0^{2\pi} d\theta \int_a^b dr$$

$$= \frac{1}{\pi(b^2-a^2)} \big[\theta\big]_0^{2\pi} \big[r\big]_a^b = \frac{1}{\pi(b^2-a^2)} (2\pi)(b-a) = \frac{2(b-a)}{(b+a)(b-a)} = \frac{2}{a+b}$$

48. The distance from a point (x,y) to the origin is $f(x,y) = \sqrt{x^2+y^2}$, so the average distance from points in D to the origin is

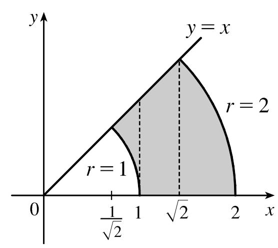
$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D \sqrt{x^2+y^2}\, dA = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{r^2}\; r\, dr\, d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} d\theta \int_0^a r^2\, dr = \frac{1}{\pi a^2} \big[\theta\big]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^a = \frac{1}{\pi a^2} \cdot 2\pi \cdot \tfrac{1}{3}a^3 = \tfrac{2}{3}a$$

49. $$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\, dy\, dx + \int_1^{\sqrt{2}} \int_0^x xy\, dy\, dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\, dy\, dx$$

$$= \int_0^{\pi/4} \int_1^2 r^3 \cos\theta \sin\theta\, dr\, d\theta = \int_0^{\pi/4} \left[\frac{r^4}{4} \cos\theta \sin\theta\right]_{r=1}^{r=2} d\theta$$

$$= \frac{15}{4} \int_0^{\pi/4} \sin\theta \cos\theta\, d\theta = \frac{15}{4} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



50. (a) $\iint_{D_a} e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_0^a re^{-r^2}\, dr\, d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty} \pi\left(1-e^{-a^2}\right) = \pi$

since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\, dA = \int_{-a}^a \int_{-a}^a e^{-x^2} e^{-y^2}\, dx\, dy = \left(\int_{-a}^a e^{-x^2}\, dx\right)\left(\int_{-a}^a e^{-y^2}\, dy\right)$ for each a.

From part (a), $\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA$, so then

$\pi = \lim\limits_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA = \lim\limits_{a\to\infty} \left(\int_{-a}^a e^{-x^2}\, dx\right)\left(\int_{-a}^a e^{-y^2}\, dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$, which is what we wish to show.

To evaluate $\lim\limits_{a\to\infty} \left(\int_{-a}^a e^{-x^2}\, dx\right)\left(\int_{-a}^a e^{-y^2}\, dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\, dx \le \int_{-\infty}^{-1} e^x\, dx + \int_{-1}^1 dx + \int_1^{\infty} e^{-x}\, dx = 2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right) dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$.

51. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^\infty x^2 e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\ &= \tfrac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx \qquad [\text{since } e^{-x^2} \text{ is an even function}]\\ &= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 50(c)]}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \quad\Rightarrow\quad dx = 2u\,du \quad\Rightarrow$

$$\int_0^\infty \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \quad \text{[by part(a)]} \;= \tfrac{1}{2}\sqrt{\pi}.$$

15.4 Applications of Double Integrals

1. $Q = \iint_D \sigma(x,y)\,dA = \int_0^5\int_2^5 (2x+4y)\,dy\,dx = \int_0^5 \left[2xy + 2y^2\right]_{y=2}^{y=5}\,dx$

$= \int_0^5 (10x + 50 - 4x - 8)\,dx = \int_0^5 (6x+42)\,dx = \left[3x^2 + 42x\right]_0^5 = 75 + 210 = 285 \text{ C}$

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\;r\,dr\,d\theta$

$= \int_0^{2\pi} d\theta\;\int_0^1 r^2\,dr = [\theta]_0^{2\pi}\;\left[\frac{1}{3}r^3\right]_0^1 = 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3}\text{ C}$

3. Since the density of the lamina is higher as $x \to 1$, we might estimate $\overline{x} = 0.7$. There is no vertical change in the density, so we would be confident that $\overline{y} = 0.5$.

$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_0^1 x^2\,dy\,dx = \int_0^1 dy\;\int_0^1 x^2\,dx = \frac{1}{3}$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_0^1 x^3\,dy\,dx = \int_0^1 dy\;\int_0^1 x^3\,dx = \frac{1}{4}$

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^1\int_0^1 x^2y\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

Hence, $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = \left(\frac{1/4}{1/3}, \frac{1/6}{1/3}\right) = \left(\frac{3}{4}, \frac{1}{2}\right)$.

4. Since the density of the lamina increases in a uniform fashion as $(x,y) \to (1,1)$, we might estimate $(x,y) = (0.7, 0.7)$.

$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x\,dx = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_0^1 x^2y\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^1\int_0^1 xy^2\,dy\,dx = \int_0^1 y^2\,dy\;\int_0^1 x\,dx = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

Hence, $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = \left(\frac{1/6}{1/4}, \frac{1/6}{1/4}\right) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

5. $m = \iint_D \rho(x,y)\,dA = \int_1^3\int_1^4 ky^2\,dy\,dx = k\int_1^3 dx\;\int_1^4 y^2\,dy = k\,[x]_1^3\;\left[\frac{1}{3}y^3\right]_1^4 = k(2)(21) = 42k$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 kxy^2\,dy\,dx = \frac{1}{42}\int_1^3 x\,dx\;\int_1^4 y^2\,dy = \frac{1}{42}\left[\frac{1}{2}x^2\right]_1^3\left[\frac{1}{3}y^3\right]_1^4 = \frac{1}{42}(4)(21) = 2$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 ky^3\,dy\,dx = \frac{1}{42}\int_1^3 dx\;\int_1^4 y^3\,dy = \frac{1}{42}\,[x]_1^3\;\left[\frac{1}{4}y^4\right]_1^4 = \frac{1}{42}(2)\left(\frac{255}{4}\right) = \frac{85}{28}$

Hence, $(\overline{x},\overline{y}) = \left(2, \frac{85}{28}\right)$.

6. $m = \iint_D \rho(x,y)\,dA = \int_0^a \int_0^b (1+x^2+y^2)\,dy\,dx = \int_0^a \left[y + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=b} dx = \int_0^a \left(b + bx^2 + \frac{1}{3}b^3\right) dx$

$= \left[bx + \frac{1}{3}bx^3 + \frac{1}{3}b^3x\right]_0^a = ab + \frac{1}{3}a^3b + \frac{1}{3}ab^3 = \frac{1}{3}ab(3+a^2+b^2),$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a \int_0^b (x + x^3 + xy^2)\,dy\,dx = \int_0^a \left[xy + x^3y + \frac{1}{3}xy^3\right]_{y=0}^{y=b} dx$

$= \int_0^a \left(bx + bx^3 + \frac{1}{3}b^3x\right) dx = \left[\frac{1}{2}bx^2 + \frac{1}{4}bx^4 + \frac{1}{6}b^3x^2\right]_0^a = \frac{1}{2}a^2b + \frac{1}{4}a^4b + \frac{1}{6}a^2b^3$

$= \frac{1}{12}a^2b(6+3a^2+2b^2)$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a \int_0^b (y + x^2y + y^3)\,dy\,dx = \int_0^a \left[\frac{1}{2}y^2 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right]_{y=0}^{y=b} dx$

$= \int_0^a \left(\frac{1}{2}b^2 + \frac{1}{2}b^2x^2 + \frac{1}{4}b^4\right) dx = \left[\frac{1}{2}b^2x + \frac{1}{6}b^2x^3 + \frac{1}{4}b^4x\right]_0^a = \frac{1}{2}ab^2 + \frac{1}{6}a^3b^2 + \frac{1}{4}ab^4$

$= \frac{1}{12}ab^2(6+2a^2+3b^2).$

Hence, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{\frac{1}{12}a^2b(6+3a^2+2b^2)}{\frac{1}{3}ab(3+a^2+b^2)}, \dfrac{\frac{1}{12}ab^2(6+2a^2+3b^2)}{\frac{1}{3}ab(3+a^2+b^2)}\right)$

$= \left(\dfrac{a(6+3a^2+2b^2)}{4(3+a^2+b^2)}, \dfrac{b(6+2a^2+3b^2)}{4(3+a^2+b^2)}\right).$

7. $m = \int_0^2 \int_{x/2}^{3-x} (x+y)\,dy\,dx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x(3-x) + \frac{1}{2}(3-x)^2 - \frac{1}{2}x^2 - \frac{1}{8}x^2\right] dx$

$= \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right) + \frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2 \int_{x/2}^{3-x} (x^2+xy)\,dy\,dx = \int_0^2 \left[x^2y + \frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \frac{9}{2},$

$M_x = \int_0^2 \int_{x/2}^{3-y} (xy+y^2)\,dy\,dx = \int_0^2 \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9 - \frac{9}{2}x\right) dx = 9.$

Hence, $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{3}{4}, \dfrac{3}{2}\right).$

8. Here $D = \left\{(x,y) \mid 0 \le y \le \frac{2}{5},\;\; y/2 \le x \le 1 - 2y\right\}.$

$m = \int_0^{2/5} \int_{y/2}^{1-2y} x\,dx\,dy = \int_0^{2/5} \left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{2}\int_0^{2/5} \left[(1-2y)^2 - \left(\frac{1}{2}y\right)^2\right] dy$

$= \frac{1}{2}\int_0^{2/5} \left(\frac{15}{4}y^2 - 4y + 1\right) dy = \frac{1}{2}\left[\frac{5}{4}y^3 - 2y^2 + y\right]_0^{2/5} = \frac{1}{2}\left[\frac{2}{25} - \frac{8}{25} + \frac{2}{5}\right] = \frac{2}{25},$

$M_y = \int_0^{2/5} \int_{y/2}^{1-2y} x \cdot x\,dx\,dy = \int_0^{2/5} \left[\frac{1}{3}x^3\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{3}\int_0^{2/5} \left[(1-2y)^3 - \left(\frac{1}{2}y\right)^3\right] dy$

$= \frac{1}{3}\int_0^{2/5} \left(-\frac{65}{8}y^3 + 12y^2 - 6y + 1\right) dy = \frac{1}{3}\left[-\frac{65}{32}y^4 + 4y^3 - 3y^2 + y\right]_0^{2/5} = \frac{1}{3}\left[-\frac{13}{250} + \frac{32}{125} - \frac{12}{25} + \frac{2}{5}\right] = \frac{31}{750},$

$M_x = \int_0^{2/5} \int_{y/2}^{1-2y} y \cdot x\,dx\,dy = \int_0^{2/5} y\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{2}\int_0^{2/5} y\left(\frac{15}{4}y^2 - 4y + 1\right) dy$

$= \frac{1}{2}\int_0^{2/5} \left(\frac{15}{4}y^3 - 4y^2 + y\right) dy = \frac{1}{2}\left[\frac{15}{16}y^4 - \frac{4}{3}y^3 + \frac{1}{2}y^2\right]_0^{2/5} = \frac{1}{2}\left[\frac{3}{125} - \frac{32}{375} + \frac{2}{25}\right] = \frac{7}{750}.$

Hence, $m = \frac{2}{25}$, $(\overline{x},\overline{y}) = \left(\frac{31/750}{2/25}, \frac{7/750}{2/25}\right) = \left(\frac{31}{60}, \frac{7}{60}\right).$

9. $m = \int_{-1}^1 \int_0^{1-x^2} ky\,dy\,dx = k\int_{-1}^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x^2} dx = \frac{1}{2}k\int_{-1}^1 (1-x^2)^2\,dx = \frac{1}{2}k\int_{-1}^1 (1 - 2x^2 + x^4)\,dx$

$= \frac{1}{2}k\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^1 = \frac{1}{2}k\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15}k,$

[continued]

$$M_y = \int_{-1}^{1}\int_0^{1-x^2} kxy\,dy\,dx = k\int_{-1}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{2}k\int_{-1}^{1} x\,(1-x^2)^2\,dx = \tfrac{1}{2}k\int_{-1}^{1}(x-2x^3+x^5)\,dx$$

$$= \tfrac{1}{2}k\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{6}x^6\right]_{-1}^{1} = \tfrac{1}{2}k\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6}\right) = 0,$$

$$M_x = \int_{-1}^{1}\int_0^{1-x^2} ky^2\,dy\,dx = k\int_{-1}^{1}\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{3}k\int_{-1}^{1}(1-x^2)^3\,dx = \tfrac{1}{3}k\int_{-1}^{1}(1-3x^2+3x^4-x^6)\,dx$$

$$= \tfrac{1}{3}k\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7\right]_{-1}^{1} = \tfrac{1}{3}k\left(1 - 1 + \tfrac{3}{5} - \tfrac{1}{7} + 1 - 1 + \tfrac{3}{5} - \tfrac{1}{7}\right) = \tfrac{32}{105}k.$$

Hence, $m = \frac{8}{15}k$, $(\overline{x},\overline{y}) = \left(0, \frac{32k/105}{8k/15}\right) = \left(0, \frac{4}{7}\right)$.

10. The boundary curves intersect when $x + 2 = x^2 \quad\Leftrightarrow\quad x^2 - x - 2 = 0 \quad\Leftrightarrow\quad x = -1, x = 2$. Thus here $D = \{(x,y) \mid -1 \le x \le 2,\ \ x^2 \le y \le x + 2\}$.

$$m = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2\,dy\,dx = k\int_{-1}^{2} x^2\left[y\right]_{y=x^2}^{y=x+2} dx = k\int_{-1}^{2}(x^3+2x^2-x^4)\,dx$$

$$= k\left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{1}{5}x^5\right]_{-1}^{2} = k\left(\tfrac{44}{15} + \tfrac{13}{60}\right) = \tfrac{63}{20}k,$$

$$M_y = \int_{-1}^{2}\int_{x^2}^{x+2} kx^3\,dy\,dx = k\int_{-1}^{2} x^3\left[y\right]_{y=x^2}^{y=x+2} dx = k\int_{-1}^{2}(x^4+2x^3-x^5)\,dx$$

$$= k\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6\right]_{-1}^{2} = k\left(\tfrac{56}{15} - \tfrac{2}{15}\right) = \tfrac{18}{5}k,$$

$$M_x = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2y\,dy\,dx = k\int_{-1}^{2} x^2\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x+2} dx = \tfrac{1}{2}k\int_{-1}^{2} x^2\left(x^2+4x+4-x^4\right)dx$$

$$= \tfrac{1}{2}k\int_{-1}^{2}(x^4+4x^3+4x^2-x^6)\,dx = \tfrac{1}{2}k\left[\tfrac{1}{5}x^5 + x^4 + \tfrac{4}{3}x^3 - \tfrac{1}{7}x^7\right]_{-1}^{2} = \tfrac{1}{2}k\left(\tfrac{1552}{105} + \tfrac{41}{105}\right) = \tfrac{531}{70}k.$$

Hence, $m = \frac{63}{20}k$, $(\overline{x},\overline{y}) = \left(\frac{18k/5}{63k/20}, \frac{531k/70}{63k/20}\right) = \left(\frac{8}{7}, \frac{118}{49}\right)$.

11. $m = \int_0^1\int_0^{e^{-x}} xy\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^{-x}} dx = \frac{1}{2}\int_0^1 x\left(e^{-x}\right)^2 dx = \frac{1}{2}\int_0^1 xe^{-2x}\,dx \quad$ [integrate by parts with $u = x, dv = e^{-2x}\,dx$]

$$= \tfrac{1}{2}\left[-\tfrac{1}{4}(2x+1)e^{-2x}\right]_0^1 = -\tfrac{1}{8}\left(3e^{-2} - 1\right) = \tfrac{1}{8} - \tfrac{3}{8}e^{-2},$$

$M_y = \int_0^1\int_0^{e^{-x}} x^2y\,dy\,dx = \int_0^1 x^2\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^{-x}} dx = \frac{1}{2}\int_0^1 x^2e^{-2x}\,dx \quad$ [integrate by parts twice]

$$= \tfrac{1}{2}\left[-\tfrac{1}{4}\left(2x^2+2x+1\right)e^{-2x}\right]_0^1 = -\tfrac{1}{8}\left(5e^{-2} - 1\right) = \tfrac{1}{8} - \tfrac{5}{8}e^{-2},$$

$$M_x = \int_0^1\int_0^{e^{-x}} xy^2\,dy\,dx = \int_0^1 x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=e^{-x}} dx = \tfrac{1}{3}\int_0^1 xe^{-3x}\,dx$$

$$= \tfrac{1}{3}\left[-\tfrac{1}{9}(3x+1)e^{-3x}\right]_0^1 = -\tfrac{1}{27}\left(4e^{-3} - 1\right) = \tfrac{1}{27} - \tfrac{4}{27}e^{-3}.$$

Hence, $m = \frac{1}{8}\left(1 - 3e^{-2}\right)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}\left(1-5e^{-2}\right)}{\frac{1}{8}\left(1-3e^{-2}\right)}, \dfrac{\frac{1}{27}\left(1-4e^{-3}\right)}{\frac{1}{8}\left(1-3e^{-2}\right)}\right) = \left(\dfrac{e^2-5}{e^2-3}, \dfrac{8\left(e^3-4\right)}{27(e^3-3e)}\right)$.

12. Note that $\cos x \ge 0$ for $-\pi/2 \le x \le \pi/2$.

$$m = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y\,dy\,dx = \int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\cos x} dx = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx = \tfrac{1}{2}\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \tfrac{\pi}{4},$$

$M_y = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_{-\pi/2}^{\pi/2} x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\cos x} dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx \quad$ [integrate by parts with $u = x, dv = \cos^2 x\,dx$]

$$= \tfrac{1}{2}\left[x\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right)dx\right]$$

$$= \tfrac{1}{2}\left(\tfrac{1}{8}\pi^2 - \tfrac{1}{8}\pi^2 - \left[\tfrac{1}{4}x^2 - \tfrac{1}{8}\cos 2x\right]_{-\pi/2}^{\pi/2}\right) = \tfrac{1}{2}\left(0 - \left[\tfrac{1}{16}\pi^2 + \tfrac{1}{8} - \tfrac{1}{16}\pi^2 - \tfrac{1}{8}\right]\right) = 0,$$

[continued]

$$M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y^2\, dy\, dx = \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=\cos x} dx = \tfrac{1}{3}\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx = \tfrac{1}{3}\int_{-\pi/2}^{\pi/2} (1-\sin^2 x)\cos x\, dx$$

[substitute $u = \sin x \quad \Rightarrow \quad du = \cos x\, dx$]

$$= \tfrac{1}{3}\left[\sin x - \tfrac{1}{3}\sin^3 x\right]_{-\pi/2}^{\pi/2} = \tfrac{1}{3}\left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) = \tfrac{4}{9}.$$

Hence, $m = \frac{\pi}{4}$, $(\overline{x}, \overline{y}) = \left(0, \frac{4/9}{\pi/4}\right) = \left(0, \frac{16}{9\pi}\right)$.

13. "The density at any point is proportional to its distance from the x-axis" $\Rightarrow \quad \rho(x,y) = ky$.

$$m = \iint_D ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^2\, dr$$
$$= k\left[-\cos\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^1 = k(1)\left(\tfrac{1}{3}\right) = \tfrac{1}{3}k,$$

$$M_y = \iint_D x \cdot ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^1 r^3\, dr$$
$$= k\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{1}{8}k,$$

$$M_x = \iint_D y \cdot ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)^2\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^1 r^3\, dr$$
$$= k\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}\right) = \tfrac{\pi}{16}k.$$

Hence, $(\overline{x}, \overline{y}) = \left(\frac{k/8}{k/3}, \frac{k\pi/16}{k/3}\right) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

14. "The density at any point is proportional to the square of its distance from the origin" $\Rightarrow$

$$\rho(x,y) = k\left(\sqrt{x^2+y^2}\right)^2 = k(x^2+y^2) = kr^2.$$

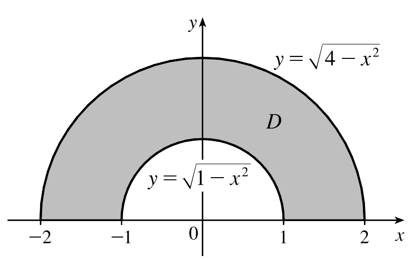
$$m = \int_0^{\pi/2}\int_0^1 kr^3\, dr\, d\theta = \tfrac{\pi}{8}k,$$

$$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\, dr\, d\theta = \tfrac{1}{5}k\int_0^{\pi/2}\cos\theta\, d\theta = \tfrac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \tfrac{1}{5}k,$$

$$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\, dr\, d\theta = \tfrac{1}{5}k\int_0^{\pi/2}\sin\theta\, d\theta = \tfrac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \tfrac{1}{5}k.$$

Hence, $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

15.



$$\rho(x,y) = k\sqrt{x^2+y^2} = kr,$$

$$m = \iint_D \rho(x,y)dA = \int_0^\pi\int_1^2 kr\cdot r\, dr\, d\theta$$
$$= k\int_0^\pi d\theta \int_1^2 r^2\, dr = k(\pi)\left[\tfrac{1}{3}r^3\right]_1^2 = \tfrac{7}{3}\pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^\pi\int_1^2 (r\cos\theta)(kr)\, r\, dr\, d\theta = k\int_0^\pi \cos\theta\, d\theta \int_1^2 r^3\, dr$$

$$= k\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(0)\left(\tfrac{15}{4}\right) = 0$$ [this is to be expected as the region and density function are symmetric about the y-axis]

[continued]

$$M_x = \iint_D y\rho(x,y)dA = \int_0^\pi \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta\,\int_1^2 r^3\,dr$$
$$= k\left[-\cos\theta\right]_0^\pi\,\left[\tfrac{1}{4}r^4\right]_1^2 = k(1+1)\left(\tfrac{15}{4}\right) = \tfrac{15}{2}k.$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

16. Now $\rho(x,y) = k\,/\sqrt{x^2+y^2} = k/r$, so

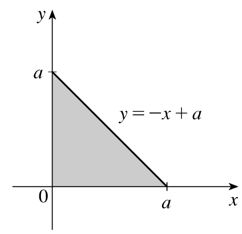
$$m = \iint_D \rho(x,y)dA = \int_0^\pi \int_1^2 (k/r)\,r\,dr\,d\theta = k\int_0^\pi d\theta\,\int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^\pi \int_1^2 (r\cos\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta\,\int_1^2 r\,dr$$
$$= k\left[\sin\theta\right]_0^\pi\,\left[\tfrac{1}{2}r^2\right]_1^2 = k(0)\left(\tfrac{3}{2}\right) = 0,$$

$$M_x = \iint_D y\rho(x,y)dA = \int_0^\pi \int_1^2 (r\sin\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta\,\int_1^2 r\,dr$$
$$= k\left[-\cos\theta\right]_0^\pi\,\left[\tfrac{1}{2}r^2\right]_1^2 = k(1+1)\left(\tfrac{3}{2}\right) = 3k.$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{3k}{\pi k}\right) = \left(0, \frac{3}{\pi}\right)$.

17. Placing the vertex opposite the hypotenuse at $(0,0)$ as in the figure,



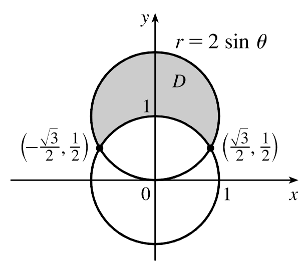
$\rho(x,y) = k(x^2+y^2)$. Then

$$m = \int_0^a \int_0^{a-x} k\left(x^2+y^2\right)dy\,dx$$
$$= k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}\,(a-x)^3\right]dx$$
$$= k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}\,(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4$$

By symmetry, $$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right]dx$$
$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence, $(\overline{x},\overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

18. $\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$.



$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1]\,d\theta$$
$$= k\left[-2\cos\theta - \theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{3\sqrt{3}}{2\left(3\sqrt{3}-\pi\right)}\right)$.

19. $I_x = \iint_D y^2\rho(x,y)dA = \int_1^3 \int_1^4 y^2 \cdot ky^2\,dy\,dx = k\int_1^3 dx\,\int_1^4 y^4\,dy = k\left[x\right]_1^3\,\left[\tfrac{1}{5}y^5\right]_1^4 = k(2)\left(\tfrac{1023}{5}\right) = 409.2k,$

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_1^3 \int_1^4 x^2 \cdot ky^2\,dy\,dx = k\int_1^3 x^2\,dx\,\int_1^4 y^2\,dy = k\left[\tfrac{1}{3}x^3\right]_1^3\,\left[\tfrac{1}{3}y^3\right]_1^4 = k\left(\tfrac{26}{3}\right)(21) = 182k,$

and $I_0 = I_x + I_y = 409.2k + 182k = 591.2k$.

20. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^{2/5}\int_{y/2}^{1-2y} y^2 \cdot x\,dx\,dy = \int_0^{2/5} y^2\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{2}\int_0^{2/5} y^2(\frac{15}{4}y^2 - 4y + 1)\,dy$

$= \frac{1}{2}\int_0^{2/5}(\frac{15}{4}y^4 - 4y^3 + y^2)\,dy = \frac{1}{2}\left[\frac{3}{4}y^5 - y^4 + \frac{1}{3}y^3\right]_0^{2/5} = \frac{16}{9375}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^{2/5}\int_{y/2}^{1-2y} x^2 \cdot x\,dx\,dy = \int_0^{2/5}\left[\frac{1}{4}x^4\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{4}\int_0^{2/5}\left[(1-2y)^4 - \frac{1}{16}y^4\right]dy$

$= \frac{1}{4}\int_0^{2/5}(\frac{255}{16}y^4 - 32y^3 + 24y^2 - 8y + 1)\,dy = \frac{1}{4}\left[\frac{51}{16}y^5 - 8y^4 + 8y^3 - 4y^2 + y\right]_0^{2/5} = \frac{78}{3125}$,

and $I_0 = I_x + I_y = \frac{16}{9375} + \frac{78}{3125} = \frac{2}{75}$.

21. As in Exercise 17, we place the vertex opposite the hypotenuse at $(0,0)$ and the equal sides along the positive axes.

$I_x = \int_0^a\int_0^{a-x} y^2 k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^2y^2+y^4)\,dy\,dx = k\int_0^a\left[\frac{1}{3}x^2y^3 + \frac{1}{5}y^5\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5\right]dx = k\left[\frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right) - \frac{1}{30}(a-x)^6\right]_0^a = \frac{7}{180}ka^6$,

$I_y = \int_0^a\int_0^{a-x} x^2 k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^4+x^2y^2)\,dy\,dx = k\int_0^a\left[x^4y + \frac{1}{3}x^2y^3\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[x^4(a-x) + \frac{1}{3}x^2(a-x)^3\right]dx = k\left[\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right)\right]_0^a = \frac{7}{180}ka^6$,

and $I_0 = I_x + I_y = \frac{7}{90}ka^6$.

22. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2(1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2}\,dx$$
$$= \tfrac{8}{3}\int_0^2(1+0.1x)\,dx = \tfrac{8}{3}\left[x + 0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[y\right]_{y=0}^{y=2}\,dx$$
$$= 2\int_0^2(x^2+0.1x^3)\,dx = 2\left[\tfrac{1}{3}x^3 + 0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

23. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^h\int_0^b \rho y^2\,dx\,dy = \rho\int_0^b dx\,\int_0^h y^2\,dy = \rho\left[x\right]_0^b\left[\frac{1}{3}y^3\right]_0^h = \rho b\left(\frac{1}{3}h^3\right) = \frac{1}{3}\rho bh^3$,

$I_y = \iint_D x^2\rho(x,y)dA = \int_0^h\int_0^b \rho x^2\,dx\,dy = \rho\int_0^b x^2\,dx\,\int_0^h dy = \rho\left[\frac{1}{3}x^3\right]_0^b\left[y\right]_0^h = \frac{1}{3}\rho b^3h$,

and $m = \rho\,(\text{area of rectangle}) = \rho bh$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{3}\rho b^3h}{\rho bh} = \dfrac{b^2}{3} \Rightarrow \overline{\overline{x}} = \dfrac{b}{\sqrt{3}}$

and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{3}\rho bh^3}{\rho bh} = \dfrac{h^2}{3} \Rightarrow \overline{\overline{y}} = \dfrac{h}{\sqrt{3}}$.

24. Here we assume $b>0$, $h>0$ but note that we arrive at the same results if $b<0$ or $h<0$. We have $D = \left\{(x,y) \mid 0 \le x \le b, 0 \le y \le h - \frac{h}{b}x\right\}$, so

$I_x = \int_0^b\int_0^{h-hx/b} y^2\rho\,dy\,dx = \rho\int_0^b\left[\frac{1}{3}y^3\right]_{y=0}^{y=h-hx/b}dx = \frac{1}{3}\rho\int_0^b\left(h - \frac{h}{b}x\right)^3dx$

$= \frac{1}{3}\rho\left[-\frac{b}{h}\left(\frac{1}{4}\right)\left(h - \frac{h}{b}x\right)^4\right]_0^b = -\frac{b}{12h}\rho(0-h^4) = \frac{1}{12}\rho bh^3$,

$I_y = \int_0^b \int_0^{h-hx/b} x^2 \rho \, dy \, dx = \rho \int_0^b x^2 \left(h - \frac{h}{b}x\right) dx = \rho \int_0^b \left(hx^2 - \frac{h}{b}x^3\right) dx$

$= \rho \left[\frac{h}{3}x^3 - \frac{h}{4b}x^4\right]_0^b = \rho\left(\frac{hb^3}{3} - \frac{hb^3}{4}\right) = \frac{1}{12}\rho b^3 h,$

and $m = \int_0^b \int_0^{h-hx/b} \rho \, dy \, dx = \rho \int_0^b \left(h - \frac{h}{b}x\right) dx = \rho \left[hx - \frac{h}{2b}x^2\right]_0^b = \frac{1}{2}\rho bh$. Hence $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho bh} = \dfrac{b^2}{6} \Rightarrow$

$\bar{\bar{x}} = \dfrac{b}{\sqrt{6}}$ and $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{12}\rho bh^3}{\frac{1}{2}\rho bh} = \dfrac{h^2}{6} \Rightarrow \bar{\bar{y}} = \dfrac{h}{\sqrt{6}}$.

25. In polar coordinates, the region is $D = \left\{(r, \theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\right\}$, so

$I_x = \iint_D y^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho (r \sin\theta)^2 \, r \, dr \, d\theta = \rho \int_0^{\pi/2} \sin^2 d\theta \int_0^a r^3 \, dr$

$= \rho \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

$I_y = \iint_D x^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho (r \cos\theta)^2 \, r \, dr \, d\theta = \rho \int_0^{\pi/2} \cos^2 d\theta \int_0^a r^3 \, dr$

$= \rho \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

and $m = \rho \cdot A(D) = \rho \cdot \frac{1}{4}\pi a^2$ since the lamina is homogeneous. Hence $\bar{\bar{x}}^2 = \bar{\bar{y}}^2 = \dfrac{\frac{1}{16}\rho a^4 \pi}{\frac{1}{4}\rho a^2 \pi} = \dfrac{a^2}{4} \Rightarrow \bar{\bar{x}} = \bar{\bar{y}} = \dfrac{a}{2}$.

26. $m = \int_0^\pi \int_0^{\sin x} \rho \, dy \, dx = \rho \int_0^\pi \sin x \, dx = \rho \big[-\cos x\big]_0^\pi = 2\rho,$

$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 \, dy \, dx = \frac{1}{3}\rho \int_0^\pi \sin^3 x \, dx = \frac{1}{3}\rho \int_0^\pi (1 - \cos^2 x) \sin x \, dx = \frac{1}{3}\rho \big[-\cos x + \frac{1}{3}\cos^3 x\big]_0^\pi = \frac{4}{9}\rho,$

$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2 \, dy \, dx = \rho \int_0^\pi x^2 \sin x \, dx = \rho \big[-x^2 \cos x + 2x \sin x + 2\cos x\big]_0^\pi$ [by integrating by parts twice]

$= \rho(\pi^2 - 4).$

Then $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\bar{\bar{y}} = \dfrac{\sqrt{2}}{3}$ and $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2 - 4}{2}$, so $\bar{\bar{x}} = \sqrt{\dfrac{\pi^2 - 4}{2}}$.

27. The right loop of the curve is given by $D = \{(r, \theta) \mid 0 \le r \le \cos 2\theta, \ -\pi/4 \le \theta \le \pi/4\}$. Using a CAS, we find $m = \iint_D \rho(x, y) \, dA = \iint_D (x^2 + y^2) \, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^2 \, r \, dr \, d\theta = \dfrac{3\pi}{64}$. Then

$\bar{x} = \dfrac{1}{m} \displaystyle\iint_D x\rho(x, y) \, dA = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r \cos\theta) \, r^2 \, r \, dr \, d\theta = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^4 \cos\theta \, dr \, d\theta = \dfrac{16384\sqrt{2}}{10395\pi}$ and

$\bar{y} = \dfrac{1}{m} \displaystyle\iint_D y\rho(x, y) \, dA = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r \sin\theta) \, r^2 \, r \, dr \, d\theta = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^4 \sin\theta \, dr \, d\theta = 0$, so

$(\bar{x}, \bar{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right)$.

The moments of inertia are

$I_x = \iint_D y^2 \rho(x, y) \, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r \sin\theta)^2 \, r^2 \, r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \sin^2\theta \, dr \, d\theta = \dfrac{5\pi}{384} - \dfrac{4}{105},$

$I_y = \iint_D x^2 \rho(x, y) \, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r \cos\theta)^2 \, r^2 \, r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \cos^2\theta \, dr \, d\theta = \dfrac{5\pi}{384} + \dfrac{4}{105}$, and

$I_0 = I_x + I_y = \dfrac{5\pi}{192}$.

28. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^2 \int_0^{xe^{-x}} x^2y^2\,dy\,dx = \frac{8}{729}(5 - 899e^{-6})$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{729}{8(5-899e^{-6})}\int_0^2\int_0^{xe^{-x}} x^3y^2\,dy\,dx = \frac{2(5e^6-1223)}{5e^6-899} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{729}{8(5-899e^{-6})}\int_0^2\int_0^{xe^{-x}} x^2y^3\,dy\,dx = \frac{729(45e^6-42037e^{-2})}{32768(5e^6-899)}, \text{ so}$$

$$(\overline{x},\overline{y}) = \left(\frac{2(5e^6-1223)}{5e^6-899}, \frac{729(45e^6-42037e^{-2})}{32768(5e^6-899)}\right).$$

The moments of inertia are $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^{xe^{-x}} x^2y^4\,dy\,dx = \frac{16}{390625}(63 - 305593e^{-10})$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^{xe^{-x}} x^4y^2\,dy\,dx = \frac{80}{2187}(7-2101e^{-6})$, and

$I_0 = I_x + I_y = \frac{16}{854296875}(13809656 - 4103515625e^{-6} - 668331891e^{-10})$.

29. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1]\times[0,2]$, we can say

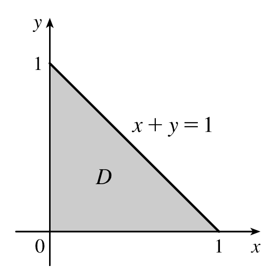
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cx(1+y)\,dy\,dx \\ &= C\int_0^1 x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=2}\,dx = C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C\end{aligned}$$

Then $2C = 1 \;\Rightarrow\; C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^1\int_{-\infty}^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\,dy\,dx$

$$= \int_0^1 \tfrac{1}{2}x\left[y+\tfrac{1}{2}y^2\right]_{y=0}^{y=1}\,dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{3}{2}\right)dx = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X+Y\le 1) = P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus



$$\begin{aligned}P(X+Y\le 1) &= \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x}\tfrac{1}{2}x(1+y)\,dy\,dx \\ &= \int_0^1 \tfrac{1}{2}x\left[y+\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x}\,dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\right)dx \\ &= \tfrac{1}{4}\int_0^1\left(x^3 - 4x^2 + 3x\right)dx = \tfrac{1}{4}\left[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\right]_0^1 \\ &= \tfrac{5}{48} \approx 0.1042\end{aligned}$$

30. (a) $f(x,y)\ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1]\times[0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^1\int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1}\,dx = \int_0^1 2x\,dx = \left.x^2\right]_0^1 = 1$.

Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\left(X\ge\tfrac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{1/2}^1\int_0^1 4xy\,dy\,dx = \int_{1/2}^1\left[2xy^2\right]_{y=0}^{y=1}\,dx = \int_{1/2}^1 2x\,dx = \left.x^2\right]_{1/2}^1 = \tfrac{3}{4}.$$

(ii) $P\left(X\ge\frac{1}{2}, Y\le\frac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx = \int_{1/2}^1\int_0^{1/2} 4xy\,dy\,dx$

$$= \int_{1/2}^1\left[2xy^2\right]_{y=0}^{y=1/2}\,dx = \int_{1/2}^1 \tfrac{1}{2}x\,dx = \tfrac{1}{2}\cdot\left.\tfrac{1}{2}x^2\right]_{1/2}^1 = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^1 \int_0^1 x(4xy)\, dy\, dx = \int_0^1 2x^2 \left[y^2\right]_{y=0}^{y=1} dx = 2\int_0^1 x^2\, dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^1 \int_0^1 y(4xy)\, dy\, dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx = \tfrac{4}{3}\int_0^1 x\, dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

31. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\begin{aligned}
\iint_{\mathbb{R}^2} f(x,y)\, dA &= \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx = 0.1\int_0^\infty \int_0^\infty e^{-0.5x}e^{-0.2y}\, dy\, dx \\
&= 0.1\int_0^\infty e^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy \\
&= 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \\
&= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1
\end{aligned}$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$\begin{aligned}
P(Y \geq 1) &= \int_{-\infty}^\infty \int_1^\infty f(x,y)\, dy\, dx = \int_0^\infty \int_1^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1\int_0^\infty e^{-0.5x}\, dx \int_1^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_1^t e^{-0.2y}\, dy \\
&= 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right] \\
&(0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187
\end{aligned}$$

(ii)
$$\begin{aligned}
P(X \leq 2, Y \leq 4) &= \int_{-\infty}^2 \int_{-\infty}^4 f(x,y)\, dy\, dx = \int_0^2 \int_0^4 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1\int_0^2 e^{-0.5x}\, dx \int_0^4 e^{-0.2y}\, dy = 0.1\left[-2e^{-0.5x}\right]_0^2 \left[-5e^{-0.2y}\right]_0^4 \\
&= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1) \\
&= (e^{-1}-1)(e^{-0.8}-1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481
\end{aligned}$$

(c) The expected value of X is given by

$$\begin{aligned}
\mu_1 &= \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^\infty \int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right] dy\, dx \\
&= 0.1\int_0^\infty xe^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t xe^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy
\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\, dx$ (or we can use Formula 96 in the Table of Integrals): $\int xe^{-0.5x}\, dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\, dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$.

Thus

$$\begin{aligned}
\mu_1 &= 0.1 \lim_{t\to\infty}\left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \\
&= 0.1 \lim_{t\to\infty}(-2)\left[(t+2)e^{-0.5t} - 2\right] \lim_{t\to\infty}(-5)\left[e^{-0.2t} - 1\right] \\
&= 0.1(-2)\left(\lim_{t\to\infty}\frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}
\end{aligned}$$

[continued]

The expected value of Y is given by

$$\begin{aligned}
\mu_2 &= \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^\infty \int_0^\infty y \left[0.1e^{-(0.5\,+0.2y)}\right] dy\, dx \\
&= 0.1 \int_0^\infty e^{-0.5x}\, dx \int_0^\infty ye^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\, dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\, dy
\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\, dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}\, dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\, dy = -5(y+5)e^{-0.2y}$. Then

$$\begin{aligned}
\mu_2 &= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t \\
&= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t}-5\right]\right) \\
&= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}
\end{aligned}$$

32. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

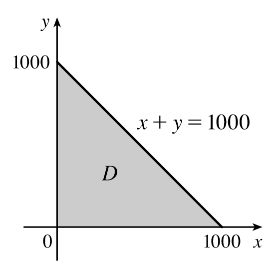
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned}
P\,(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\, dy\, dx = \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000}\, dy\, dx \\
&= 10^{-6} \int_0^{1000} e^{-x/1000}\, dx \int_0^{1000} e^{-y/1000}\, dy \\
&= 10^{-6} \left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000} \\
&= \left(e^{-1}-1\right)^2 \approx 0.3996
\end{aligned}$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure. Then



$$\begin{aligned}
P(X+Y \leq 1000) &= \iint_D f\,(x,y)\; dA \\
&= \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\, dy\, dx \\
&= 10^{-6} \int_0^{1000} \left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000}\right) dx \\
&= -10^{-3} \left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642
\end{aligned}$$

33. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}$ and

$f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$f(x,y) = f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}\,\frac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02} = \tfrac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2}.$$

Then $P(40 \le X \le 50, 20 \le Y \le 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx = \frac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$.

Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50, 20 \le Y \le 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2-4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

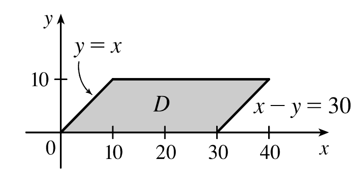
$$\iint_D \tfrac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2}\,dA = \tfrac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}}\int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \le 2) \approx 0.632$.

34. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50} e^{-x} y & \text{if } x \ge 0,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \ge Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \le 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$\begin{aligned} P((X,Y) \in D) &= \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \tfrac{1}{50} e^{-x} y\,dx\,dy \\ &= \tfrac{1}{50}\int_0^{10} y\big[-e^{-x}\big]_{x=y}^{x=y+30}\,dy = \tfrac{1}{50}\int_0^{10} y\big(-e^{-(y+30)} + e^{-y}\big)\,dy \\ &= \tfrac{1}{50}\big(1-e^{-30}\big)\int_0^{10} y e^{-y}\,dy \end{aligned}$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

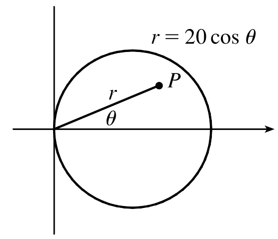
$\frac{1}{50}\big(1-e^{-30}\big)\big[-(y+1)e^{-y}\big]_0^{10} = \frac{1}{50}(1-e^{-30})(1-11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

35. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P, A)\,dA = k\iint_D \tfrac{1}{20}\left[20 - d(P, A)\right]\,dA = k\iint_D \left[1 - \tfrac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$$

(b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k\iint_D \left[1 - \tfrac{1}{20}\sqrt{x^2 + y^2}\right] dA \\ &= k\int_0^{2\pi}\int_0^{10}\left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = 2\pi k\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_0^{10} \\ &= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_{r=0}^{r=20\cos\theta} d\theta \\ &= k\textstyle\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta \\ &= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right] \\ &= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

15.5 Surface Area

1. Here $z = f(x, y) = 10 + x + y^2$ and D is the triangle with vertices $(0, 0)$, $(0, -2)$, and $(2, -2)$. By Formula 2, the area of the surface is

$$\begin{aligned} A(S) &= \textstyle\iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \int_{-2}^0\int_0^{-y}\sqrt{1^2 + (2y)^2 + 1}\,dx\,dy \\ &= \textstyle\int_{-2}^0\int_0^{-y}\sqrt{2 + 4y^2}\,dx\,dy = \int_{-2}^0\sqrt{2 + 4y^2}\,[x]_{x=0}^{x=-y}\,dy = -\int_{-2}^0 y\sqrt{2 + 4y^2}\,dy \\ &= -\frac{1}{8}\cdot\left[\frac{2}{3}(2 + 4y^2)^{3/2}\right]_{-2}^0 = \frac{18^{3/2} - 2^{3/2}}{12} = \frac{54\sqrt{2} - 2\sqrt{2}}{12} = \frac{13\sqrt{2}}{3} \end{aligned}$$

2. Here $z = f(x, y) = 3 + xy$ and D is the circle $x^2 + y^2 \le 1$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \iint_{x^2+y^2\le 1}\sqrt{y^2 + x^2 + 1}\,dA \qquad \text{[Switch to polar coordinates]}$$

$$= \textstyle\int_0^{2\pi}\int_0^1\sqrt{r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^1\sqrt{r^2 + 1}\,r\,dr = 2\pi\cdot\frac{1}{2}\left[\frac{2}{3}(r^2 + 1)^{3/2}\right]_{r=0}^{r=1} = \frac{2\pi}{3}\left(2^{3/2} - 1\right)$$

3. Here $z = f(x,y) = 5x + 3y + 6$ and D is the rectangle $[1,4] \times [2,6]$. By Formula 2, the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{5^2 + 3^2 + 1}\, dA = \sqrt{35} \iint_D dA \\ &= \sqrt{35}\, A(D) = \sqrt{35}\,(4-1)(6-2) = 12\sqrt{35} \end{aligned}$$

4. Here $z = f(x,y) = \frac{1}{2} - 3x - 2y$ and D is the disk $x^2 + y^2 \le 25$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\,(\pi \cdot 5^2) = 25\sqrt{14}\,\pi$$

5. The surface S is given by $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \;\middle|\; 0 \le x \le 2,\, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 2, the surface area of S is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

6. $z = f(x,y) = \frac{1}{4}x^2 - \frac{1}{2}y + \frac{5}{4}$, and D is the triangular region given by $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2x\}$. By Formula 2,

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\tfrac{1}{2}x\right)^2 + \left(-\tfrac{1}{2}\right)^2 + 1}\, dA = \int_0^2 \int_0^{2x} \sqrt{\tfrac{1}{4}x^2 + \tfrac{5}{4}}\, dy\, dx = \int_0^2 \tfrac{1}{2}\sqrt{x^2 + 5}\, \big[\, y\, \big]_{y=0}^{y=2x}\, dx \\ &= \tfrac{1}{2} \int_0^2 2x\sqrt{x^2 + 5}\, dx = \tfrac{1}{2} \cdot \tfrac{2}{3}(x^2 + 5)^{3/2}\Big]_0^2 = \tfrac{1}{3}(9^{3/2} - 5^{3/2}) = 9 - \tfrac{5}{3}\sqrt{5} \end{aligned}$$

7. The paraboloid intersects the plane $z = -2$ when $1 - x^2 - y^2 = -2 \quad\Leftrightarrow\quad x^2 + y^2 = 3$, so $D = \left\{(x,y) \mid x^2 + y^2 \le 3\right\}$. Here $z = f(x,y) = 1 - x^2 - y^2 \quad\Rightarrow\quad f_x = -2x,\ f_y = -2y$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2 + y^2) + 1}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\sqrt{4r^2 + 1}\, dr = \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2 + 1)^{3/2}\right]_0^{\sqrt{3}} = 2\pi \cdot \tfrac{1}{12}\left(13^{3/2} - 1\right) = \tfrac{\pi}{6}\left(13\sqrt{13} - 1\right) \end{aligned}$$

8. $x^2 + z^2 = 4 \quad\Rightarrow\quad z = \sqrt{4 - x^2}$ (since $z \ge 0$), so $f_x = -x(4 - x^2)^{-1/2}$, $f_y = 0$ and

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{[-x(4 - x^2)^{-1/2}]^2 + 0^2 + 1}\; dy\, dx = \int_0^1 \int_0^1 \sqrt{\frac{x^2}{4 - x^2} + 1}\; dy\, dx \\ &= \int_0^1 \frac{2}{\sqrt{4 - x^2}}\, dx \int_0^1 dy = \left[2 \sin^{-1} \frac{x}{2}\right]_0^1 \big[\, y\, \big]_0^1 = \left(2 \cdot \tfrac{\pi}{6} - 0\right)(1) = \tfrac{\pi}{3} \end{aligned}$$

9. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{4r^2 + 1}\, dr \\ &= \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2 + 1)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

10. $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x + y + 1}\, dy\, dx = \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx \\ &= \tfrac{2}{3} \int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx = \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 \\ &= \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

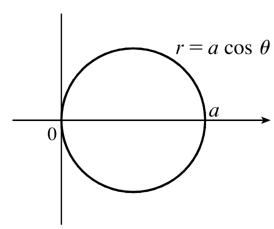
11. $z = f(x,y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

12. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x,y) \mid x^2 + y^2 \le 3\}$ and $z = f(x,y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2 + 4 - r^2}{4 - r^2}}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

13. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,



$$A(S) = \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2 - r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a\right) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{1 - \cos^2\theta}\right) d\theta$$
$$= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2 \int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)$$

14. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \{(x,y) \mid x^2 + y^2 \le 3\}$.

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta$$
$$= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

15. $z = f(x,y) = (1+x^2+y^2)^{-1}$, $f_x = -2x(1+x^2+y^2)^{-2}$, $f_y = -2y(1+x^2+y^2)^{-2}$. Then

$$\begin{aligned} A(S) &= \iint_{x^2+y^2\le 1} \sqrt{[-2x(1+x^2+y^2)^{-2}]^2 + [-2y(1+x^2+y^2)^{-2}]^2 + 1}\, dA \\ &= \iint_{x^2+y^2\le 1} \sqrt{4(x^2+y^2)(1+x^2+y^2)^{-4}+1}\, dA \end{aligned}$$

Converting to polar coordinates we have

$$\begin{aligned} A(S) &= \int_0^{2\pi}\int_0^1 \sqrt{4r^2(1+r^2)^{-4}+1}\; r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{4r^2(1+r^2)^{-4}+1}\,dr \\ &= 2\pi\int_0^1 r\sqrt{4r^2(1+r^2)^{-4}+1}\,dr \approx 3.6258 \text{ using a calculator.} \end{aligned}$$

16. $z = f(x,y) = \cos(x^2+y^2)$, $f_x = -2x\sin(x^2+y^2)$, $f_y = -2y\sin(x^2+y^2)$.

$$A(S) = \iint_{x^2+y^2\le 1} \sqrt{4x^2\sin^2(x^2+y^2) + 4y^2\sin^2(x^2+y^2)+1}\,dA = \iint_{x^2+y^2\le 1}\sqrt{4(x^2+y^2)\sin^2(x^2+y^2)+1}\,dA.$$

Converting to polar coordinates gives

$$\begin{aligned} A(S) &= \int_0^{2\pi}\int_0^1 \sqrt{4r^2\sin^2(r^2)+1}\; r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\,dr \\ &= 2\pi\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\,dr \approx 4.1073 \quad \text{using a calculator.} \end{aligned}$$

17. (a) The midpoints of the four squares are $\left(\frac14,\frac14\right)$, $\left(\frac14,\frac34\right)$, $\left(\frac34,\frac14\right)$, and $\left(\frac34,\frac34\right)$. Here $f(x,y) = x^2+y^2$, so the Midpoint Rule gives

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dA = \iint_D\sqrt{(2x)^2+(2y)^2+1}\,dA \\ &\approx \tfrac14\Bigg(\sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1} + \sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1} \\ &\qquad + \sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1} + \sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1}\Bigg) \\ &= \tfrac14\left(\sqrt{\tfrac32}+2\sqrt{\tfrac72}+\sqrt{\tfrac{11}{2}}\right) \approx 1.8279 \end{aligned}$$

(b) A CAS estimates the integral to be $A(S) = \iint_D \sqrt{1+(2x)^2+(2y)^2}\,dA = \int_0^1\int_0^1\sqrt{1+4x^2+4y^2}\,dy\,dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

18. (a) With $m = n = 2$ we have four squares with midpoints $\left(\frac12,\frac12\right)$, $\left(\frac12,\frac32\right)$, $\left(\frac32,\frac12\right)$, and $\left(\frac32,\frac32\right)$. Since $z = xy+x^2+y^2$, the Midpoint Rule gives

$$\begin{aligned} A(S) &= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D\sqrt{1+(y+2x)^2+(x+2y)^2}\,dA \\ &\approx 1\left(\sqrt{1+\left(\tfrac32\right)^2+\left(\tfrac32\right)^2}+\sqrt{1+\left(\tfrac52\right)^2+\left(\tfrac72\right)^2}+\sqrt{1+\left(\tfrac72\right)^2+\left(\tfrac52\right)^2}+\sqrt{1+\left(\tfrac92\right)^2+\left(\tfrac92\right)^2}\right) \\ &= \tfrac{\sqrt{22}}{2}+\tfrac{\sqrt{78}}{2}+\tfrac{\sqrt{78}}{2}+\tfrac{\sqrt{166}}{2}\approx 17.619 \end{aligned}$$

(b) Using a CAS, we have

$A(S) = \iint_D\sqrt{1+(y+2x)^2+(x+2y)^2}\,dA = \int_0^2\int_0^2\sqrt{1+(y+2x)^2+(x+2y)^2}\,dy\,dx \approx 17.7165$. This is within about 0.1 of the Midpoint Rule estimate.

19. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3 + 8y)^2}\, dy\, dx = \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx.$$

Using a CAS, we have $\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx = \frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \frac{15}{16}\ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$

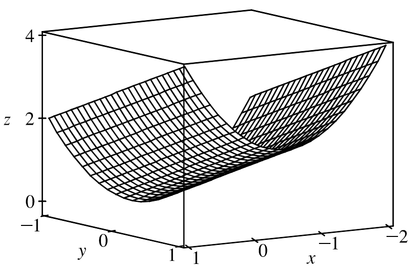
or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\dfrac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

20. $f(x, y) = 1 + x + y + x^2 \quad\Rightarrow\quad f_x = 1 + 2x$, $f_y = 1$. We use a CAS to calculate the integral

$$\begin{aligned} A(S) &= \textstyle\int_{-2}^{1} \int_{-1}^{1} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \\ &= \textstyle\int_{-2}^{1} \int_{-1}^{1} \sqrt{(1 + 2x)^2 + 2}\, dy\, dx = 2\int_{-2}^{1} \sqrt{4x^2 + 4x + 3}\, dx \end{aligned}$$

and find that $A(S) = 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or

$A(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.

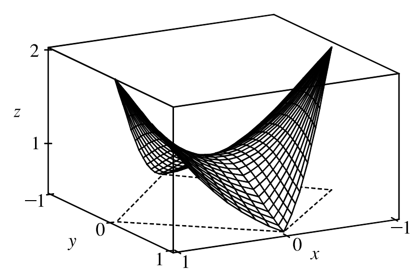


21. $f(x, y) = 1 + x^2y^2 \quad\Rightarrow\quad f_x = 2xy^2$, $f_y = 2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$$A(S) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dy\, dx,$$ and find that $A(S) \approx 3.3213$.

22. Let $f(x, y) = \dfrac{1 + x^2}{1 + y^2}$. Then $f_x = \dfrac{2x}{1 + y^2}$,

$f_y = \left(1 + x^2\right)\left[-\dfrac{2y}{(1 + y^2)^2}\right] = -\dfrac{2y\left(1 + x^2\right)}{(1 + y^2)^2}$. We use a CAS to estimate $\int_{-1}^{1} \int_{-(1-|x|)}^{1-|x|} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1 - |x|) \le y \le 1 - |x|$ as the y-range in our plot command.



23. Here $z = f(x, y) = ax + by + c$, $f_x(x, y) = a$, $f_y(x, y) = b$, so

$A(S) = \iint_D \sqrt{a^2 + b^2 + 1}\, dA = \sqrt{a^2 + b^2 + 1} \iint_D dA = \sqrt{a^2 + b^2 + 1}\, A(D)$.

24. Let S be the upper hemisphere. Then $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, so

$$\begin{aligned} A(S) &= \textstyle\iint_D \sqrt{[-x(a^2 - x^2 - y^2)^{-1/2}]^2 + [-y(a^2 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA \\ &= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA = \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta \\ &= \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta = 2\pi \lim_{t \to a^-} \left[-a\sqrt{a^2 - r^2}\right]_0^t = 2\pi \lim_{t \to a^-} -a\left[\sqrt{a^2 - t^2} - a\right] \\ &= 2\pi(-a)(-a) = 2\pi a^2. \end{aligned}$$ Thus the surface area of the entire sphere is $4\pi a^2$.

25. If we project the surface onto the xz-plane, then the surface lies "above" the disk $x^2 + z^2 \le 25$ in the xz-plane.

We have $y = f(x, z) = x^2 + z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2+z^2\le 25} \sqrt{[f_x(x,z)]^2 + [f_z(x,z)]^2 + 1}\, dA = \iint_{x^2+z^2\le 25} \sqrt{4x^2 + 4z^2 + 1}\, dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$A(S) = \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^5 r(4r^2+1)^{1/2}\, dr = \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^5 = \tfrac{\pi}{6}\left(101\sqrt{101} - 1\right)$$

26. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 25, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2 + z^2 \le 1$. Then $y = f(x, z) = \sqrt{1 - z^2}$ and the area is

$$\begin{aligned} A(S) &= \iint_{x^2+z^2\le 1} \sqrt{[f_x(x,z)]^2 + [f_z(x,z)]^2 + 1}\, dA = \iint_{x^2+z^2\le 1} \sqrt{0 + \left(\frac{-z}{\sqrt{1-z^2}}\right)^2 + 1}\, dA \\ &= \iint_{x^2+z^2\le 1} \sqrt{\frac{z^2}{1-z^2} + 1}\, dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz \\ &= 4\int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz \qquad \text{[by the symmetry of the surface]} \end{aligned}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\, dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

15.6 Triple Integrals

1. $$\begin{aligned} \iiint_B xyz^2\, dV &= \int_0^1\int_0^3\int_{-1}^2 xyz^2\, dy\, dz\, dx = \int_0^1\int_0^3 \left[\tfrac12 xy^2z^2\right]_{y=-1}^{y=2} dz\, dx = \int_0^1\int_0^3 \tfrac32 xz^2\, dz\, dx \\ &= \int_0^1 \left[\tfrac12 xz^3\right]_{z=0}^{z=3} dx = \int_0^1 \tfrac{27}{2}\, x\, dx = \tfrac{27}{4}x^2\Big]_0^1 = \tfrac{27}{4} \end{aligned}$$

2. There are six different possible orders of integration.

$$\begin{aligned} \iiint_E (xy + z^2)\, dV &= \int_0^2\int_0^1\int_0^3 (xy+z^2)\, dz\, dy\, dx = \int_0^2\int_0^1 \left[xyz + \tfrac13 z^3\right]_{z=0}^{z=3} dy\, dx = \int_0^2\int_0^1 (3xy+9)\, dy\, dx \\ &= \int_0^2 \left[\tfrac32 xy^2 + 9y\right]_{y=0}^{y=1} dx = \int_0^2 \left(\tfrac32 x + 9\right) dx = \left[\tfrac34 x^2 + 9x\right]_0^2 = 21 \end{aligned}$$

$$\begin{aligned} \iiint_E (xy + z^2)\, dV &= \int_0^1\int_0^2\int_0^3 (xy+z^2)\, dz\, dx\, dy = \int_0^1\int_0^2 \left[xyz + \tfrac13 z^3\right]_{z=0}^{z=3} dx\, dy = \int_0^1\int_0^2 (3xy+9)\, dx\, dy \\ &= \int_0^1 \left[\tfrac32 x^2y + 9x\right]_{x=0}^{x=2} dy = \int_0^1 (6y+18)\, dy = \left[3y^2 + 18y\right]_0^1 = 21 \end{aligned}$$

$$\begin{aligned} \iiint_E (xy + z^2)\, dV &= \int_0^2\int_0^3\int_0^1 (xy+z^2)\, dy\, dz\, dx = \int_0^2\int_0^3 \left[\tfrac12 xy^2 + yz^2\right]_{y=0}^{y=1} dz\, dx = \int_0^2\int_0^3 \left(\tfrac12 x + z^2\right) dz\, dx \\ &= \int_0^2 \left[\tfrac12 xz + \tfrac13 z^3\right]_{z=0}^{z=3} dx = \int_0^2 \left(\tfrac32 x + 9\right) dx = \left[\tfrac34 x^2 + 9x\right]_0^2 = 21 \end{aligned}$$

[continued]

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^2\int_0^1 (xy+z^2)\,dy\,dx\,dz = \int_0^3\int_0^2 \left[\tfrac{1}{2}xy^2+yz^2\right]_{y=0}^{y=1} dx\,dz = \int_0^3\int_0^2 \left(\tfrac{1}{2}x+z^2\right) dx\,dz$$
$$= \int_0^3 \left[\tfrac{1}{4}x^2+xz^2\right]_{x=0}^{x=2} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^1\int_0^3\int_0^2 (xy+z^2)\,dx\,dz\,dy = \int_0^1\int_0^3 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dz\,dy = \int_0^1\int_0^3 \left(2y+2z^2\right) dz\,dy$$
$$= \int_0^1 \left[2yz+\tfrac{2}{3}z^3\right]_{z=0}^{z=3} dy = \int_0^1 (6y+18)\,dy = \left[3y^2+18y\right]_0^1 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^1\int_0^2 (xy+z^2)\,dx\,dy\,dz = \int_0^3\int_0^1 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dy\,dz = \int_0^3\int_0^1 \left(2y+2z^2\right) dy\,dz$$
$$= \int_0^3 \left[y^2+2yz^2\right]_{y=0}^{y=1} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

3. $$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz = \int_0^2\int_0^{z^2}\left[x^2-xy\right]_{x=0}^{x=y-z} dy\,dz = \int_0^2\int_0^{z^2}\left[(y-z)^2-(y-z)y\right] dy\,dz$$
$$= \int_0^2\int_0^{z^2}\left(z^2-yz\right) dy\,dz = \int_0^2 \left[yz^2-\tfrac{1}{2}y^2z\right]_{y=0}^{y=z^2} dz = \int_0^2\left(z^4-\tfrac{1}{2}z^5\right) dz$$
$$= \left[\tfrac{1}{5}z^5-\tfrac{1}{12}z^6\right]_0^2 = \tfrac{32}{5}-\tfrac{64}{12} = \tfrac{16}{15}$$

4. $$\int_0^1\int_y^{2y}\int_0^{x+y} 6xy\,dz\,dx\,dy = \int_0^1\int_y^{2y}\left[6xyz\right]_{z=0}^{z=x+y} dx\,dy = \int_0^1\int_y^{2y} 6xy(x+y)\,dx\,dy = \int_0^1\int_y^{2y}(6x^2y+6xy^2)\,dx\,dy$$
$$= \int_0^1 \left[2x^3y+3x^2y^2\right]_{x=y}^{x=2y} dy = \int_0^1 23y^4\,dy = \tfrac{23}{5}y^5\Big]_0^1 = \tfrac{23}{5}$$

5. $$\int_1^2\int_0^{2z}\int_0^{\ln x} xe^{-y}\,dy\,dx\,dz = \int_1^2\int_0^{2z}\left[-xe^{-y}\right]_{y=0}^{y=\ln x} dx\,dz = \int_1^2\int_0^{2z}\left(-xe^{-\ln x}+xe^0\right) dx\,dz$$
$$= \int_1^2\int_0^{2z}(-1+x)\,dx\,dz = \int_1^2\left[-x+\tfrac{1}{2}x^2\right]_{x=0}^{x=2z} dz$$
$$= \int_1^2\left(-2z+2z^2\right) dz = \left[-z^2+\tfrac{2}{3}z^3\right]_1^2 = -4+\tfrac{16}{3}+1-\tfrac{2}{3} = \tfrac{5}{3}$$

6. $$\int_0^{\pi/2}\int_0^{2x}\int_0^{x+z}\cos(x-2y+z)\,dy\,dz\,dx = -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[\sin(x-2y+z)\right]_{y=0}^{y=x+z} dz\,dx$$
$$= -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[\sin(-x-z)-\sin(x+z)\right] dz\,dx$$
$$= -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[-2\sin(x+z)\right] dz\,dx$$
$$= \int_0^{\pi/2}\int_0^{2x}\sin(x+z)\,dz\,dx = -\int_0^{\pi/2}\left[\cos(x+z)\right]_{z=0}^{z=2x} dx$$
$$= -\int_0^{\pi/2}(\cos 3x-\cos x)\,dx = \left[\sin x-\tfrac{1}{3}\sin 3x\right]_0^{\pi/2}$$
$$= 1-\tfrac{1}{3}(-1) = \tfrac{4}{3}$$

7. $$\int_1^3\int_{-1}^2\int_{-y}^z \frac{z}{y}\,dx\,dz\,dy = \int_1^3\int_{-1}^2 \frac{z}{y}\Big[x\Big]_{x=-y}^{x=z} dz\,dy = \int_1^3\int_{-1}^2\left(\frac{z^2}{y}+z\right) dz\,dy = \int_1^3\left[\frac{z^3}{3y}+\frac{z^2}{2}\right]_{z=-1}^{z=2} dy$$
$$= \int_1^3\left(\frac{3}{y}+\frac{3}{2}\right) dy = \left[3\ln|y|+\frac{3}{2}y\right]_1^3 = 3\ln 3+3$$

8. $$\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dz\,dy\,dx = \int_0^1\int_0^1\left[xye^z\right]_{z=0}^{z=2-x^2-y^2} dy\,dx = \int_0^1\int_0^1(xye^{2-x^2-y^2}-xy)\,dy\,dx$$
$$= \int_0^1\left[-\tfrac{1}{2}xe^{2-x^2-y^2}-\tfrac{1}{2}xy^2\right]_{y=0}^{y=1} dx = \int_0^1\left(-\tfrac{1}{2}xe^{1-x^2}-\tfrac{1}{2}x+\tfrac{1}{2}xe^{2-x^2}\right) dx$$
$$= \left[\tfrac{1}{4}e^{1-x^2}-\tfrac{1}{4}x^2-\tfrac{1}{4}e^{2-x^2}\right]_0^1 = \tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}e-\tfrac{1}{4}e+0+\tfrac{1}{4}e^2 = \tfrac{1}{4}e^2-\tfrac{1}{2}e$$

9. (a) The solid region E can be described as $E = \{(x, y, z) \mid -1 \le x \le 1, 0 \le y \le 2 - z, 0 \le z \le 1 - x^2\}$.

Thus, $\iiint_E x\, dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} x\, dy\, dz\, dx$.

(b) $$\int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} x\, dy\, dz\, dx = \int_{-1}^{1} \int_0^{1-x^2} x \Big[y \Big]_{y=0}^{y=2-z} dz\, dx = \int_{-1}^{1} \int_0^{1-x^2} (2x - xz)\, dz\, dx$$

$$= \int_{-1}^{1} \left[2xz - x\frac{z^2}{2} \right]_{z=0}^{z=1-x^2} dx = \int_{-1}^{1} \left(\frac{3}{2}x - x^3 - \frac{x^5}{2} \right) dx$$

$$= \left[\frac{3x^2}{4} - \frac{x^4}{4} - \frac{x^6}{12} \right]_{-1}^{1} = 0$$

10. (a) The solid region E can be described as $E = \{(x, y, z) \mid 0 \le x \le y, 0 \le y \le 2, 0 \le z \le 4 - y^2\}$.

Thus, $\iiint_E xy\, dV = \int_0^2 \int_0^y \int_0^{4-y^2} xy\, dz\, dx\, dy$.

(b) $\int_0^2 \int_0^y \int_0^{4-y^2} xy\, dz\, dx\, dy = \int_0^2 \int_0^y xy \,[z]_{z=0}^{z=4-y^2}\, dx\, dy = \int_0^2 \int_0^y xy(4 - y^2)\, dx\, dy = \int_0^2 \int_0^y x(4y - y^3)\, dx\, dy$

$= \int_0^2 (4y - y^3) \left[\frac{x^2}{2} \right]_{x=0}^{x=y} dy = \frac{1}{2} \int_0^2 (4y^3 - y^5)\, dy = \frac{1}{2} \left[y^4 - \frac{y^6}{6} \right]_0^2 = \frac{8}{3}$

11. (a) The solid region E can be described as $E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le x^2, 0 \le z \le 2 - x\}$.

Thus, $\iiint_E (x + y)\, dV = \int_0^2 \int_0^{2-x} \int_0^{x^2} (x + y)\, dy\, dz\, dx$.

(b) $$\int_0^2 \int_0^{2-x} \int_0^{x^2} (x + y)\, dy\, dz\, dx = \int_0^2 \int_0^{2-x} \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x^2} dz\, dx = \int_0^2 \int_0^{2-x} \left(x^3 + \frac{x^4}{2} \right) dz\, dx$$

$$= \int_0^2 \left(x^3 + \frac{x^4}{2} \right) \Big[z \Big]_{z=0}^{z=2-x} dx = \int_0^2 \left(2x^3 - \frac{x^5}{2} \right) dx = \left[\frac{x^4}{2} - \frac{x^6}{12} \right]_0^2 = \frac{8}{3}$$

12. (a) The solid region E can be described as $E = \{(x, y, z) \mid z - 4 \le x \le 4 - z, -2 \le y \le 2, 0 \le z \le 4 - y^2\}$.

Thus, $\iiint_E 2\, dV = \int_{-2}^{2} \int_0^{4-y^2} \int_{z-4}^{4-z} 2\, dx\, dz\, dy$.

(b) $\int_{-2}^{2} \int_0^{4-y^2} \int_{z-4}^{4-z} 2\, dx\, dz\, dy = \int_{-2}^{2} \int_0^{4-y^2} 2\Big[x \Big]_{x=z-4}^{x=4-z} dz\, dy = 2\int_{-2}^{2} \int_0^{4-y^2} (8 - 2z)\; dz\, dy$

$= 2\int_{-2}^{2} \Big[8z - z^2 \Big]_{z=0}^{z=4-y^2} dy = 2\int_{-2}^{2} (16 - y^4)\, dy = 2\Big[16y - \frac{y^5}{5} \Big]_{-2}^{2} = \frac{512}{5}$

13. $\iiint_E y\, dV = \int_0^3 \int_0^x \int_{x-y}^{x+y} y\, dz\, dy\, dx = \int_0^3 \int_0^x \,[yz]_{z=x-y}^{z=x+y}\, dy\, dx = \int_0^3 \int_0^x 2y^2\, dy\, dx$

$= \int_0^3 \left[\frac{2}{3} y^3 \right]_{y=0}^{y=x} dx = \int_0^3 \frac{2}{3} x^3\, dx = \frac{1}{6} x^4 \Big]_0^3 = \frac{81}{6} = \frac{27}{2}$

14. $\iiint_E e^{z/y}\, dV = \int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\, dz\, dx\, dy = \int_0^1 \int_y^1 \left[ye^{z/y} \right]_{z=0}^{z=xy} dx\, dy$

$= \int_0^1 \int_y^1 (ye^x - y)\, dx\, dy = \int_0^1 \,[ye^x - xy]_{x=y}^{x=1}\, dy = \int_0^1 \left(ey - y - ye^y + y^2 \right) dy$

$= \left[\frac{1}{2}ey^2 - \frac{1}{2}y^2 - (y - 1)e^y + \frac{1}{3}y^3 \right]_0^1$ [integrate by parts]

$= \frac{1}{2}e - \frac{1}{2} + \frac{1}{3} - 1 = \frac{1}{2}e - \frac{7}{6}$

15. $$\iiint_E \frac{1}{x^3}\,dV = \int_0^1 \int_0^{y^2} \int_1^{z+1} \frac{1}{x^3}\,dx\,dz\,dy = \int_0^1 \int_0^{y^2} -\frac{1}{2}\left[\frac{1}{x^2}\right]_{x=1}^{x=z+1} dz\,dy$$
$$= -\frac{1}{2}\int_0^1 \int_0^{y^2} \left(\frac{1}{(z+1)^2} - 1\right) dz\,dy = -\frac{1}{2}\int_0^1 \left[-\frac{1}{z+1} - z\right]_{z=0}^{z=y^2} dy$$
$$= \frac{1}{2}\int_0^1 \left(\frac{1}{y^2+1} + y^2 - 1\right) dy = \frac{1}{2}\left[\tan^{-1} y + \frac{y^3}{3} - y\right]_0^1 = \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{3} - 1\right) - 0\right]$$
$$= \frac{\pi}{8} - \frac{1}{3}$$

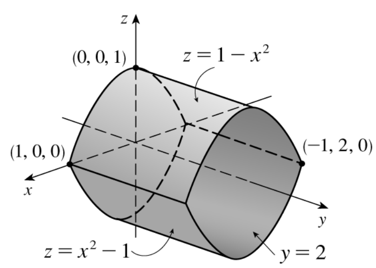
16. Here $E = \{(x,y,z) \mid 0 \le x \le \pi, 0 \le y \le \pi - x, 0 \le z \le x\}$, so

$$\iiint_E \sin y\,dV = \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y\,dz\,dy\,dx = \int_0^\pi \int_0^{\pi-x} \left[z \sin y\right]_{z=0}^{z=x} dy\,dx = \int_0^\pi \int_0^{\pi-x} x \sin y\,dy\,dx$$
$$= \int_0^\pi \left[-x\cos y\right]_{y=0}^{y=\pi-x} dx = \int_0^\pi \left[-x\cos(\pi - x) + x\right] dx$$
$$= \left[x\sin(\pi - x) - \cos(\pi - x) + \tfrac{1}{2}x^2\right]_0^\pi \qquad \text{[integrate by parts]}$$
$$= 0 - 1 + \tfrac{1}{2}\pi^2 - 0 - 1 - 0 = \tfrac{1}{2}\pi^2 - 2$$

17. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$
$$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \tfrac{3}{4}x^4 + \tfrac{4}{7}x^{7/2}\right]_0^1 = \tfrac{65}{28}$$

18.

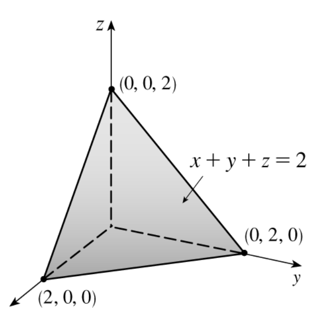


Here $E = \left\{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 2,\ x^2 - 1 \le z \le 1 - x^2\right\}$.

Thus,

$$\iiint_E (x - y)\,dV = \int_{-1}^1 \int_0^2 \int_{x^2-1}^{1-x^2} (x-y)\,dz\,dy\,dx$$
$$= \int_{-1}^1 \int_0^2 (x-y)(1 - x^2 - (x^2 - 1))\,dy\,dx$$
$$= \int_{-1}^1 \int_0^2 (2x - 2x^3 - 2y + 2x^2y)\,dy\,dx$$
$$= \int_{-1}^1 \left[2xy - 2x^3y - y^2 + x^2y^2\right]_{y=0}^{y=2} dx$$
$$= \int_{-1}^1 (4x - 4x^3 - 4 + 4x^2)\,dx$$
$$= \left[2x^2 - x^4 - 4x + \tfrac{4}{3}x^3\right]_{-1}^1 = -\tfrac{5}{3} - \tfrac{11}{3} = -\tfrac{16}{3}$$

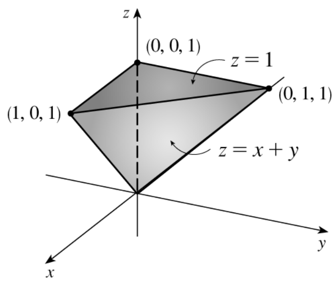
19.



Here $T = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2 - x,\ 0 \le z \le 2 - x - y\}$. Thus,

$$\iiint_T y^2\,dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y^2\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} y^2(2 - x - y)\,dy\,dx$$
$$= \int_0^2 \int_0^{2-x} \left[(2-x)y^2 - y^3\right] dy\,dx$$
$$= \int_0^2 \left[(2-x)\left(\tfrac{1}{3}y^3\right) - \tfrac{1}{4}y^4\right]_{y=0}^{y=2-x} dx$$
$$= \int_0^2 \left[\tfrac{1}{3}(2-x)^4 - \tfrac{1}{4}(2-x)^4\right] dx = \int_0^2 \tfrac{1}{12}(2-x)^4\,dx$$
$$= \left[\tfrac{1}{12}\left(-\tfrac{1}{5}\right)(2-x)^5\right]_0^2 = -\tfrac{1}{60}(0 - 32) = \tfrac{8}{15}$$

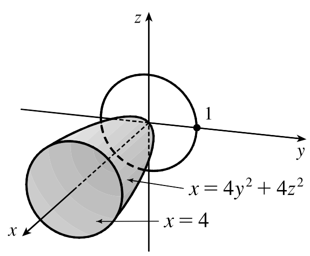
20.



The projection of T onto the xz-plane is the triangle bounded by the lines $z = x$, $x = 0$, and $z = 1$. Then $T = \{(x, y, z) \mid 0 \le x \le 1,\ x \le z \le 1,\ 0 \le y \le z - x\}$, and

$$\iiint_T xz\,dV = \int_0^1 \int_x^1 \int_0^{z-x} xz\,dy\,dz\,dx = \int_0^1 \int_x^1 xz(z-x)\,dz\,dx$$
$$= \int_0^1 \int_x^1 (xz^2 - x^2 z)\,dz\,dx = \int_0^1 \left[\tfrac{1}{3}xz^3 - \tfrac{1}{2}x^2z^2\right]_{z=x}^{z=1} dx$$
$$= \int_0^1 \left(\tfrac{1}{3}x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{2}x^4\right) dx$$
$$= \left[\tfrac{1}{6}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{30}x^5\right]_0^1 = \tfrac{1}{6} - \tfrac{1}{6} + \tfrac{1}{30} = \tfrac{1}{30}$$

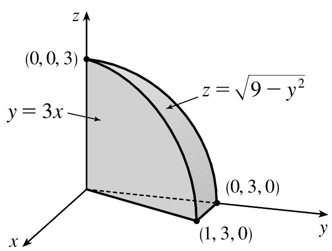
21.



The projection of E onto the yz-plane is the disk $y^2 + z^2 \le 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

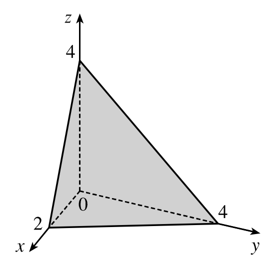
$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx\right] dA = \tfrac{1}{2}\iint_D \left[4^2 - (4y^2 + 4z^2)^2\right] dA$$
$$= 8\int_0^{2\pi}\int_0^1 (1 - r^4)\,r\,dr\,d\theta = 8\int_0^{2\pi} d\theta \int_0^1 (r - r^5)\,dr$$
$$= 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

22.



$$\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1 \int_{3x}^3 \tfrac{1}{2}(9 - y^2)\,dy\,dx$$
$$= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$
$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx$$
$$= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

23. The plane $2x + y + z = 4$ intersects the xy-plane when $2x + y + 0 = 4 \;\Rightarrow\; y = 4 - 2x$, so $E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and



$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx$$
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$
$$= \int_0^2 \left[4(4 - 2x) - 2x(4 - 2x) - \tfrac{1}{2}(4 - 2x)^2\right] dx$$
$$= \int_0^2 (2x^2 - 8x + 8)\,dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$

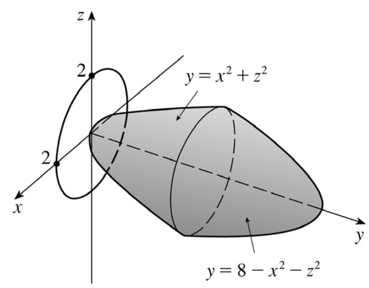
24. The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2 \;\Leftrightarrow\; x^2 + z^2 = 4$, thus the intersection is the circle $x^2 + z^2 = 4$, $y = 4$. The projection of E onto the xz-plane is the disk $x^2 + z^2 \le 4$, so $E = \left\{(x, y, z) \mid x^2 + z^2 \le y \le 8 - x^2 - z^2, x^2 + z^2 \le 4\right\}$. Let $D = \left\{(x, z) \mid x^2 + z^2 \le 4\right\}$. Then

using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have

$V = \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right) dA = \iint_D (8 - 2x^2 - 2z^2)\,dA$

$= \int_0^{2\pi}\int_0^2 (8 - 2r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\,dr$

$= \big[\theta\big]_0^{2\pi} \left[4r^2 - \frac{1}{2}r^4\right]_0^2 = 2\pi(16 - 8) = 16\pi$



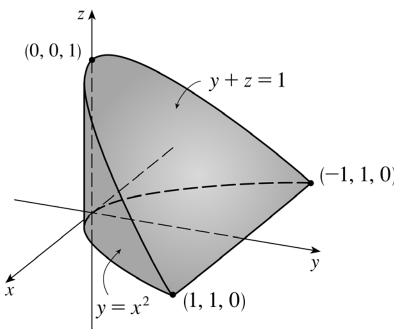
25. The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so

$E = \left\{(x, y, z) \mid -1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\right\}$ and

$V = \iiint_E dV = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx = \int_{-1}^1 \int_{x^2}^1 (1 - y)\,dy\,dx$

$= \int_{-1}^1 \left[y - \frac{1}{2}y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^1 \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_{-1}^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{8}{15}$



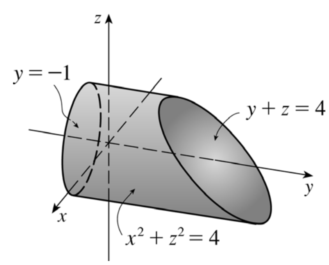
26. Here $E = \left\{(x, y, z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\right\}$, so

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-1}^{4-z} dy\,dz\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\,dz\,dx$$

$$= \int_{-2}^{2}\left[5z - \tfrac{1}{2}z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^{2} 10\sqrt{4 - x^2}\,dx$$

$= 10\left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^2$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 20\pi$



Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5 - z)\,dz\,dx = \int_0^{2\pi}\int_0^2 (5 - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{5}{2}r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(10 - \tfrac{8}{3}\sin\theta\right) d\theta$$

$$= 10\theta + \tfrac{8}{3}\cos\theta\Big]_0^{2\pi} = 20\pi$$

27. (a) The wedge can be described as the region

$D = \left\{(x, y, z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\right\}$

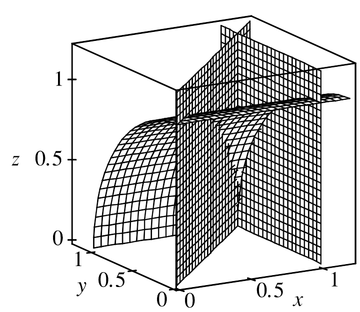
$= \left\{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1 - y^2}\right\}$

So the integral expressing the volume of the wedge is

$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx.$

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)



28. Divide B into 8 cubes of size $\Delta V = 8$. With $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, the Midpoint Rule gives

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+z^2}\,dV &\approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= 8[f(1,1,1)+f(1,1,3)+f(1,3,1)+f(1,3,3)+f(3,1,1) \\ &\qquad + f(3,1,3)+f(3,3,1)+f(3,3,3)] \\ &\approx 239.64\end{aligned}$$

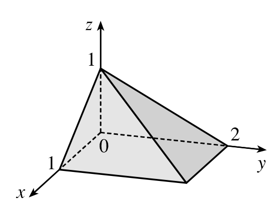
29. Here $f(x,y,z) = \cos(xyz)$ and $\Delta V = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= \tfrac{1}{8}\left[f\left(\tfrac14,\tfrac14,\tfrac14\right)+f\left(\tfrac14,\tfrac14,\tfrac34\right)+f\left(\tfrac14,\tfrac34,\tfrac14\right)+f\left(\tfrac14,\tfrac34,\tfrac34\right)\right. \\ &\qquad \left.+f\left(\tfrac34,\tfrac14,\tfrac14\right)+f\left(\tfrac34,\tfrac14,\tfrac34\right)+f\left(\tfrac34,\tfrac34,\tfrac14\right)+f\left(\tfrac34,\tfrac34,\tfrac34\right)\right] \\ &= \tfrac{1}{8}\left[\cos\tfrac{1}{64}+\cos\tfrac{3}{64}+\cos\tfrac{3}{64}+\cos\tfrac{9}{64}+\cos\tfrac{3}{64}+\cos\tfrac{9}{64}+\cos\tfrac{9}{64}+\cos\tfrac{27}{64}\right] \approx 0.985\end{aligned}$$

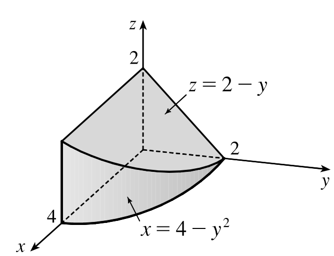
30. Here $f(x,y,z) = \sqrt{x}\,e^{xyz}$ and $\Delta V = 2\cdot\frac{1}{2}\cdot 1 = 1$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= 1\left[f\left(1,\tfrac14,\tfrac12\right)+f\left(1,\tfrac14,\tfrac32\right)+f\left(1,\tfrac34,\tfrac12\right)+f\left(1,\tfrac34,\tfrac32\right)\right. \\ &\qquad \left.+f\left(3,\tfrac14,\tfrac12\right)+f\left(3,\tfrac14,\tfrac32\right)+f\left(3,\tfrac34,\tfrac12\right)+f\left(3,\tfrac34,\tfrac32\right)\right] \\ &= e^{1/8}+e^{3/8}+e^{3/8}+e^{9/8}+\sqrt{3}e^{3/8}+\sqrt{3}e^{9/8}+\sqrt{3}e^{9/8}+\sqrt{3}e^{27/8} \approx 70.932\end{aligned}$$

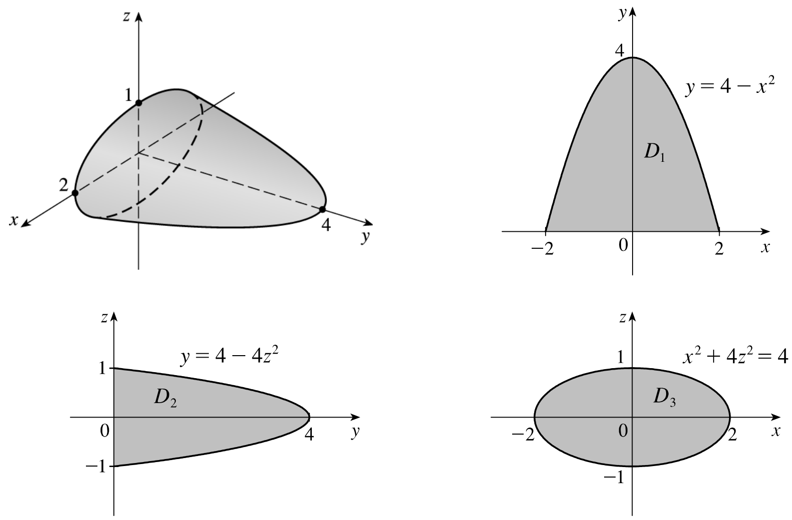
31. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1-x, 0 \le y \le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1-x$, $y = 2-2z$.



32. $E = \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le 2-y, 0 \le x \le 4-y^2\right\}$, the solid bounded by the three coordinate planes, the plane $z = 2-y$, and the cylindrical surface $x = 4-y^2$.



33.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\right\} = \left\{(x,y) \mid 0 \le y \le 4, \ -\sqrt{4-y} \le x \le \sqrt{4-y}\,\right\}$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, \ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\right\} = \left\{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\right\}$$

$$D_3 = \left\{(x,z) \mid x^2 + 4z^2 \le 4\right\}$$

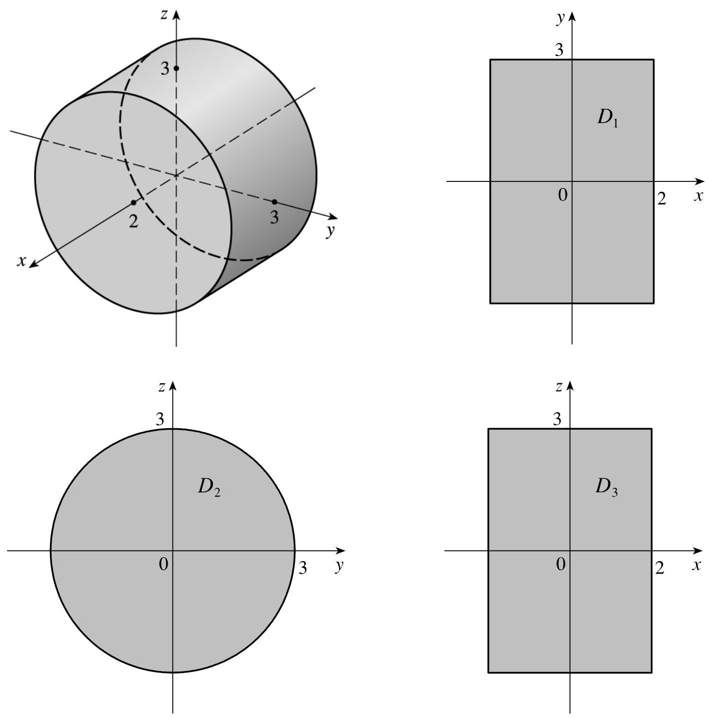
Therefore

$$\begin{aligned}
E &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2, \ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\,\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4, \ -\sqrt{4-y} \le x \le \sqrt{4-y}, \ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\,\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2, \ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\,\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4, \ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}, \ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\,\right\} \\
&= \left\{(x,y,z) \mid -2 \le x \le 2, \ -\tfrac{1}{2}\sqrt{4 - x^2} \le z \le \tfrac{1}{2}\sqrt{4 - x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1, \ -\sqrt{4 - 4z^2} \le x \le \sqrt{4 - 4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dx\,dy = \int_{-1}^{1}\int_0^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^4\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx \\
&= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

34.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -2 \le x \le 2,\ -3 \le y \le 3\}$$

$$D_2 = \left\{(y,z) \mid y^2 + z^2 \le 9\right\}$$

$$D_3 = \{(x,z) \mid -2 \le x \le 2,\ -3 \le z \le 3\}$$

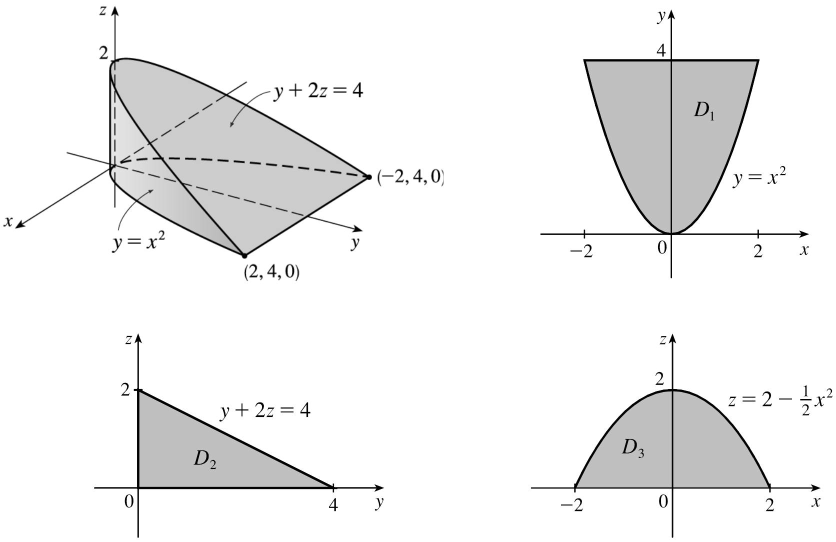
Therefore

$$\begin{aligned}
E &= \left\{(x,y,z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2}\right\} \\
&= \left\{(x,y,z) \mid -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x,y,z) \mid -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x,y,z) \mid -2 \le x \le 2,\ -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dx\,dy \\
&= \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dz\,dy = \int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dy\,dz \\
&= \int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx = \int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

35.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, x^2 \le y \le 4\right\} = \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}\right\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} = \left\{(y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z\right\}, \text{ and}$$

$$D_3 = \left\{(x,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2\right\} = \left\{(x,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}\right\}$$

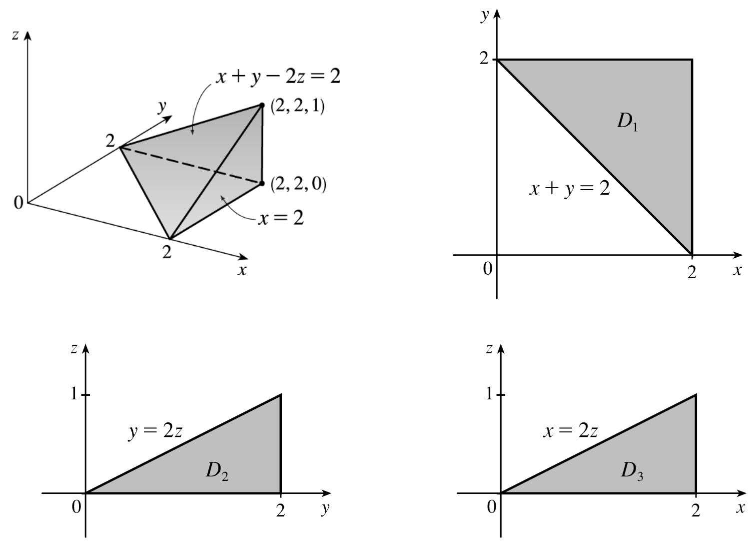
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{y} \le x \le \sqrt{y}, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y,\ -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z,\ -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2, x^2 \le y \le 4 - 2z\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2,\ -\sqrt{4-2z} \le x \le \sqrt{4-2z}, x^2 \le y \le 4 - 2z\right\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy = \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx = \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

36.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid 0 \le x \le 2, 2 - x \le y \le 2\} = \{(x, y) \mid 0 \le y \le 2, 2 - y \le x \le 2\},$$

$$D_2 = \left\{(y, z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y\right\} = \{(y, z) \mid 0 \le z \le 1, 2z \le y \le 2\}, \text{ and}$$

$$D_3 = \left\{(x, z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x\right\} = \{(x, z) \mid 0 \le z \le 1, 2z \le x \le 2\}$$

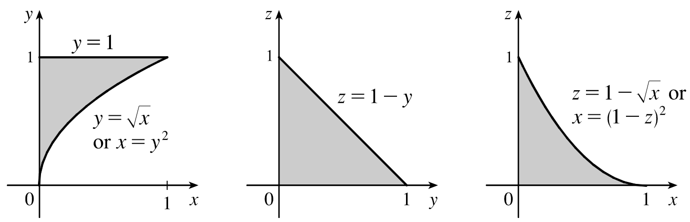
Therefore

$$\begin{aligned}
E &= \left\{(x, y, z) \mid 0 \le x \le 2, 2 - x \le y \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\
&= \left\{(x, y, z) \mid 0 \le y \le 2, 2 - y \le x \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\
&= \left\{(x, y, z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y, 2 - y + 2z \le x \le 2\right\} \\
&= \{(x, y, z) \mid 0 \le z \le 1, 2z \le y \le 2, 2 - y + 2z \le x \le 2\} \\
&= \left\{(x, y, z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x, 2 - x + 2z \le y \le 2\right\} \\
&= \{(x, y, z) \mid 0 \le z \le 1, 2z \le x \le 2, 2 - x + 2z \le y \le 2\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \int_0^2 \int_{2-x}^2 \int_0^{(x+y-2)/2} f(x, y, z)\, dz\, dy\, dx \\
&= \int_0^2 \int_{2-y}^2 \int_0^{(x+y-2)/2} f(x, y, z)\, dz\, dx\, dy \\
&= \int_0^2 \int_0^{y/2} \int_{2-y+2z}^2 f(x, y, z)\, dx\, dz\, dy \\
&= \int_0^1 \int_{2z}^2 \int_{2-y+2z}^2 f(x, y, z)\, dx\, dy\, dz \\
&= \int_0^2 \int_0^{x/2} \int_{2-x+2z}^2 f(x, y, z)\, dy\, dz\, dx \\
&= \int_0^1 \int_{2z}^2 \int_{2-x+2z}^2 f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$

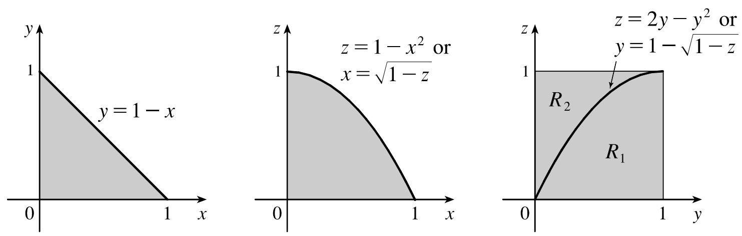
37.



The diagrams show the projections of E onto the xy-, yz-, and xz-planes. Therefore

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

38.



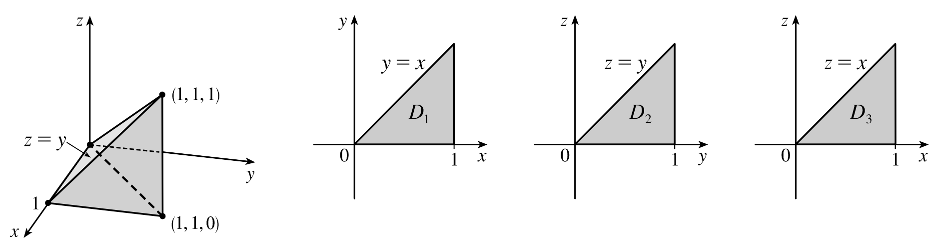
The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$
$$= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y, z) in R_1, $0 \le x \le 1 - y$ and for (y, z) in R_2, $0 \le x \le \sqrt{1 - z}$, and so the given integral is also equal to

$$\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

39.



$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E onto the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\},$$

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, and

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

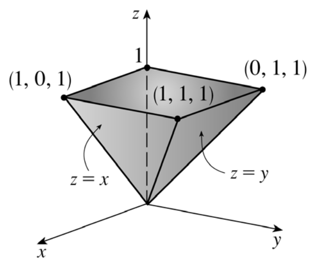
Thus we also have

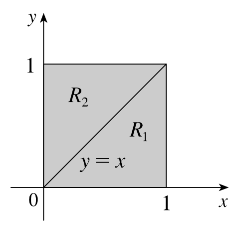
$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}. \end{aligned}$$

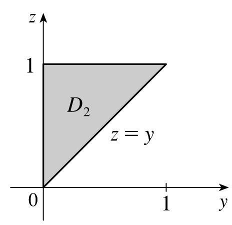
Then

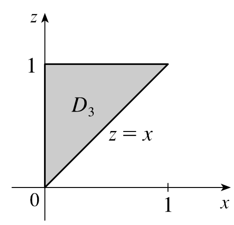
$$\begin{aligned} \int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1\int_z^1\int_z^x f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

40.









$\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le z, y \le z \le 1, 0 \le y \le 1\}$.

Notice that E is bounded below by two different surfaces, so we must split the projection of E onto the xy-plane into two regions as in the second diagram. If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$\begin{aligned} D_1 = R_1 \cup R_2 &= \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\} \\ &= \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} \cup \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\}, \end{aligned}$$

$D_2 = \{(y,z) \mid 0 \le y \le 1, y \le z \le 1\} = \{(y,z) \mid 0 \le z \le 1, 0 \le y \le z\}$, and

$$D_3 = \{(x,z) \mid 0 \le x \le 1, x \le z \le 1\} = \{(x,z) \mid 0 \le z \le 1, 0 \le x \le z\}.$$

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le x \le 1, x \le y \le 1, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le y \le 1, 0 \le x \le y, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le y \le z, 0 \le x \le z\} = \{(x,y,z) \mid 0 \le x \le 1, x \le z \le 1, 0 \le y \le z\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le z, 0 \le y \le z\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy &= \int_0^1\int_0^x\int_x^1 f(x,y,z)\,dz\,dy\,dx + \int_0^1\int_x^1\int_y^1 f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^1\int_y^1\int_x^1 f(x,y,z)\,dz\,dx\,dy + \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^1\int_0^z\int_0^z f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_x^1\int_0^z f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1\int_0^z\int_0^z f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

41. The region C is the solid bounded by a circular cylinder of radius 2 with axis the z-axis for $-2 \le z \le 2$. We can write $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV + \iiint_C 5x^2yz^2\,dV$, but $f(x,y,z) = 5x^2yz^2$ is an odd function with respect to y. Since C is symmetrical about the xz-plane, we have $\iiint_C 5x^2yz^2\,dV = 0$. Thus $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV = 4 \cdot V(E) = 4 \cdot \pi(2)^2(4) = 64\pi$.

42. We can write $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B z^3\,dV + \iiint_B \sin y\,dV + \iiint_B 3\,dV$. But z^3 is an odd function with respect to z and the region B is symmetric about the xy-plane, so $\iiint_B z^3\,dV = 0$. Similarly, $\sin y$ is an odd function with respect to y and B is symmetric about the xz-plane, so $\iiint_B \sin y\,dV = 0$. Thus $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B 3\,dV = 3 \cdot V(B) = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi$.

43. The projection of E onto the xy-plane is the disk $D = \{(x,y) \mid x^2 + y^2 \le 1\}$.

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3\,dz\right] dA = \iint_D 3(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr \\ &= 3\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = 3\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{3}{2}\pi \end{aligned}$$

$$\begin{aligned} M_{yz} &= \iiint_E x\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3x\,dz\right] dA = \iint_D 3x(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\cos\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \cos\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[\sin\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xz} &= \iiint_E y\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3y\,dz\right] dA = \iint_D 3y(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\sin\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \sin\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[-\cos\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3z\,dz\right] dA = \iint_D \left[\tfrac{3}{2}z^2\right]_{z=0}^{z=1-x^2-y^2} dA \\ &= \tfrac{3}{2}\iint_D (1 - x^2 - y^2)^2\,dA = \tfrac{3}{2}\int_0^1 \int_0^{2\pi} (1 - r^2)^2\,r\,dr\,d\theta \\ &= \tfrac{3}{2}\int_0^{2\pi} d\theta \int_0^1 (r - 2r^3 + r^5)\,dr = \tfrac{3}{2}\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{2}r^4 + \tfrac{1}{6}r^6\right]_0^1 \\ &= \tfrac{3}{2}\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{2}\pi \end{aligned}$$

Thus the mass is $\frac{3}{2}\pi$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \dfrac{1}{3}\right)$.

44. $m = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1 \int_0^{1-y^2} (1 - z)\,dz\,dy = 4\int_{-1}^1 \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy = 2\int_{-1}^1 (1 - y^4)\,dy = \frac{16}{5}$,

$$\begin{aligned} M_{yz} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1 \int_0^{1-y^2} (1 - z)^2\,dz\,dy = 2\int_{-1}^1 \left[-\tfrac{1}{3}(1 - z)^3\right]_{z=0}^{z=1-y^2} dy \\ &= \tfrac{2}{3}\int_{-1}^1 (1 - y^6)\,dy = \left(\tfrac{4}{3}\right)\left(\tfrac{6}{7}\right) = \tfrac{24}{21} \end{aligned}$$

[continued]

$$M_{xz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2} 4y(1-z)\,dz\,dy$$

$$= \int_{-1}^{1}\left[4y(1-y^2) - 2y(1-y^2)^2\right]dy = \int_{-1}^{1}(2y-2y^5)\,dy = 0 \quad \text{[the integrand is odd]}$$

$$M_{xy} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2}(4z-4z^2)\,dz\,dy = 2\int_{-1}^{1}\left[(1-y^2)^2 - \tfrac{2}{3}(1-y^2)^3\right]dy$$

$$= 2\int_{-1}^{1}\left[\tfrac{1}{3} - y^4 + \tfrac{2}{3}y^6\right]dy = \left[\tfrac{4}{3}y - \tfrac{4}{5}y^5 + \tfrac{8}{21}y^7\right]_0^1 = \tfrac{96}{105} = \tfrac{32}{35}$$

Thus, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

45. $m = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^a\left[\tfrac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a}dy\,dz = \int_0^a\int_0^a\left(\tfrac{1}{3}a^3 + ay^2 + az^2\right)dy\,dz$

$$= \int_0^a\left[\tfrac{1}{3}a^3y + \tfrac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a}dz = \int_0^a\left(\tfrac{2}{3}a^4 + a^2z^2\right)dz = \left[\tfrac{2}{3}a^4z + \tfrac{1}{3}a^2z^3\right]_0^a = \tfrac{2}{3}a^5 + \tfrac{1}{3}a^5 = a^5$$

$$M_{yz} = \int_0^a\int_0^a\int_0^a\left[x^3 + x(y^2+z^2)\right]dx\,dy\,dz = \int_0^a\int_0^a\left[\tfrac{1}{4}a^4 + \tfrac{1}{2}a^2(y^2+z^2)\right]dy\,dz$$

$= \int_0^a\left(\tfrac{1}{4}a^5 + \tfrac{1}{6}a^5 + \tfrac{1}{2}a^3z^2\right)dz = \tfrac{1}{4}a^6 + \tfrac{1}{3}a^6 = \tfrac{7}{12}a^6 = M_{xz} = M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

46. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$$= \int_0^1\left[\tfrac{1}{2}(1-x)^3 - \tfrac{1}{3}(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1(1-x)^3\,dx = \tfrac{1}{24}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y - xy^2\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{2}x(1-x)^3 - \tfrac{1}{3}x(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1\left(x - 3x^2 + 3x^3 - x^4\right)dx = \tfrac{1}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right) = \tfrac{1}{120}$$

$$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{3}(1-x)^4 - \tfrac{1}{4}(1-x)^4\right]dx = \tfrac{1}{12}\left[-\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{60}$$

$$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\tfrac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$= \tfrac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx = \tfrac{1}{2}\int_0^1\left[\tfrac{1}{2}(1-x)^4 - \tfrac{2}{3}(1-x)^4 + \tfrac{1}{4}(1-x)^4\right]dx$$

$$= \tfrac{1}{24}\int_0^1(1-x)^4\,dx = -\tfrac{1}{24}\left[\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{120}$$

Hence, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

47. $I_x = \int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx = k\int_0^L\int_0^L\left(Ly^2 + \tfrac{1}{3}L^3\right)dy\,dx = k\int_0^L \tfrac{2}{3}L^4\,dx = \tfrac{2}{3}kL^5$

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

48. $I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} k(y^2+z^2)\,dx\,dy\,dz = ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$

$$= ak\int_{-c/2}^{c/2}\left[\tfrac{1}{3}y^3 + z^2y\right]_{y=-b/2}^{y=b/2}dz = ak\int_{-c/2}^{c/2}\left(\tfrac{1}{12}b^3 + bz^2\right)dz = ak\left[\tfrac{1}{12}b^3z + \tfrac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\tfrac{1}{12}b^3c + \tfrac{1}{12}bc^3\right) = \tfrac{1}{12}kabc(b^2+c^2)$$

By symmetry, $I_y = \frac{1}{12}kabc(a^2+c^2)$ and $I_z = \frac{1}{12}kabc(a^2+b^2)$.

49. $I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le a^2} \left[\int_0^h k(x^2+y^2)\,dz\right] dA = \iint\limits_{x^2+y^2\le a^2} k(x^2+y^2)h\,dA$

$= kh\int_0^{2\pi}\int_0^a (r^2)\,r\,dr\,d\theta = kh\int_0^{2\pi} d\theta\int_0^a r^3\,dr = kh(2\pi)\left[\frac{1}{4}r^4\right]_0^a = 2\pi kh\cdot\frac{1}{4}a^4 = \frac{1}{2}\pi kha^4$

50. $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le h^2}\left[\int_{\sqrt{x^2+y^2}}^h k(x^2+y^2)\,dz\right] dA$

$= \iint\limits_{x^2+y^2\le h^2} k(x^2+y^2)\left(h-\sqrt{x^2+y^2}\right) dA = k\int_0^{2\pi}\int_0^h r^2(h-r)\,r\,dr\,d\theta$

$= k\int_0^{2\pi} d\theta\int_0^h \left(r^3h-r^4\right) dr = k(2\pi)\left[\frac{1}{4}r^4h-\frac{1}{5}r^5\right]_0^h = 2\pi k\left(\frac{1}{4}h^5-\frac{1}{5}h^5\right) = \frac{1}{10}\pi kh^5$

51. (a) $m = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} x\sqrt{x^2+y^2}\,dz\,dy\,dx$, $\overline{y} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$, and

$\overline{z} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx.$

(c) $I_z = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)^{3/2}\,dz\,dy\,dx$

52. (a) $m = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{y} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{z} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(c) $I_z = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}(x^2+y^2)(1+x+y+z)\,dz\,dx\,dy$

53. (a) $m = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y(1+x+y+z)\,dz\,dy\,dx = \frac{3\pi}{32}+\frac{11}{24}$

(b) $(\overline{x},\overline{y},\overline{z}) = \Big(m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y x(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y y(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\,dz\,dy\,dx\Big)$

$$= \left(\frac{28}{9\pi+44}, \frac{30\pi+128}{45\pi+220}, \frac{45\pi+208}{135\pi+660}\right)$$

(c) $I_z = \displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y(x^2+y^2)(1+x+y+z)\,dz\,dy\,dx = \frac{68+15\pi}{240}$

54. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)\,dz\,dy\,dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2+y^2)\,dz\,dy\,dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2+y^2)\,dz\,dy\,dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2+y^2)\,dz\,dy\,dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\,dz\,dy\,dx = \frac{10{,}464}{175} \approx 59.79$

55. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^2\int_0^2\int_0^2 Cxyz\,dz\,dy\,dx \\ &= C\int_0^2 x\,dx \int_0^2 y\,dy \int_0^2 z\,dz = C\left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}z^2\right]_0^2 = 8C\end{aligned}$$

Then we must have $8C = 1 \quad\Rightarrow\quad C = \frac{1}{8}$.

(b)
$$\begin{aligned}P(X \le 1, Y \le 1, Z \le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \tfrac{1}{8}xyz\,dz\,dy\,dx \\ &= \tfrac{1}{8}\int_0^1 x\,dx \int_0^1 y\,dy \int_0^1 z\,dz = \tfrac{1}{8}\left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}\end{aligned}$$

(c) $P(X+Y+Z \le 1) = P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. The plane $x+y+z=1$ meets the xy-plane in the line $x+y=1$, so we have

$$\begin{aligned}P(X+Y+Z \le 1) &= \iiint_E f(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \tfrac{1}{8}xyz\,dz\,dy\,dx \\ &= \tfrac{1}{8}\int_0^1\int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\,dx = \tfrac{1}{16}\int_0^1\int_0^{1-x} xy(1-x-y)^2\,dy\,dx \\ &= \tfrac{1}{16}\int_0^1\int_0^{1-x}\left[(x^3-2x^2+x)y + (2x^2-2x)y^2 + xy^3\right] dy\,dx \\ &= \tfrac{1}{16}\int_0^1 \left[(x^3-2x^2+x)\tfrac{1}{2}y^2 + (2x^2-2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{192}\int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\,dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760}\end{aligned}$$

56. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= C\int_0^{\infty} e^{-0.5x}\,dx \int_0^{\infty} e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\ &= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy \lim_{t\to\infty}\int_0^t e^{-0.1z}\,dz \\ &= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \\ &= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right] \\ &= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C\end{aligned}$$

So we must have $100C = 1 \quad\Rightarrow\quad C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned} P(X \leq 1, Y \leq 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^\infty \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^\infty e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\ &= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132 \end{aligned}$$

(c) $P(X \leq 1, Y \leq 1, Z \leq 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \frac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$

$$\begin{aligned} &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1\left[-10e^{-0.1z}\right]_0^1 \\ &= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787 \end{aligned}$$

57. $V(E) = L^3 \quad \Rightarrow \quad f_{\text{avg}} = \dfrac{1}{L^3}\displaystyle\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz = \dfrac{1}{L^3}\int_0^L x\,dx\int_0^L y\,dy\int_0^L z\,dz$

$$= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L\left[\frac{y^2}{2}\right]_0^L\left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3}\frac{L^2}{2}\frac{L^2}{2}\frac{L^2}{2} = \frac{L^3}{8}$$

58. The height of each point is given by its z-coordinate, so the average height of the points in

$E = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1,\ z \geq 0\}$ is

$$\frac{1}{V(E)}\iiint_E z\,dV$$

Here $V(E) = \frac{1}{2}\cdot\frac{4}{3}\pi(1)^3 = \frac{2}{3}\pi$ [half the volume of a sphere], so

$$\begin{aligned} \tfrac{1}{V(E)}\iiint_E z\,dV &= \tfrac{1}{2\pi/3}\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx = \tfrac{3}{2\pi}\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}}dy\,dx \\ &= \tfrac{3}{2\pi}\cdot\tfrac{1}{2}\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx = \tfrac{3}{4\pi}\int_0^{2\pi}\int_0^1(1-r^2)\,r\,dr\,d\theta \\ &= \tfrac{3}{4\pi}\int_0^{2\pi}d\theta\int_0^1(r-r^3)\,dr = \tfrac{3}{4\pi}(2\pi)\left[\tfrac{1}{2}r^2-\tfrac{1}{4}r^4\right]_0^1 = \tfrac{3}{2}\left(\tfrac{1}{4}\right) = \tfrac{3}{8} \end{aligned}$$

59. (a) The triple integral will attain its maximum when the integrand $1-x^2-2y^2-3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2+2y^2+3z^2 \leq 1$. This describes the region bounded by the ellipsoid $x^2+2y^2+3z^2=1$.

(b) The maximum value of $\iiint_E(1-x^2-2y^2-3z^2)\,dV$ occurs when E is the solid region bounded by the ellipsoid $x^2+2y^2+3z^2=1$. The projection of E on the xy-plane is the planar region bounded by the ellipse $x^2+2y^2=1$, so

$E = \left\{(x,y,z) \mid -1 \leq x \leq 1, -\sqrt{\frac{1}{2}(1-x^2)} \leq y \leq \sqrt{\frac{1}{2}(1-x^2)}, -\sqrt{\frac{1}{3}(1-x^2-2y^2)} \leq z \leq \sqrt{\frac{1}{3}(1-x^2-2y^2)}\right\}$

and

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\, dV = \int_{-1}^{1} \int_{-\sqrt{\frac{1}{2}(1-x^2)}}^{\sqrt{\frac{1}{2}(1-x^2)}} \int_{-\sqrt{\frac{1}{3}(1-x^2-2y^2)}}^{\sqrt{\frac{1}{3}(1-x^2-2y^2)}} (1 - x^2 - 2y^2 - 3z^2)\, dz\, dy\, dx = \frac{4\sqrt{6}}{45}\,\pi$$

using a CAS.

DISCOVERY PROJECT Volumes of Hyperspheres

In this project we use V_n to denote the n-dimensional volume of an n-dimensional hypersphere.

1. The interior of the circle is the set of points $\left\{(x, y) \mid -r \le y \le r,\ -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 from the Table of Integrals to evaluate the integral, we get

$$\begin{aligned} V_2(r) &= \int_{-r}^{r} \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} dx\, dy = \int_{-r}^{r} 2\sqrt{r^2 - y^2}\, dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1 - \sin^2\theta}\ (r\cos\theta\, d\theta) \\ &= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2r^2 \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2 \end{aligned}$$

2. The region of integration is

$$\left\{(x, y, z) \mid -r \le z \le r, -\sqrt{r^2 - z^2} \le y \le \sqrt{r^2 - z^2}, -\sqrt{r^2 - z^2 - y^2} \le x \le \sqrt{r^2 - z^2 - y^2}\right\}.$$

Substituting $y = \sqrt{r^2 - z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$\begin{aligned} V_3(r) &= \int_{-r}^{r} \int_{-\sqrt{r^2 - z^2}}^{\sqrt{r^2 - z^2}} \int_{-\sqrt{r^2 - z^2 - y^2}}^{\sqrt{r^2 - z^2 - y^2}} dx\, dy\, dz = \int_{-r}^{r} \int_{-\sqrt{r^2 - z^2}}^{\sqrt{r^2 - z^2}} 2\sqrt{r^2 - z^2 - y^2}\, dy\, dz \\ &= \int_{-r}^{r} \int_{-\pi/2}^{\pi/2} 2\sqrt{r^2 - z^2}\sqrt{1 - \sin^2\theta}\left(\sqrt{r^2 - z^2}\cos\theta\, d\theta\right) dz \\ &= 2\left[\int_{-r}^{r} (r^2 - z^2)\, dz\right]\left[\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3} \end{aligned}$$

3. The formula for 4-dimensional hypersphere is $x^2 + y^2 + z^2 + w^2 = r^2$. Here we substitute $y = \sqrt{r^2 - w^2 - z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2} \cos^p\theta\, d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 7.1.55–56, we have

$$\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}\,\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k + 1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2} \cos^{2k} x\, dx = 2\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k - 1)\pi}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k} \qquad \textbf{(1)}$$

$$\int_{-\pi/2}^{\pi/2} \cos^{2k+1} x\, dx = 2\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k + 1)} \qquad \textbf{(2)}$$

[continued]

Thus $$V_4(r) = \int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx\,dy\,dz\,dw$$

$$= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}} \sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw$$

$$= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\pi/2}^{\pi/2}(r^2-w^2-z^2)\cos^2\theta\,d\theta\,dz\,dw \qquad \left[\begin{aligned} y &= \sqrt{r^2-w^2-z^2}\,\sin\theta, \\ dy &= \sqrt{r^2-w^2-z^2}\,\cos\theta\,d\theta\end{aligned}\right]$$

$$= 2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}(r^2-w^2-z^2)\,dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right]$$

$$= 2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}(r^2-w^2)^{3/2}\,dw\right] = \pi\left(\tfrac{4}{3}\right)\int_{-\pi/2}^{\pi/2} r^4\cos^4\phi\,d\phi \qquad \left[\begin{aligned} w &= r\sin\phi, \\ dw &= r\cos\phi\,d\phi\end{aligned}\right]$$

$$= \frac{4\pi}{3}r^4\cdot\frac{1\cdot 3\cdot\pi}{2\cdot 4} = \frac{\pi^2 r^4}{2}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\,\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$V_n(r) = \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\,dx_2\cdots dx_{n-1}\,dx_n$$

$$= 2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^{n}\theta_n\,d\theta_n\right]r^n$$

$$= \begin{cases} \left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot 2}{1\cdot 3}\cdot\dfrac{1\cdot 3\pi}{2\cdot 4}\right]\left[\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\dfrac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n \text{ even} \\[2ex] 2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot 2}{1\cdot 3}\right]\left[\dfrac{1\cdot 3\pi}{2\cdot 4}\cdot\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n \text{ odd}\end{cases}$$

By canceling within each set of brackets, we find that

$$V_n(r) = \begin{cases} \dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n = \dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n \text{ even} \\[2ex] 2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n = \dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n \text{ odd}\end{cases}$$

5. We need to show that $\lim\limits_{n\to\infty} V_n(1) = 0$. We'll consider the cases of n even and n odd separately.

For $n = 2k$ and $r = 1$: $$V_n(1) = \frac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n = \frac{\pi^{2k/2}}{\left(\frac{1}{2}\cdot 2k\right)!} = \frac{\pi^k}{k!}$$

Then $$0 \le \pi\cdot\frac{\pi}{2}\cdot\frac{\pi}{3}\cdots\frac{\pi}{k} \le \pi\cdot\frac{\pi}{2}\cdot\frac{\pi}{3}\cdot\frac{\pi}{k} = \frac{\pi^4}{6k} \qquad [\text{for } k > 3]$$

$\dfrac{\pi^4}{6k} \to 0$ as $k\to\infty$ $\Rightarrow$ $V_n(1)\to 0$ as $k\to\infty$ for n even by the Squeeze Theorem.

[continued]

For $n = 2k + 1$ and $r = 1$:

$$V_n(1) = \frac{2^n \left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}\,r^n = \frac{2^{2k+1}\,k!\,\pi^k}{(2k+1)!}$$

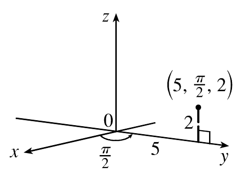
$$0 \le 2 \cdot \frac{2^2\pi}{3\cdot 2} \cdot \frac{2^2(2)\pi}{5\cdot 4} \cdot \frac{2^2(3)\pi}{7\cdot 6} \cdot \cdots \cdot \frac{2^2(k)\pi}{(2k+1)(2k)} \le 2 \cdot \frac{2\pi}{3} \cdot \frac{2\pi}{5} \cdot \frac{2\pi}{2k+1} = \frac{2^4\pi^3}{15(2k+1)} \qquad [\text{for } k > 3]$$

$\dfrac{2^4\pi^3}{15(2k+1)} \to 0$ as $k \to \infty \quad \Rightarrow \quad V_n(1) \to 0$ as $k \to \infty$ for n odd by the Squeeze Theorem.

Thus, $\lim\limits_{n\to\infty} V_n(1) = 0$.

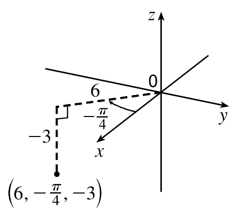
15.7 Triple Integrals in Cylindrical Coordinates

1. (a)



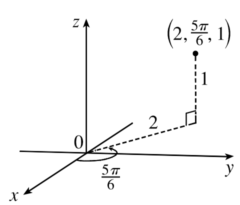
From Equations 1, $x = r\cos\theta = 5\cos\frac{\pi}{2} = 5 \cdot 0 = 0$, $y = r\sin\theta = 5\sin\frac{\pi}{2} = 5 \cdot 1 = 5$, and $z = 2$, so the point is $(0, 5, 2)$ in rectangular coordinates.

(b)



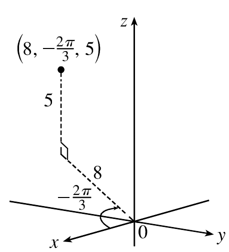
From Equations 1, $x = r\cos\theta = 6\cos\left(-\frac{\pi}{4}\right) = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$, $y = r\sin\theta = 6\sin\left(-\frac{\pi}{4}\right) = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$, and $z = -3$, so the point is $\left(3\sqrt{2}, -3\sqrt{2}, -3\right)$ in rectangular coordinates.

2. (a)



From Equations 1, $x = r\cos\theta = 2\cos\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$, $y = r\sin\theta = 2\sin\frac{5\pi}{6} = 2 \cdot \frac{1}{2} = 1$, and $z = 1$, so the point is $\left(-\sqrt{3}, 1, 1\right)$ in rectangular coordinates.

(b)



From Equations 1, $x = r\cos\theta = 8\cos\left(-\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4$, $y = r\sin\theta = 8\sin\left(-\frac{2\pi}{3}\right) = 8\left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$, and $z = 5$, so the point is $\left(-4, -4\sqrt{3}, 5\right)$ in rectangular coordinates.

3. (a) $(4, 4, -3)$. From Equations 2, we have $r^2 = x^2 + y^2 = 4^2 + 4^2 = 32$, so $r = \sqrt{32}$. $\tan\theta = \frac{y}{x} = \frac{4}{4} = 1$ and the point $(4, 4)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(4\sqrt{2}, \frac{\pi}{4}, -3\right)$.

(b) $\left(5\sqrt{3}, -5, \sqrt{3}\right)$. $r^2 = \left(5\sqrt{3}\right)^2 + (-5)^2 = 100$, so $r = 10$. $\tan\theta = \frac{-5}{5\sqrt{3}} = -\frac{1}{\sqrt{3}}$ and the point $\left(5\sqrt{3}, -5\right)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{11\pi}{6} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(10, -\frac{\pi}{6}, \sqrt{3}\right)$.

4. (a) $(0, -2, 9)$. $r^2 = 0^2 + (-2)^2 = 4$, so $r = 2$. $\tan\theta = 0/(-2)$ is undefined and $y < 0$, so $\theta = \frac{3\pi}{2} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{3\pi}{2}, 9\right)$.

(b) $\left(-1, \sqrt{3}, 6\right)$. $r^2 = (-1)^2 + \left(\sqrt{3}\right)^2 = 4$, so $r = 2$. $\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and the point $\left(-1, \sqrt{3}\right)$ is in the second quadrant of the xy-plane, so $\theta = \frac{2\pi}{3} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{2\pi}{3}, 6\right)$.

5. Since $r = 2$, the distance from any point to the z-axis is 2. Because θ and z may vary, the surface is a circular cylinder with radius 2 and axis the z-axis. (See Figure 4.)

Also, $x^2 + y^2 = r^2 = 4$, which we recognize as an equation of this cylinder.

6. Since $\theta = \frac{\pi}{6}$ but r and z may vary, the surface is a vertical plane including the z-axis and intersecting the xy-plane in the line $y = \frac{1}{\sqrt{3}}x$. (Here we are assuming that r can be negative; if we restrict $r \geq 0$, then we get a half-plane.)

7. Since $r^2 + z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 4$, a sphere centered at the origin with radius 2.

8. $r = 2\sin\theta \quad \Rightarrow \quad r^2 = 2r\sin\theta \quad \Rightarrow \quad x^2 + y^2 = 2y \quad \Leftrightarrow \quad x^2 + (y-1)^2 = 1$. z doesn't appear in the equation, so any horizontal trace in $z = k$ is the circle $x^2 + (y-1)^2 = 1$, $z = k$, which has center $(0, 1, k)$ and radius 1. Thus the surface is a circular cylinder with radius 1 and axis the vertical line $x = 0$, $y = 1$.

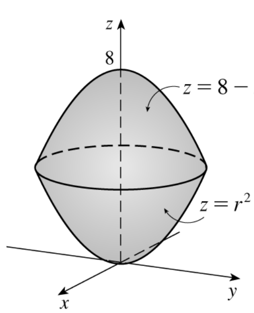
9. (a) Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$, the equation $x^2 - x + y^2 + z^2 = 1$ becomes $r^2 - r\cos\theta + z^2 = 1$ or $z^2 = 1 + r\cos\theta - r^2$.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $z = x^2 - y^2$ becomes $z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

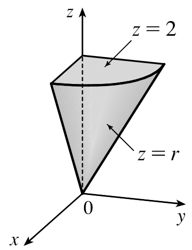
10. (a) The equation $2x^2 + 2y^2 - z^2 = 4$ can be written as $2(x^2 + y^2) - z^2 = 4$ which becomes $2r^2 - z^2 = 4$ or $z^2 = 2r^2 - 4$ in cylindrical coordinates.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $2x - y + z = 1$ becomes $2r\cos\theta - r\sin\theta + z = 1$ or $z = 1 + r(\sin\theta - 2\cos\theta)$.

11.



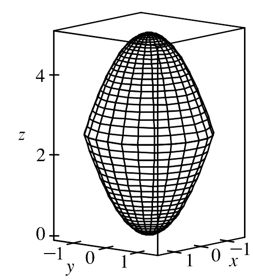
$z = r^2 \quad \Leftrightarrow \quad z = x^2 + y^2$, a circular paraboloid opening upward with vertex the origin, and $z = 8 - r^2 \quad \Leftrightarrow \quad z = 8 - (x^2 + y^2)$, a circular paraboloid opening downward with vertex $(0, 0, 8)$. The paraboloids intersect when $r^2 = 8 - r^2 \quad \Leftrightarrow \quad r^2 = 4$. Thus $r^2 \leq z \leq 8 - r^2$ describes the solid above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 8 - x^2 - y^2$ for $x^2 + y^2 \leq 4$.

12. 

$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \le z \le 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \le \theta \le \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

13. We can position the cylindrical shell vertically so that its axis coincides with the z-axis and its base lies in the xy-plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \le r \le 7$, $0 \le \theta \le 2\pi$, $0 \le z \le 20$.

14. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$. The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \le r \le \sqrt{5/2}$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `RevolutionPlot3D` or `ParametricPlot3D`.



15. (a) In cylindrical coordinates, the region can be described as $E = \{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le z \le 2 - r^2\}$.

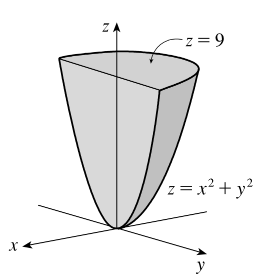
Thus, $\iiint_E (x^2 + y^2)\, dV = \int_0^\pi \int_0^1 \int_0^{2-r^2} r^2 \cdot r\, dz\, dr\, d\theta$.

(b) $$\int_0^\pi \int_0^1 \int_0^{2-r^2} r^3\, dz\, dr\, d\theta = \int_0^\pi \int_0^1 r^3 \Big[z \Big]_{z=0}^{z=2-r^2} dr\, d\theta = \int_0^\pi \int_0^1 (2r^3 - r^5)\, dr\, d\theta$$
$$= \int_0^\pi d\theta \int_0^1 (2r^3 - r^5)\, dr = \Big[\theta \Big]_{\theta=0}^{\theta=\pi} \cdot \left[\frac{r^4}{2} - \frac{r^6}{6} \right]_{r=0}^{r=1} = \frac{\pi}{3}$$

16. (a) In cylindrical coordinates, the region E is bounded above by the paraboloid $z = 6 - r^2$ and below by the cone $z = r$. The paraboloid and cone intersect when $6 - r^2 = r \;\Rightarrow\; r^2 + r - 6 = 0 \;\Rightarrow\; r = 2$ $(r > 0)$, so the region can be described as $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 6 - r^2\}$. Then

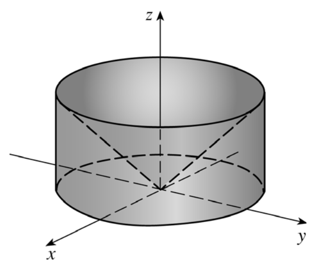
$\iiint_E (xy)\, dV = \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r\cos\theta \cdot r\sin\theta \cdot r\, dz\, dr\, d\theta$.

(b) $$\int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r^3 \cos\theta \sin\theta\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta \Big[z \Big]_{z=r}^{z=6-r^2} dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta\, (6r^3 - r^4 - r^5)\, dr\, d\theta$$
$$= \int_0^{2\pi} \cos\theta \sin\theta\, d\theta \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0 \cdot \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0$$

17. 

The region of integration represents the solid enclosed by the paraboloid $z = r^2$, $(z = x^2 + y^2)$, below the plane $z = 9$ in the second and third quadrants.

$$\int_{\pi/2}^{3\pi/2} \int_0^3 \int_{r^2}^9 r\, dz\, dr\, d\theta = \int_{\pi/2}^{3\pi/2} \int_0^3 [rz]_{z=r^2}^{z=9}\, dr\, d\theta = \int_{\pi/2}^{3\pi/2} \int_0^3 \left(9r - r^3\right)\, dr\, d\theta$$
$$= \int_{\pi/2}^{3\pi/2} d\theta \int_0^3 \left(9r - r^3\right)\, dr = \pi \left[\tfrac{9}{2} r^2 - \tfrac{r^4}{4} \right]_0^3 = \tfrac{81\pi}{4}$$

18. 

The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le r\}$. This represents the solid region enclosed by the circular cylinder $r = 2$, bounded above by the cone $z = r$, and bounded below by the xy-plane.

$$\int_0^2 \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} \left[rz\right]_{z=0}^{z=r} d\theta\,dr = \int_0^2 \int_0^{2\pi} r^2\,d\theta\,dr$$
$$= \int_0^2 r^2\,dr \int_0^{2\pi} d\theta = \left[\tfrac{1}{3}r^3\right]_0^2 \left[\theta\right]_0^{2\pi} = \tfrac{8}{3} \cdot 2\pi = \tfrac{16}{3}\pi$$

19. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2 + y^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^4 \left[z\right]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

20. The paraboloid $z = x^2 + y^2 = r^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \;\Rightarrow\; r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le 2\pi, 0 \le r \le 2, r^2 \le z \le 4\right\}$. Thus

$$\iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[\tfrac{1}{2}rz^2\right]_{z=r^2}^{z=4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr = 2\pi \left[4r^2 - \tfrac{1}{12}r^6\right]_0^2$$
$$= 2\pi\left(16 - \tfrac{16}{3}\right) = \tfrac{64}{3}\pi$$

21. The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \;\Rightarrow\; r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. Thus

$$\iiint_E (x + y + z)\,dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2}rz^2\right]_{z=0}^{z=4-r^2} dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2}r(4 - r^2)^2\right] dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\left(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2}$$
$$= \tfrac{64}{15}(1 - 0) + \tfrac{16}{3} \cdot \tfrac{\pi}{2} - \tfrac{64}{15}(0 - 1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}$$

22. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\sin\theta + 4$ and the cylinders $r = 1$ and $r = 4$, so E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 4,\ 0 \le z \le r\sin\theta + 4\}$. Thus

$$\iiint_E (x - y)\,dV = \int_0^{2\pi} \int_1^4 \int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)\left[z\right]_{z=0}^{z=r\sin\theta+4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)(r\sin\theta + 4)\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 \left[r^3(\sin\theta\cos\theta - \sin^2\theta) + 4r^2(\cos\theta - \sin\theta)\right] dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{4}r^4(\sin\theta\cos\theta - \sin^2\theta) + \tfrac{4}{3}r^3(\cos\theta - \sin\theta)\right]_{r=1}^{r=4} d\theta$$
$$= \int_0^{2\pi} \left[\left(64 - \tfrac{1}{4}\right)(\sin\theta\cos\theta - \sin^2\theta) + \left(\tfrac{256}{3} - \tfrac{4}{3}\right)(\cos\theta - \sin\theta)\right] d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{255}{4}(\sin\theta\cos\theta - \sin^2\theta) + 84(\cos\theta - \sin\theta)\right] d\theta$$
$$= \left[\tfrac{255}{4}\left(\tfrac{1}{2}\sin^2\theta - \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right) + 84\left(\sin\theta + \cos\theta\right)\right]_0^{2\pi} = \tfrac{255}{4}(-\pi) + 84(1) - 0 - 84(1) = -\tfrac{255}{4}\pi$$

23. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\,z\right]_{z=0}^{z=2r} dr\,d\theta = \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1} d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta = \tfrac{2}{5}\int_0^{2\pi}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{5}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \tfrac{2\pi}{5}$$

24. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\right\}$. Thus the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta\int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8 - 3^{3/2})$$

25. In cylindrical coordinates, E is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The cone and the sphere intersect when $2r^2 = 2 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{2-r^2}\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi} d\theta\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^1$$
$$= 2\pi\left(-\tfrac{1}{3}\right)\left(1 + 1 - 2^{3/2}\right) = -\tfrac{2}{3}\pi\left(2 - 2\sqrt{2}\right) = \tfrac{4}{3}\pi\left(\sqrt{2} - 1\right)$$

26. In cylindrical coordinates, E is bounded below by the paraboloid $z = r^2$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The paraboloid and the sphere intersect when $r^2 + r^4 = 2 \quad\Rightarrow\quad (r^2+2)(r^2-1) = 0 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r^2 \le z \le \sqrt{2-r^2}\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r^2}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr\,d\theta$$
$$= \int_0^{2\pi} d\theta\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{4}r^4\right]_0^1$$
$$= 2\pi(-\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{3}\cdot 2^{3/2} - 0) = 2\pi\left(-\tfrac{7}{12} + \tfrac{2}{3}\sqrt{2}\right) = \left(-\tfrac{7}{6} + \tfrac{4}{3}\sqrt{2}\right)\pi$$

27. (a) In cylindrical coordinates, E is bounded above by the paraboloid $z = 24 - r^2$ and below by the cone $z = 2\sqrt{r^2}$ or $z = 2r$ $(r \ge 0)$. The surfaces intersect when $24 - r^2 = 2r \quad\Rightarrow\quad r^2 + 2r - 24 = 0 \quad\Rightarrow\quad (r+6)(r-4) = 0 \quad\Rightarrow\quad r = 4$, so $E = \left\{(r, \theta, z) \mid 2r \le z \le 24 - r^2, 0 \le r \le 4, 0 \le \theta \le 2\pi\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r\left(24 - r^2 - 2r\right)dr\,d\theta = \int_0^{2\pi} d\theta\int_0^4\left(24r - r^3 - 2r^2\right)dr$$
$$= 2\pi\left[12r^2 - \tfrac{1}{4}r^4 - \tfrac{2}{3}r^3\right]_0^4 = 2\pi\left(192 - 64 - \tfrac{128}{3}\right) = \tfrac{512}{3}\pi$$

(b) For constant density K, $m = KV = \frac{512}{3}\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

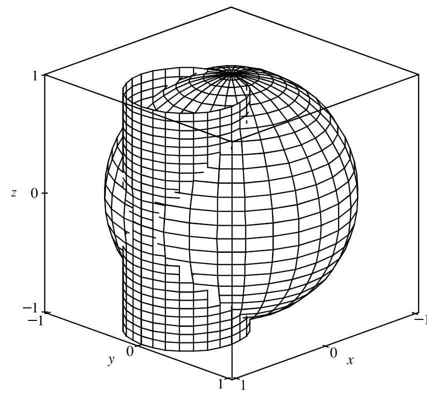
$$M_{xy} = \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2}(zK)\,r\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^4 r\left[\tfrac{1}{2}z^2\right]_{z=2r}^{z=24-r^2}dr\,d\theta$$
$$= \tfrac{K}{2}\int_0^{2\pi}\int_0^4 r[(24-r^2)^2 - 4r^2]\,dr\,d\theta = \tfrac{K}{2}\int_0^{2\pi}d\theta\int_0^4(576r - 52r^3 + r^5)\,dr$$
$$= \tfrac{K}{2}(2\pi)\left[288r^2 - 13r^4 + \tfrac{1}{6}r^6\right]_0^4 = \pi K\left(4608 - 3328 + \tfrac{2048}{3}\right) = \tfrac{5888}{3}\pi K$$

Thus $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right) = \left(0,0,\frac{5888\pi K/3}{512\pi K/3}\right) = \left(0,0,\frac{23}{2}\right)$.

28. (a) $V = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2-r^2)^{3/2}\right]_{r=0}^{r=a\cos\theta}d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2 - a^2\cos^2\theta)^{3/2} - a^3\right]d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2} - a^3\right]d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta$$
$$= -\frac{4a^3}{3}\int_0^{\pi/2}\left[\sin\theta\,(1-\cos^2\theta) - 1\right]d\theta$$
$$= -\frac{4a^3}{3}\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta - \theta\right]_0^{\pi/2} = -\frac{4a^3}{3}\left(-\tfrac{\pi}{2}+\tfrac{2}{3}\right) = \tfrac{2}{9}a^3(3\pi - 4)$$

(b)



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d(cos(theta),theta=-Pi/2..Pi/2,z=-1..1,coords=cylindrical):
with(plots):
display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3D[1,{phi,0,Pi},{theta,0,2Pi}]
cylinder=ParametricPlot3D[{(Cos[theta])^2,Cos[theta]*Sin[theta],z},
                    {theta,-Pi/2,Pi/2},{z,-1,1}]
Show[sphere,cylinder]
```

29. The paraboloid $z = 4x^2 + 4y^2$ intersects the plane $z = a$ when $a = 4x^2 + 4y^2$ or $x^2 + y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \left\{(r,\theta,z) \mid 0 \le r \le \frac{1}{2}\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\right\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}(ar - 4r^3)\,dr\,d\theta$$
$$= K\int_0^{2\pi}\left[\tfrac{1}{2}ar^2 - r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac{1}{8}a^2\pi K$$

[continued]

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K \int_0^{2\pi} \int_0^{\sqrt{a}/2} \left(\tfrac{1}{2}a^2 r - 8r^5\right) dr\,d\theta$$

$$= K \int_0^{2\pi} \left[\tfrac{1}{4}a^2 r^2 - \tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K \int_0^{2\pi} \tfrac{1}{24}a^3\,d\theta = \tfrac{1}{12}a^3 \pi K$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

30. Since density is proportional to the distance from the z-axis, we can say $\rho(x, y, z) = K\sqrt{x^2 + y^2}$. Then

$$m = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} (Kr)\,r\,dz\,dr\,d\theta = 2\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K \int_0^{2\pi} \int_0^a r^2\sqrt{a^2 - r^2}\,dr\,d\theta$$

$$= 2K \int_0^{2\pi} \left[\tfrac{1}{8}r(2r^2 - a^2)\sqrt{a^2 - r^2} + \tfrac{1}{8}a^4 \sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta = 2K \int_0^{2\pi} \left[\left(\tfrac{1}{8}a^4\right)\left(\tfrac{\pi}{2}\right)\right] d\theta = \tfrac{1}{4}a^4\pi^2 K$$

31. The region of integration is the region above the cone $z = \sqrt{x^2 + y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \le y \le 2$ with $-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy = \int_0^{2\pi} \int_0^2 \int_r^2 (r\cos\theta)\,z\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^2(\cos\theta)\,z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^2(\cos\theta)\left[\tfrac{1}{2}z^2\right]_{z=r}^{z=2} dr\,d\theta = \tfrac{1}{2}\int_0^{2\pi} \int_0^2 r^2(\cos\theta)\left(4 - r^2\right) dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \cos\theta\,d\theta \int_0^2 \left(4r^2 - r^4\right) dr = \tfrac{1}{2}\left[\sin\theta\right]_0^{2\pi}\left[\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right]_0^2 = 0$$

32. The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9 - x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0, 0)$. Thus,

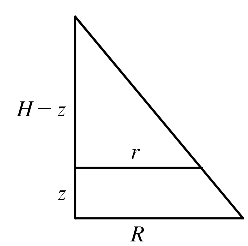
$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}\,dz\,dy\,dx = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^3 r^2\left(9 - r^2\right) dr\,d\theta = \int_0^{\pi} d\theta \int_0^3 \left(9r^2 - r^4\right) dr$$

$$= \left[\,\theta\,\right]_0^{\pi}\left[3r^3 - \tfrac{1}{5}r^5\right]_0^3 = \pi\left(81 - \tfrac{243}{5}\right) = \tfrac{162}{5}\pi$$

33. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\,dV$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200\text{ lb/ft}^3$ at all points P in C. We use cylindrical coordinates:



$$\frac{r}{R} = \frac{H - z}{H} = 1 - \frac{z}{H}$$

$$W = \int_0^{2\pi} \int_0^H \int_0^{R(1-z/H)} z \cdot 200r\,dr\,dz\,d\theta = 2\pi \int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)} dz$$

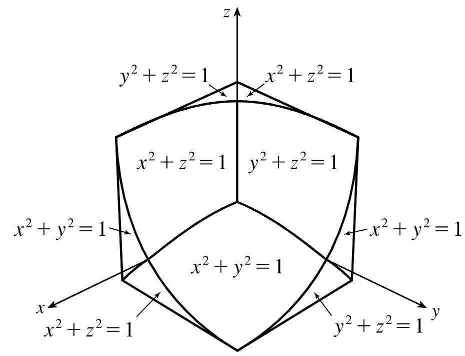
$$= 400\pi \int_0^H z\,\frac{R^2}{2}\left(1 - \frac{z}{H}\right)^2 dz = 200\pi R^2 \int_0^H \left(z - \frac{2z^2}{H} + \frac{z^3}{H^2}\right) dz$$

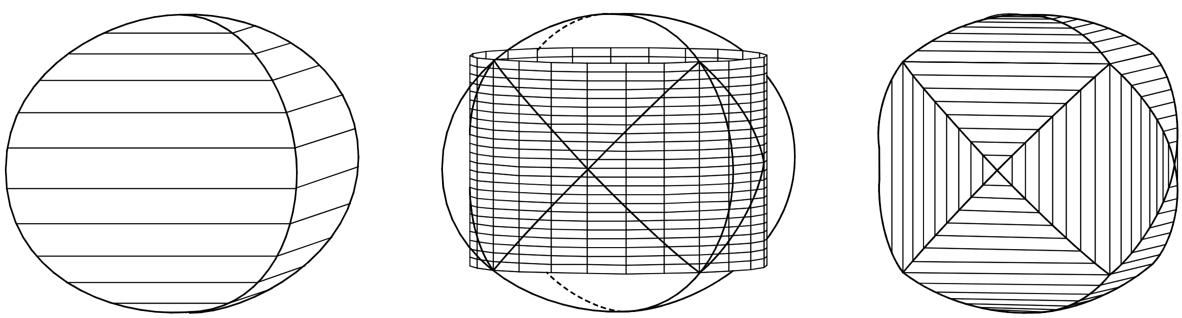
$$= 200\pi R^2 \left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2}\right]_0^H = 200\pi R^2 \left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4}\right)$$

$$= \tfrac{50}{3}\pi R^2 H^2 = \tfrac{50}{3}\pi(62{,}000)^2(12{,}400)^2 \approx 3.1 \times 10^{19}\text{ ft-lb}$$

DISCOVERY PROJECT The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \le s \le 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. The resulting solid has twelve curved faces bounded by "edges" which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.



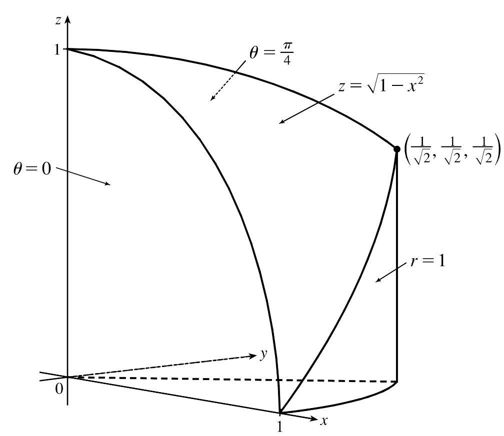


2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \le \theta \le \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!) This piece is described by

$\left\{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le \sqrt{1 - x^2}\right\}$,

and so, substituting $x = r\cos\theta$, the volume of the entire solid is

$$\begin{aligned} V &= 16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-x^2}} r\,dz\,dr\,d\theta \\ &= 16\int_0^{\pi/4}\int_0^1 r\sqrt{1 - r^2\cos^2\theta}\,dr\,d\theta \\ &= 16 - 8\sqrt{2} \approx 4.6863 \end{aligned}$$

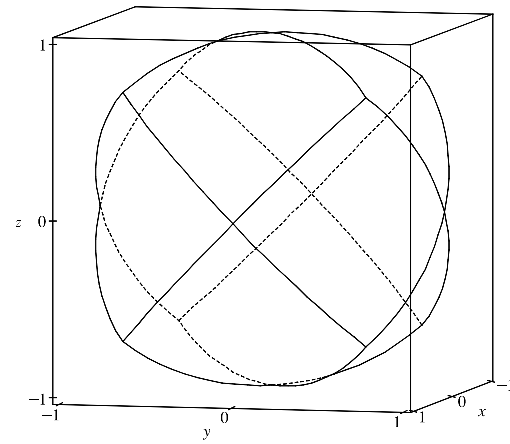


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$x = r,\ y = \pm r,\ z = \pm\sqrt{1 - r^2},\ -\frac{1}{\sqrt{2}} \le r \le \frac{1}{\sqrt{2}};$

$x = \pm s,\ y = \pm\sqrt{1 - s^2},\ z = s,\ -\frac{1}{\sqrt{2}} \le s \le \frac{1}{\sqrt{2}};$

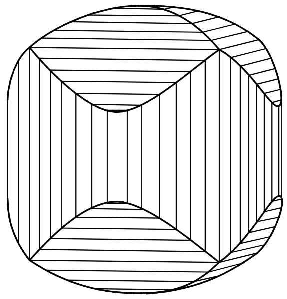
$x = \pm\sqrt{1 - t^2},\ y = t,\ z = \pm t,\ -\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}.$

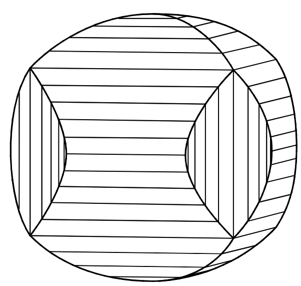


4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

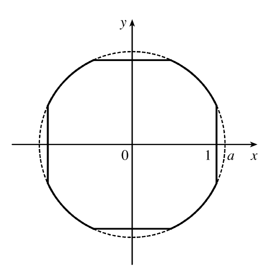
If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.

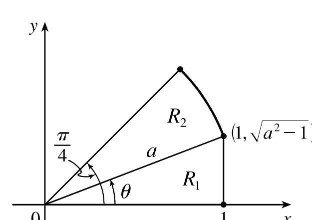

$a = 0.95,\ b = c = 1$


$a = 1.1,\ b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\left\{(r, \theta) \mid 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{4}\right\}$ in the xy-plane and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}$. Thus, the total volume is $V = 16\int_0^{\pi/4}\int_0^a \sqrt{1 - r^2\cos^2\theta}\, r\, dr\, d\theta$.

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed at the right. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}$.





[continued]

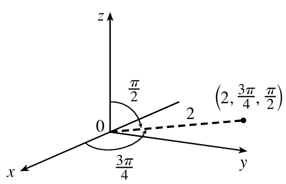
We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = a$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 15.5.26. Its volume is $V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$.

15.8 Triple Integrals in Spherical Coordinates

1. (a)



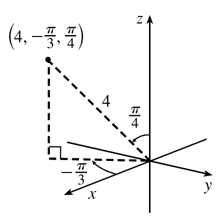
From Equations 1 with $(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{\pi}{2}\right)$,

$x = \rho\sin\phi\cos\theta = 2\sin\frac{\pi}{2}\cos\frac{3\pi}{4} = 2(1)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$,

$y = \rho\sin\phi\sin\theta = 2\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 2(1)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$, and

$z = \rho\cos\phi = 2\cos\frac{\pi}{2} = 2 \cdot 0 = 0$, so the point is $\left(-\sqrt{2}, \sqrt{2}, 0\right)$ in rectangular coordinates.

(b)



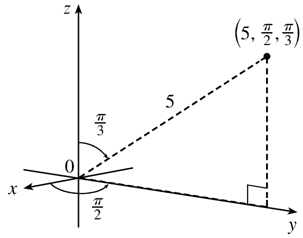
From Equations 1 with $(\rho, \theta, \phi) = \left(4, -\frac{\pi}{3}, \frac{\pi}{4}\right)$,

$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\left(-\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{2}$,

$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\left(-\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{6}$, and

$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$, so the point is $\left(\sqrt{2}, -\sqrt{6}, 2\sqrt{2}\right)$ in rectangular coordinates.

2. (a)



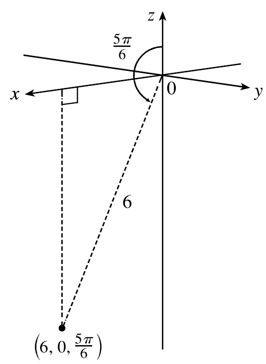
From Equations 1 with $(\rho, \theta, \phi) = \left(5, \frac{\pi}{2}, \frac{\pi}{3}\right)$,

$x = \rho\sin\phi\cos\theta = 5\sin\frac{\pi}{3}\cos\frac{\pi}{2} = 5 \cdot \frac{\sqrt{3}}{2} \cdot 0 = 0$,

$y = \rho\sin\phi\sin\theta = 5\sin\frac{\pi}{3}\sin\frac{\pi}{2} = 5 \cdot \frac{\sqrt{3}}{2} \cdot 1 = \frac{5\sqrt{3}}{2}$, and

$z = \rho\cos\phi = 5\cos\frac{\pi}{3} = 5 \cdot \frac{1}{2} = \frac{5}{2}$, so the point is $\left(0, \frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$ in rectangular coordinates.

(b)



From Equations 1 with $(\rho, \theta, \phi) = \left(6, 0, \frac{5\pi}{6}\right)$,

$x = \rho \sin\phi \cos\theta = 6 \sin\frac{5\pi}{6} \cos 0 = 6 \cdot \frac{1}{2} \cdot 1 = 3$,

$y = \rho \sin\phi \sin\theta = 6 \sin\frac{5\pi}{6} \sin 0 = 6 \cdot \frac{1}{2} \cdot 0 = 0$, and

$z = \rho \cos\phi = 6 \cos\frac{5\pi}{6} = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$, so the point is

$\left(3, 0, -3\sqrt{3}\right)$.

3. (a) From Equations 1 and 2 with $(x, y, z) = (3, 3, 0)$, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 3^2 + 0^2} = 3\sqrt{2}$,

$$\cos\phi = \frac{z}{\rho} = \frac{0}{3\sqrt{2}} = 0 \quad\Rightarrow\quad \phi = \tfrac{\pi}{2} \text{ and } \cos\theta = \frac{x}{\rho\sin\phi} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \frac{\pi}{4} \quad [\text{since } x > 0 \text{ and } y > 0].$$

Thus, spherical coordinates are $\left(3\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

(b) $\rho = \sqrt{1^2 + (-\sqrt{3}\,)^2 + (2\sqrt{3}\,)^2} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \dfrac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{4\sin\frac{\pi}{6}} = \dfrac{1}{2} \quad\Rightarrow\quad \theta = -\dfrac{\pi}{3}$ [since $x > 0$ and $y < 0$]. Thus, spherical coordinates are $\left(4, -\frac{\pi}{3}, \frac{\pi}{6}\right)$.

4. (a) $\rho = \sqrt{0^2 + 4^2 + (-4)^2} = 4\sqrt{2}$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{-4}{4\sqrt{2}} = \dfrac{-1}{\sqrt{2}} \quad\Rightarrow\quad \phi = \dfrac{3\pi}{4}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{4\sqrt{2}\sin\frac{3\pi}{4}} = 0 \quad\Rightarrow\quad \theta = \dfrac{\pi}{2}$ [since $y > 0$]. Thus, spherical coordinates are $\left(4\sqrt{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right)$.

(b) $\rho = \sqrt{(-2)^2 + 2^2 + (2\sqrt{6}\,)^2} = 4\sqrt{2}$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{6}}{4\sqrt{2}} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \frac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{-2}{4\sqrt{2}\sin\frac{\pi}{6}} = -\dfrac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \frac{3\pi}{4}$ [since $x < 0$ and $y > 0$]. Thus, spherical coordinates are

$\left(4\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)$.

5. Since $\phi = \frac{3\pi}{4}$, but ρ and θ can vary, the surface is the bottom half of a right circular cone with vertex at the origin and axis the negative z-axis. (See Figure 4.)

6. $\rho^2 - 3\rho + 2 = 0 \quad\Rightarrow\quad (\rho - 1)(\rho - 2) = 0 \quad\Rightarrow\quad \rho = 1$ or $\rho = 2$. Thus the equation represents two surfaces. In the case $\rho = 1$, the distance from any point to the origin is 1. Because θ and ϕ can vary, the surface is a sphere centered at the origin with radius 1. (See Figure 2.) Similarly, $\rho = 2$ is a sphere centered at the origin with radius 2.

Also, $\rho = 1 \quad\Rightarrow\quad \rho^2 = 1 \quad\Rightarrow\quad x^2 + y^2 + z^2 = 1$ which we recognize as the equation of the unit sphere, and similarly, $\rho = 2 \quad\Rightarrow\quad \rho^2 = 4 \quad\Rightarrow\quad x^2 + y^2 + z^2 = 4$.

7. From Equations 1 we have $z = \rho\cos\phi$, so $\rho\cos\phi = 1 \quad\Leftrightarrow\quad z = 1$, and the surface is the horizontal plane $z = 1$.

8. $\rho = \cos\phi \Rightarrow \rho^2 = \rho\cos\phi \Leftrightarrow x^2 + y^2 + z^2 = z \Leftrightarrow x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4} \Leftrightarrow x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$.

Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$.

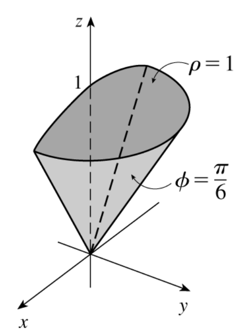
9. (a) From Equation 2 we have $\rho^2 = x^2 + y^2 + z^2$, so $x^2 + y^2 + z^2 = 9 \Leftrightarrow \rho^2 = 9 \Rightarrow \rho = 3$ (since $\rho \geq 0$).

(b) From Equations 1 we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $x^2 - y^2 - z^2 = 1$ becomes $(\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2 - (\rho\cos\phi)^2 = 1 \Leftrightarrow (\rho^2\sin^2\phi)(\cos^2\theta - \sin^2\theta) - \rho^2\cos^2\phi = 1 \Leftrightarrow$ $\rho^2(\sin^2\phi\cos 2\theta - \cos^2\phi) = 1$.

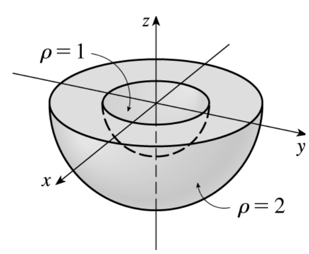
10. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z = x^2 + y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi$ or $\rho = \cos\phi\csc^2\phi$ or $\rho = \cot\phi\csc\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

(b) The equation $z = x^2 - y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2(\sin^2\phi)(\cos^2\theta - \sin^2\theta) \Leftrightarrow \rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi\cos 2\theta$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

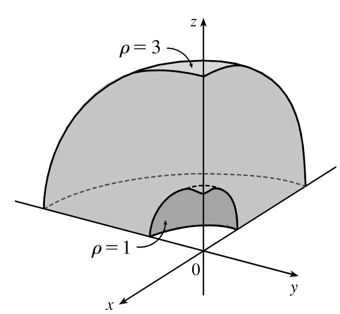
11. $\rho \leq 1$ represents the (solid) unit ball. $0 \leq \phi \leq \frac{\pi}{6}$ restricts the solid to that portion on or above the cone $\phi = \frac{\pi}{6}$, and $0 \leq \theta \leq \pi$ further restricts the solid to that portion on or to the right of the xz-plane.



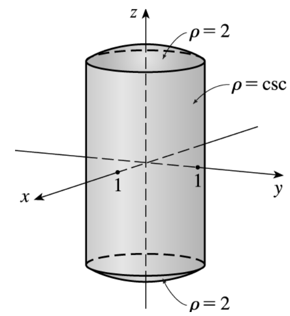
12. $1 \leq \rho \leq 2$ represents the solid region between and including the spheres of radii 1 and 2, centered at the origin. $\frac{\pi}{2} \leq \phi \leq \pi$ restricts the solid to that portion on or below the xy-plane.



13. $1 \leq \rho \leq 3$ represents the solid region between and including the spheres of radii 1 and 3, centered at the origin. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion on or above the xy-plane. $\pi \leq \theta \leq \frac{3\pi}{2}$ further restricts the solid to the portion over the third quadrant.



14. $\rho \le 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Then $\rho = \csc\phi \quad\Rightarrow\quad \rho\sin\phi = 1 \quad\Rightarrow\quad \rho^2\sin^2\phi = x^2 + y^2 = 1$, so $\rho \le \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.

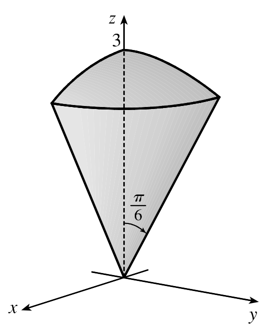


15. $x^2 + y^2 + z^2 = 4z \quad\Leftrightarrow\quad x^2 + y^2 + z^2 - 4z + 4 = 4 \quad\Leftrightarrow\quad x^2 + y^2 + (z-2)^2 = 2^2$, which is a sphere with radius 2 centered at $(0, 0, 2)$. In spherical coordinates, we have $\rho^2 = 4\rho\cos\phi \quad\Leftrightarrow\quad \rho^2 - 4\rho\cos\phi = 0 \quad\Leftrightarrow\quad \rho = 0$ or $\rho = 4\cos\phi$, so "inside the sphere" is described by $0 \le \rho \le 4\cos\phi$. The cone $z = \sqrt{x^2 + y^2}$ (see Figure 15.7.13) is described by $\phi = \frac{\pi}{4}$, so "outside the cone" is described by $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$.

16. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane. This restricts ϕ from 0 to $\pi/2$ and the hemisphere can be described by $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$.

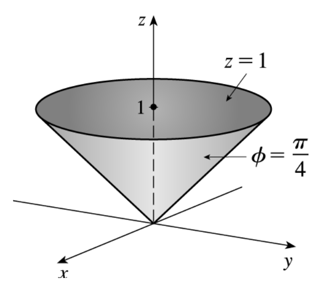
17.



The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned}
\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\
&= \left[-\cos\phi\right]_0^{\pi/6} \left[\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}\rho^3\right]_0^3 \\
&= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}\left(2 - \sqrt{3}\right)
\end{aligned}$$

18. The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le \sec\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}$. $\rho = \sec\phi \quad\Leftrightarrow\quad \rho\cos\phi = 1 \quad\Leftrightarrow\quad z = 1$, so E is the solid region above the cone $\phi = \pi/4$ and below the plane $z = 1$.



$$\begin{aligned}
\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/4} \int_0^{2\pi} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_{\rho=0}^{\rho=\sec\phi} d\theta \, d\phi \\
&= \int_0^{\pi/4} \int_0^{2\pi} \tfrac{1}{3} \sec^3\phi \sin\phi \, d\theta \, d\phi \\
&= \tfrac{1}{3} \int_0^{\pi/4} \sec^3\phi \sin\phi \, d\phi \int_0^{2\pi} d\theta = \tfrac{1}{3} \int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi \int_0^{2\pi} d\theta \\
&= \tfrac{1}{3} \left[\tfrac{1}{2}\tan^2\phi\right]_0^{\pi/4} \left[\theta\right]_0^{2\pi} = \tfrac{1}{3}\left(\tfrac{1}{2} - 0\right)(2\pi) = \tfrac{\pi}{3}
\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:

$E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 3,\ 0 \le z \le 2\right\}$. Then

$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$.

20. The solid E is most conveniently described if we use spherical coordinates:

$E = \left\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2,\ \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}\right\}$. Then

$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

21. (a) The solid can be described in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 2 \le \rho \le 3, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \frac{\pi}{2} \le \phi \le \pi\}$.

Thus, $\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV = \int_{\pi/2}^{\pi} \int_{\pi/2}^{3\pi/2} \int_2^3 \rho \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

(b) $\int_{\pi/2}^{\pi} \int_{\pi/2}^{3\pi/2} \int_2^3 \rho^3 \sin\phi\, d\rho\, d\theta\, d\phi = \int_{\pi/2}^{\pi} \sin\phi\, d\phi \int_{\pi/2}^{3\pi/2} d\theta \int_2^3 \rho^3\, d\rho$

$$= \Big[-\cos\phi\Big]_{\phi=\pi/2}^{\phi=\pi} \Big[\theta\Big]_{\theta=\pi/2}^{\theta=3\pi/2} \left[\frac{\rho^4}{4}\right]_{\rho=2}^{\rho=3} = (1)(\pi) \cdot \frac{1}{4}(81 - 16) = \frac{65\pi}{4}$$

22. (a) The solid can be described in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 2\sqrt{2}, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}\}$.

Thus, $\iiint_E xy\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{2\sqrt{2}} \rho\sin\phi\cos\theta \cdot \rho\sin\phi\sin\theta \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

(b) $\int_0^{\pi/4} \int_0^{2\pi} \int_0^{2\sqrt{2}} \rho^4 \sin^3\phi \cos\theta \sin\theta\, d\rho\, d\theta\, d\phi = \int_0^{\pi/4} \sin^3\phi\, d\phi \int_0^{2\pi} \cos\theta\sin\theta\, d\theta \int_0^{2\sqrt{2}} \rho^4\, d\rho$. Since

$\int_0^{2\pi} \cos\theta\sin\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta = \frac{1}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{2\pi} = -\frac{1}{4}(1 - 1) = 0$, the original iterated integral equals 0.

23. In spherical coordinates, B is represented by $\{(\rho, \theta, \phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned} \iiint_B (x^2 + y^2 + z^2)^2\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^5 (\rho^2)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^5 \rho^6\, d\rho \\ &= \big[-\cos\phi\big]_0^{\pi} \big[\,\theta\,\big]_0^{2\pi} \big[\tfrac{1}{7}\rho^7\big]_0^5 = (2)(2\pi)\left(\tfrac{78{,}125}{7}\right) \\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7 \end{aligned}$$

24. In spherical coordinates, E is represented by $\left\{(\rho, \theta, \phi) \,\middle|\, 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3}\right\}$. Thus

$$\begin{aligned} \iiint_E y^2 z^2\, dV &= \int_0^{\pi/3} \int_0^{2\pi} \int_0^1 (\rho\sin\phi\sin\theta)^2 (\rho\cos\phi)^2\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= \int_0^{\pi/3} \sin^3\phi \cos^2\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \int_0^1 \rho^6\, d\rho \\ &= \int_0^{\pi/3} (1 - \cos^2\phi) \cos^2\phi \sin\phi\, d\phi \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta \int_0^1 \rho^6\, d\rho \\ &= \left[\tfrac{1}{5}\cos^5\phi - \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/3} \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{7}\rho^7\right]_0^1 \\ &= \left[\tfrac{1}{5}\left(\tfrac{1}{2}\right)^5 - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - \tfrac{1}{5} + \tfrac{1}{3}\right] (\pi - 0) \left(\tfrac{1}{7} - 0\right) = \tfrac{47}{480} \cdot \pi \cdot \tfrac{1}{7} = \tfrac{47}{3360}\pi \end{aligned}$$

25. In spherical coordinates, E is represented by $\{(\rho, \theta, \phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and

$x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) = \rho^2 \sin^2\phi$. Thus

[continued]

$$\iiint_E (x^2+y^2)\,dV = \int_0^\pi \int_0^{2\pi} \int_2^3 (\rho^2 \sin^2\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} d\theta \int_2^3 \rho^4\,d\rho$$

$$= \int_0^\pi (1-\cos^2\phi)\sin\phi\,d\phi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{5}\rho^5\big]_2^3 = \big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\big]_0^\pi\,(2\pi)\cdot\tfrac{1}{5}(243-32)$$

$$= \big(1-\tfrac{1}{3}+1-\tfrac{1}{3}\big)(2\pi)\big(\tfrac{211}{5}\big) = \tfrac{1688\pi}{15}$$

26. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 3, 0\le\theta\le\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_E y^2\,dV = \int_0^\pi \int_0^\pi \int_0^3 (\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin^3\phi\,d\phi \int_0^\pi \sin^2\theta\,d\theta \int_0^3 \rho^4\,d\rho$$

$$= \int_0^\pi (1-\cos^2\phi)\sin\phi\,d\phi \int_0^\pi \tfrac{1}{2}(1-\cos 2\theta)\,d\theta \int_0^3 \rho^4\,d\rho$$

$$= \big[-\cos\phi+\tfrac{1}{3}\cos^3\phi\big]_0^\pi\,\big[\tfrac{1}{2}\big(\theta-\tfrac{1}{2}\sin 2\theta\big)\big]_0^\pi\,\big[\tfrac{1}{5}\rho^5\big]_0^3$$

$$= \big(\tfrac{2}{3}+\tfrac{2}{3}\big)\big(\tfrac{1}{2}\pi\big)\big(\tfrac{1}{5}(243)\big) = \big(\tfrac{4}{3}\big)\big(\tfrac{\pi}{2}\big)\big(\tfrac{243}{5}\big) = \tfrac{162\pi}{5}$$

27. In spherical coordinates, E is represented by $\big\{(\rho,\theta,\phi)\,\big|\,0\le\rho\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\big\}$. Thus

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\int_0^1\rho^3e^{\rho^2}\,d\rho$$

$$= \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\phi)\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\left(\tfrac{1}{2}\rho^2e^{\rho^2}\Big]_0^1 - \int_0^1\rho e^{\rho^2}\,d\rho\right)$$

$$\left[\text{integrate by parts with } u=\rho^2,\, dv=\rho e^{\rho^2}d\rho\right]$$

$$= \big[\tfrac{1}{2}\phi-\tfrac{1}{4}\sin 2\phi\big]_0^{\pi/2}\,[\sin\theta]_0^{\pi/2}\,\Big[\tfrac{1}{2}\rho^2e^{\rho^2}-\tfrac{1}{2}e^{\rho^2}\Big]_0^1 = \big(\tfrac{\pi}{4}-0\big)(1-0)\big(0+\tfrac{1}{2}\big) = \tfrac{\pi}{8}$$

28. In spherical coordinates, the cone $z=\sqrt{x^2+y^2}$ is equivalent to $\phi=\pi/4$ (as in Example 4) and E is represented by $\{(\rho,\theta,\phi)\mid 1\le\rho\le 2,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/4\}$. Also $\sqrt{x^2+y^2+z^2}=\sqrt{\rho^2}=\rho$, so

$$\iiint_E\sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_1^2\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/4}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_1^2\rho^3\,d\rho$$

$$= [-\cos\phi]_0^{\pi/4}\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{4}\rho^4\big]_1^2 = \left(-\tfrac{\sqrt{2}}{2}+1\right)(2\pi)\cdot\tfrac{1}{4}(16-1) = \tfrac{15}{2}\pi\left(1-\tfrac{\sqrt{2}}{2}\right)$$

29. The solid region is given by $E=\big\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le 2\pi, \frac{\pi}{6}\le\phi\le\frac{\pi}{3}\big\}$ and its volume is

$$V=\iiint_E dV = \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^a\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/3}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^2\,d\rho$$

$$= [-\cos\phi]_{\pi/6}^{\pi/3}\,[\theta]_0^{2\pi}\,\big[\tfrac{1}{3}\rho^3\big]_0^a = \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)(2\pi)\big(\tfrac{1}{3}a^3\big) = \tfrac{\sqrt{3}-1}{3}\pi a^3$$

30. If we center the ball at the origin, then the ball is given by $B=\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$ and the distance from any point (x,y,z) in the ball to the center $(0,0,0)$ is $\sqrt{x^2+y^2+z^2}=\rho$. Thus the average distance is

$$\frac{1}{V(B)}\iiint_B\rho\,dV = \frac{1}{\frac{4}{3}\pi a^3}\int_0^\pi\int_0^{2\pi}\int_0^a\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3}\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^3\,d\rho$$

$$= \frac{3}{4\pi a^3}\,\big[-\cos\phi\big]_0^\pi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{4}\rho^4\big]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\big(\tfrac{1}{4}a^4\big) = \tfrac{3}{4}a$$

31. (a) Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi \quad\Leftrightarrow\quad x^2+y^2+z^2 = 4z \quad\Leftrightarrow\quad x^2+y^2+(z-2)^2 = 4$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(\tfrac{64}{3}\cos^3\phi\right)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16}-1\right) d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3 \cos\phi\sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \cos\phi\sin\phi\left(64\cos^4\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi} 64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \tfrac{21}{2} \, d\theta = 21\pi$$

Hence, $(\overline{x},\overline{y},\overline{z}) = (0,0,21\pi/(10\pi)) = (0,0,2.1)$.

32. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \frac{\pi}{4}$ (as in Example 4). Thus, the solid is given by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\right\}$ and

$$V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\,\pi}{3}$$

33. (a) By the symmetry of the region, $M_{yz} = 0$ and $M_{xz} = 0$. Assuming constant density K, $m = \iiint_E K \, dV = K \iiint_E dV = \frac{\pi}{8}K$ (from Example 4). Then

$$M_{xy} = \iiint_E z\,K \, dV = K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$

$$= \tfrac{1}{4}K\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi\left(\cos^4\phi\right) d\phi \, d\theta = \tfrac{1}{4}K\int_0^{2\pi} d\theta \int_0^{\pi/4} \cos^5\phi\sin\phi \, d\phi$$

$$= \tfrac{1}{4}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi\right]_0^{\pi/4} = \tfrac{1}{4}K(2\pi)\left(-\tfrac{1}{6}\right)\left[\left(\tfrac{\sqrt{2}}{2}\right)^6 - 1\right] = -\tfrac{\pi}{12}K\left(-\tfrac{7}{8}\right) = \tfrac{7\pi}{96}K$$

Thus, the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \dfrac{7\pi K/96}{\pi K/8}\right) = \left(0,0,\tfrac{7}{12}\right)$.

(b) As in Exercise 25, $x^2+y^2 = \rho^2\sin^2\phi$ and

$$I_z = \iiint_E (x^2+y^2)\,K \, dV = K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (\rho^2\sin^2\phi)\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi\left[\tfrac{1}{5}\rho^5\right]_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$

$$= \tfrac{1}{5}K\int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi\cos^5\phi \, d\phi \, d\theta = \tfrac{1}{5}K\int_0^{2\pi} d\theta\int_0^{\pi/4} \cos^5\phi\left(1-\cos^2\phi\right)\sin\phi \, d\phi$$

$$= \tfrac{1}{5}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi + \tfrac{1}{8}\cos^8\phi\right]_0^{\pi/4}$$

$$= \tfrac{1}{5}K(2\pi)\left[-\tfrac{1}{6}\left(\tfrac{\sqrt{2}}{2}\right)^6 + \tfrac{1}{8}\left(\tfrac{\sqrt{2}}{2}\right)^8 + \tfrac{1}{6} - \tfrac{1}{8}\right] = \tfrac{2\pi}{5}K\left(\tfrac{11}{384}\right) = \tfrac{11\pi}{960}K$$

34. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \, d\phi \int_0^a \rho^3 \, d\rho$$

$$= K\left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta \\ &= K \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi \int_0^a \rho^4 \, d\rho \\ &= K \big[\, \theta \,\big]_0^{2\pi} \left[\tfrac{1}{2} \sin^2 \phi\right]_0^{\pi/2} \left[\tfrac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{5}\pi K a^5 \end{aligned}$$

Hence, $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{5}a\right)$.

(c) $I_z = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho^3 \sin\phi)(\rho^2 \sin^2\phi) \, d\rho \, d\phi \, d\theta = K \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3\phi \, d\phi \int_0^a \rho^5 \, d\rho$

$= K \big[\, \theta \,\big]_0^{2\pi} \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/2} \left[\frac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\frac{2}{3}\right)\left(\frac{1}{6}a^6\right) = \frac{2}{9}\pi K a^6$

35. (a) The density function is $\rho(x, y, z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^3 \sin\phi \, \cos\phi \, d\rho \, d\phi \, d\theta = \frac{1}{2}\pi K a^4 \int_0^{\pi/2} \sin\phi \, \cos\phi \, d\phi = \frac{1}{8}\pi K a^4$. But the mass is

$K \cdot (\text{volume of the hemisphere}) = \frac{2}{3}\pi K a^3$, so the centroid is $\left(0, 0, \frac{3}{8}a\right)$.

(b) Place the center of the base at $(0, 0, 0)$; the density function is $\rho(x, y, z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$\begin{aligned} I_x &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho^2 \sin\phi) \, \rho^2 \, (\sin^2\phi \, \sin^2\theta + \cos^2\phi) \, d\rho \, d\phi \, d\theta \\ &= K \int_0^{2\pi} \int_0^{\pi/2} (\sin^3\phi \, \sin^2\theta + \sin\phi \, \cos^2\phi) \left(\tfrac{1}{5}a^5\right) d\phi \, d\theta \\ &= \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\sin^2\theta \left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) + \left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2} d\theta = \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\right] d\theta \\ &= \tfrac{1}{5}Ka^5 \left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{3}\theta\right]_0^{2\pi} = \tfrac{1}{5}Ka^5 \left[\tfrac{2}{3}(\pi - 0) + \tfrac{1}{3}(2\pi - 0)\right] = \tfrac{4}{15}\pi K a^5 \end{aligned}$$

36. Place the center of the base at $(0, 0, 0)$, then the density is $\rho(x, y, z) = Kz$, K a constant. Then

$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho\cos\phi) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi K \int_0^{\pi/2} \cos\phi \sin\phi \cdot \frac{1}{4}a^4 \, d\phi = \frac{1}{2}\pi K a^4 \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \frac{\pi}{4}Ka^4$.

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2}{5}\pi K a^5 \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi = \frac{2}{5}\pi K a^5 \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{15}\pi K a^5$.

Hence, $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{8}{15}a\right)$.

37. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\phi = \frac{\pi}{4}$ (as in Example 4). Then

$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^1 \rho^2 \, d\rho = 2\pi \left(-\frac{\sqrt{2}}{2} + 1\right)\left(\frac{1}{3}\right) = \frac{1}{3}\pi\left(2 - \sqrt{2}\right)$,

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4} \left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence, $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \dfrac{3}{8\left(2 - \sqrt{2}\right)}\right)$.

38. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$V = \int_0^{\pi/6} \int_0^{\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a \rho^2 \, d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3.$

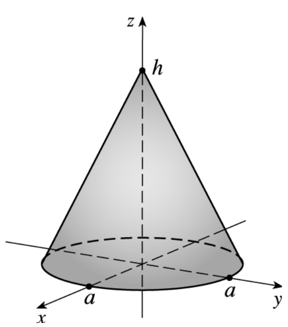
39. (a) If we orient the cylinder so that its axis is the z-axis and its base lies in the xy-plane, then the cylinder is described, in cylindrical coordinates, by $E = \{(r, \theta, z) \mid 0 \le r \le a, \ 0 \le \theta \le 2\pi, \ 0 \le z \le h\}$. Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^h K(r^2)\, r \, dz \, dr \, d\theta = K \int_0^{2\pi} d\theta \int_0^a r^3 \, dr \int_0^h dz \\ &= K \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^a \left[z\right]_0^h = K\,(2\pi)\left(\tfrac{1}{4}a^4\right)(h) = \tfrac{1}{2}\pi K a^4 h \end{aligned}$$

(b) By symmetry, the moments of inertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned} I_x &= \iiint_E (y^2 + z^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^h K(r^2 \sin^2\theta + z^2)\, r \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \int_0^a \int_0^h r^3 \sin^2\theta \, dz \, dr \, d\theta + K \int_0^{2\pi} \int_0^a \int_0^h r z^2 \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^a r^3 \, dr \int_0^h dz + K \int_0^{2\pi} d\theta \int_0^a r \, dr \int_0^h z^2 \, dz \\ &= K \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^a \left[z\right]_0^h + K \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^a \left[\tfrac{1}{3}z^3\right]_0^h \\ &= K\,(\pi)\left(\tfrac{1}{4}a^4\right)(h) + K\,(2\pi)\left(\tfrac{1}{2}a^2\right)\left(\tfrac{1}{3}h^3\right) = \tfrac{1}{12}\pi K a^2 h (3a^2 + 4h^2) \end{aligned}$$

40.



Orient the cone so that its axis is the z-axis and its base lies in the xy-plane, as shown in the figure. (Then the z-axis is the axis of the cone and the x-axis contains a diameter of the base.) A right circular cone with axis the z-axis and vertex at the origin has equation $z^2 = c^2(x^2 + y^2)$. Here we have the bottom frustum, shifted upward h units, and with $c^2 = h^2/a^2$ so that the cone includes the point $(a, 0, 0)$. Thus an equation of the cone in rectangular coordinates is $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \le z \le h$. In cylindrical coordinates, the cone is described by

$$E = \left\{(r, \theta, z) \mid 0 \le r \le a, \ 0 \le \theta \le 2\pi, \ 0 \le z \le h\left(1 - \tfrac{1}{a}r\right)\right\}$$

(a) Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^{h(1-r/a)} K(r^2)\, r \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \int_0^a \left[r^3 z\right]_{z=0}^{z=h(1-r/a)} dr \, d\theta = K \int_0^{2\pi} \int_0^a r^3 h\left(1 - \tfrac{1}{a}r\right) dr \, d\theta \\ &= Kh \int_0^{2\pi} d\theta \int_0^a \left(r^3 - \tfrac{1}{a}r^4\right) dr = Kh \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4 - \tfrac{1}{5a}r^5\right]_0^a \\ &= Kh\,(2\pi)\left(\tfrac{1}{4}a^4 - \tfrac{1}{5}a^4\right) = \tfrac{1}{10}\pi K a^4 h \end{aligned}$$

(b) By symmetry, the moments of intertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned}
I_x &= \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)} K(r^2\sin^2\theta+z^2)\,r\,dz\,dr\,d\theta \\
&= K\int_0^{2\pi}\int_0^a \left[(r^3\sin^2\theta)z+\tfrac{1}{3}rz^3\right]_{z=0}^{z=h(1-r/a)}\,dr\,d\theta \\
&= K\int_0^{2\pi}\int_0^a \left[(r^3\sin^2\theta)\left(h\left(1-\tfrac{1}{a}r\right)\right)+\tfrac{1}{3}r\left(h\left(1-\tfrac{1}{a}r\right)\right)^3\right]dr\,d\theta \\
&= Kh\int_0^{2\pi}\int_0^a (r^3\sin^2\theta)\left(1-\tfrac{1}{a}r\right)\,dr\,d\theta + Kh^3\int_0^{2\pi}\int_0^a \tfrac{1}{3}r\left(1-\tfrac{1}{a}r\right)^3\,dr\,d\theta \\
&= Kh\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^a\left(r^3-\tfrac{1}{a}r^4\right)\,dr+\tfrac{1}{3}Kh^3\int_0^{2\pi}d\theta\int_0^a\left(r-\tfrac{3}{a}r^2+\tfrac{3}{a^2}r^3-\tfrac{1}{a^3}r^4\right)dr \\
&= Kh\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4-\tfrac{1}{5a}r^5\right]_0^a+\tfrac{1}{3}Kh^3\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{2}r^2-\tfrac{1}{a}r^3+\tfrac{3}{4a^2}r^4-\tfrac{1}{5a^3}r^5\right]_0^a \\
&= Kh\,(\pi)\left(\tfrac{1}{4}a^4-\tfrac{1}{5}a^4\right)+\tfrac{1}{3}Kh^3\,(2\pi)\left(\tfrac{1}{2}a^2-a^2+\tfrac{3}{4}a^2-\tfrac{1}{5}a^2\right) \\
&= \pi Kh\left(\tfrac{1}{20}a^4\right)+\tfrac{2}{3}\pi Kh^3\left(\tfrac{1}{20}a^2\right)=\pi Ka^2h\left(\tfrac{1}{20}a^2+\tfrac{1}{30}h^2\right)
\end{aligned}$$

41. In cylindrical coordinates the paraboloid is given by $z=r^2$ and the plane by $z=2r\sin\theta$ and the projection of the intersection onto the xy-plane is the circle $r=2\sin\theta$. Then $\iiint_E z\,dV=\int_0^\pi\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta=\frac{5\pi}{6}$ [using a CAS].

42. (a) The region enclosed by the torus is $\{(\rho,\theta,\phi)\mid 0\le\theta\le 2\pi,\,0\le\phi\le\pi,\,0\le\rho\le\sin\phi\}$, so its volume is

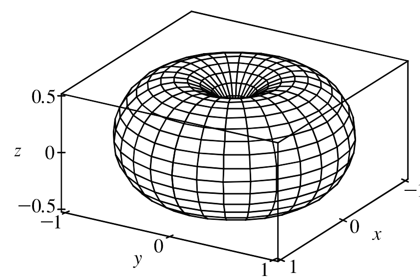
$V=\int_0^{2\pi}\int_0^\pi\int_0^{\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=2\pi\int_0^\pi\frac{1}{3}\sin^4\phi\,d\phi=\frac{2}{3}\pi\left[\frac{3}{8}\phi-\frac{1}{4}\sin 2\phi+\frac{1}{16}\sin 4\phi\right]_0^\pi=\frac{1}{4}\pi^2.$

(b) In Maple, we can plot the torus using the command

```
plot3d(sin(phi),theta=0..2*Pi,
       phi=0..Pi,coords=spherical);.
```

In Mathematica, use

```
SphericalPlot3D[Sin[phi],{phi,0,Pi},{theta,0,2Pi}].
```



43. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$. The region E of integration is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=2$ in the first octant. Because E is in the first octant we have $0\le\theta\le\frac{\pi}{2}$. The cone has equation $\phi=\frac{\pi}{4}$ (as in Example 4), so $0\le\phi\le\frac{\pi}{4}$, and $0\le\rho\le\sqrt{2}$. Then the integral becomes

$$\begin{aligned}
&\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}}(\rho\sin\phi\cos\theta)\,(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&\quad=\int_0^{\pi/4}\sin^3\phi\,d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\sqrt{2}}\rho^4\,d\rho=\left(\int_0^{\pi/4}\left(1-\cos^2\phi\right)\sin\phi\,d\phi\right)\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}} \\
&\quad=\left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{5}\left(\sqrt{2}\right)^5=\left[\tfrac{\sqrt{2}}{12}-\tfrac{\sqrt{2}}{2}-\left(\tfrac{1}{3}-1\right)\right]\cdot\tfrac{2\sqrt{2}}{5}=\tfrac{4\sqrt{2}-5}{15}
\end{aligned}$$

44. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dz\,dx\,dy$. The region of integration is the solid sphere $x^2+y^2+z^2\le a^2$, so $0\le\theta\le 2\pi$, $0\le\phi\le\pi$, and $0\le\rho\le a$. Also $x^2z+y^2z+z^3=(x^2+y^2+z^2)z=\rho^2z=\rho^3\cos\phi$, so the integral becomes

$$\int_0^\pi\int_0^{2\pi}\int_0^a\left(\rho^3\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^\pi\sin\phi\cos\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^5\,d\rho=\left[\tfrac12\sin^2\phi\right]_0^\pi\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac16\rho^6\right]_0^a=0$$

45. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$. The region of integration is the solid sphere $x^2+y^2+(z-2)^2\le 4$, or equivalently, $\rho^2\sin^2\phi+(\rho\cos\phi-2)^2=\rho^2-4\rho\cos\phi+4\le 4\quad\Rightarrow\quad\rho\le 4\cos\phi$, so $0\le\theta\le 2\pi$, $0\le\phi\le\frac{\pi}{2}$, and $0\le\rho\le 4\cos\phi$. Also $(x^2+y^2+z^2)^{3/2}=(\rho^2)^{3/2}=\rho^3$, so the integral becomes

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos\phi}\left(\rho^3\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi&=\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left[\tfrac16\rho^6\right]_{\rho=0}^{\rho=4\cos\phi}d\theta\,d\phi\\&=\tfrac16\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left(4096\cos^6\phi\right)d\theta\,d\phi\\&=\tfrac16(4096)\int_0^{\pi/2}\cos^6\phi\sin\phi\,d\phi\int_0^{2\pi}d\theta=\tfrac{2048}{3}\left[-\tfrac17\cos^7\phi\right]_0^{\pi/2}\left[\,\theta\,\right]_0^{2\pi}\\&=\tfrac{2048}{3}\left(\tfrac17\right)(2\pi)=\tfrac{4096\pi}{21}\end{aligned}$$

46. The solid region between the ground and an altitude of 5 km (5000 m) is given by

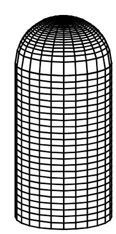
$$E=\left\{(\rho,\theta,\phi)\mid 6.370\times10^6\le\rho\le 6.375\times10^6,0\le\theta\le2\pi,0\le\phi\le\pi\right\}.$$

Then the mass of the atmosphere in this region is

$$\begin{aligned}m=\iiint_E\delta\,dV&=\int_0^{2\pi}\int_0^\pi\int_{6.370\times10^6}^{6.375\times10^6}(619.09-0.000097\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\&=\int_0^{2\pi}d\theta\int_0^\pi\sin\phi\,d\phi\int_{6.370\times10^6}^{6.375\times10^6}\left(619.09\rho^2-0.000097\rho^3\right)d\rho\\&=\left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_0^\pi\left[\tfrac{619.09}{3}\rho^3-\tfrac{0.000097}{4}\rho^4\right]_{6.370\times10^6}^{6.375\times10^6}\\&=(2\pi)(2)\left[\tfrac{619.09}{3}\left((6.375\times10^6)^3-(6.370\times10^6)^3\right)-\tfrac{0.000097}{4}\left((6.375\times10^6)^4-(6.370\times10^6)^4\right)\right]\\&\approx4\pi\left(1.944\times10^{17}\right)\approx2.44\times10^{18}\text{ kg}\end{aligned}$$

47. In cylindrical coordinates, the equation of the cylinder is $r=3$, $0\le z\le 10$.

The hemisphere is the upper part of the sphere radius 3, center $(0,0,10)$, equation $r^2+(z-10)^2=3^2$, $z\ge10$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `ParametricPlot3D`.



48. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho=3960$ mi	$\rho=3960$ mi
$\theta=360^\circ-\ 73.60^\circ=286.40^\circ$	$\theta=360^\circ-\ 118.25^\circ=241.75^\circ$
$\phi=90^\circ-\ 45.50^\circ=44.50^\circ$	$\phi=90^\circ-\ 34.06^\circ=55.94^\circ$

[continued]

Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the earth). In particular:

Montréal: $\langle 783.67, -2662.67, 2824.47\rangle$ Los Angeles: $\langle -1552.80, -2889.91, 2217.84\rangle$

To find the angle γ between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2\cos\gamma \quad\Rightarrow\quad \cos\gamma \approx 0.8126 \quad\Rightarrow$ $\gamma \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\gamma \approx 3960(0.6223) \approx 2464$ mi.

49. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta\,\sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta\sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by $V(E) = \iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1 + (\sin 6\theta\sin 5\phi)/5} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{136\pi}{99}$ [using a CAS].

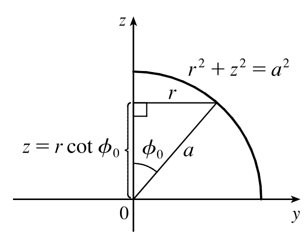
50.
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = \lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \lim_{R\to\infty}\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi \int_0^{R}\rho^3 e^{-\rho^2}\,d\rho$$

Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty} 2\pi(2)\left(\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\Big]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\,d\rho\right) = \lim_{R\to\infty} 4\pi\left(-\tfrac{1}{2}R^2e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)$$
$$= 4\pi\lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi$$

(Note that $R^2e^{-R^2} \to 0$ as $R \to \infty$ by l'Hospital's Rule.)

51. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus

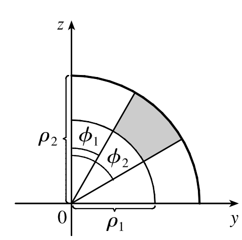


$$V = \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right)dr$$
$$= \tfrac{2\pi}{3}\left[-(a^2-r^2)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$
$$= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right]$$
$$= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3(1 - \cos\phi_0)$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$.

Letting

V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)



and letting V be the volume of the wedge, we have

[continued]

$$\begin{aligned}V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1-\cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)\right] \\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right]\end{aligned}$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1 \sin\phi_1}^{\rho_2 \sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

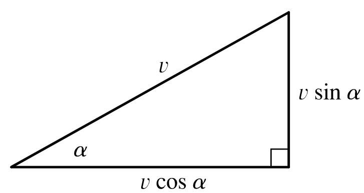
(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_2^3 - \rho_1^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly, with $f(\phi) = \cos\phi$ there exists some $\tilde{\phi}$ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives $\Delta V = \frac{1}{3}(\theta_2 - \theta_1)(3\tilde{\rho}^2\,\Delta\rho)(\sin\tilde{\phi})\;\Delta\phi = \tilde{\rho}^2 \sin\tilde{\phi}\,\Delta\rho\,\Delta\theta\,\Delta\phi$.

APPLIED PROJECT Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad\Rightarrow\quad 2gh = v^2 + \dfrac{I}{m}\left(\dfrac{v}{r}\right)^2 \quad\Rightarrow\quad 2gh = v^2\left(1 + \dfrac{I}{mr^2}\right) \quad\Rightarrow\quad v^2 = \dfrac{2gh}{1+I^*}$.

2. The vertical component of the speed is $dy/dt = v\sin\alpha$, so by the same reasoning as used in Problem 1,

$$\frac{dy}{dt} = \sqrt{\frac{2gy}{1+I^*}}\sin\alpha = \sqrt{\frac{2g}{1+I^*}}\sin\alpha\,\sqrt{y}.$$



3. Solving the separable differential equation, we get $\dfrac{dy}{\sqrt{y}} = \sqrt{\dfrac{2g}{1+I^*}}\sin\alpha\,dt \quad\Rightarrow\quad 2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)t + C$.

But $y = 0$ when $t = 0$, so $C = 0$ and we have $2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)t$. Solving for t when $y = h$ gives

$$T = \frac{2\sqrt{h}}{\sin\alpha}\sqrt{\frac{1+I^*}{2g}} = \sqrt{\frac{2h(1+I^*)}{g\sin^2\alpha}}.$$

4. Assume that the length of each cylinder is ℓ. Then the density of the solid cylinder is $\dfrac{m}{\pi r^2\ell}$, and from Formulas 15.6.16, its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint (x^2+y^2)\frac{m}{\pi r^2\ell}\,dV = \frac{m}{\pi r^2\ell}\int_0^\ell\int_0^{2\pi}\int_0^r R^2 R\,dR\,d\theta\,dz = \frac{m}{\pi r^2\ell}\cdot\ell\cdot 2\pi\cdot\left[\tfrac{1}{4}R^4\right]_0^r = \frac{mr^2}{2}$$

and so $I^* = \dfrac{I_z}{mr^2} = \dfrac{1}{2}$.

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \dfrac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral: $I_z = \displaystyle\iint (x^2+y^2)\frac{m}{2\pi r\ell}\,dA = \frac{mr^2}{2\pi r\ell}\iint dA = mr^2$, so $I^* = \dfrac{I_z}{mr^2} = 1$.

5. The volume of such a ball is $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi[r^3 - (br)^3] = \frac{4}{3}\pi r^3(1 - b^3)$, and so its density is $\dfrac{m}{\frac{4}{3}\pi r^3(1-b^3)}$. Now

$$\begin{aligned} I_z &= \iiint (x^2+y^2)\,\frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\,dV \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \int_a^r \int_0^{2\pi} \int_0^{\pi} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\,d\phi\,d\theta\,d\rho \qquad \text{[from Formula 15.8.3 and Exercise 15.8.25]} \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \left[\frac{\rho^5}{5}\right]_a^r \cdot \Big[\theta\Big]_0^{2\pi} \cdot \left[-\frac{(2+\sin^2\phi)\cos\phi}{3}\right]_0^{\pi} \qquad \text{[from Formula 67 in the Table of Integrals]} \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \cdot \frac{r^5-a^5}{5} \cdot 2\pi \cdot \frac{4}{3} = \frac{2mr^5(1-b^5)}{5r^3(1-b^3)} = \frac{2(1-b^5)mr^2}{5(1-b^3)} \end{aligned}$$

Therefore, $I^* = \dfrac{I_z}{mr^2} = \dfrac{2(1-b^5)}{5(1-b^3)}$. Since a represents the inner radius, $a \to 0$ corresponds to a solid ball, and $a \to r$ corresponds to a hollow ball.

6. For a solid ball, $a \to 0 \;\Rightarrow\; b \to 0$, so $I^* = \lim\limits_{b\to 0} \dfrac{2(1-b^5)}{5(1-b^3)} = \dfrac{2}{5}$. For a hollow ball, $a \to r \;\Rightarrow\; b \to 1$, so

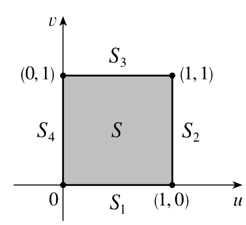
$$I^* = \lim_{b\to 1} \frac{2(1-b^5)}{5(1-b^3)} = \frac{2}{5} \lim_{b\to 1} \frac{-5b^4}{-3b^2} = \frac{2}{5}\left(\frac{5}{3}\right) = \frac{2}{3} \qquad \text{[by l'Hospital's Rule]}$$

Note: We could instead have calculated $I^* = \lim\limits_{b\to 1} \dfrac{2(1-b)(1+b+b^2+b^3+b^4)}{5(1-b)(1+b+b^2)} = \dfrac{2\cdot 5}{5\cdot 3} = \dfrac{2}{3}$.

Thus the objects finish in the following order: solid ball $\left(I^* = \frac{2}{5}\right)$, solid cylinder $\left(I^* = \frac{1}{2}\right)$, hollow ball $\left(I^* = \frac{2}{3}\right)$, hollow cylinder $(I^* = 1)$.

15.9 Change of Variables in Multiple Integrals

1. For Exercise 1(a)–(f), we refer to the figure. Each transformation maps the boundary of S to the boundary of one of the images (I–VI).



(a) Along S_1, $v = 0$, $0 \le u \le 1$, so $x = u + v = u$ and $y = u - v = u \;\Rightarrow\;$ $y = x$, $0 \le x \le 1$. [Note that only images V and VI have $y = x$ as a boundary.] Along S_2, $u = 1$, $0 \le v \le 1$, so $x = 1 + v$ and $y = 1 - v$. Eliminating v gives $x + y = 2$, $1 \le x \le 2$. Along S_3, $v = 1$, $0 \le u \le 1 \;\Rightarrow\; (x)x = u + 1$ and $y = u - 1$. Eliminating u gives $y - x = -2$, $1 \le x \le 2$. Finally, along S_4, $u = 0$, $0 \le v \le 1 \;\Rightarrow\; x = v$ and $y = -v \;\Rightarrow\; y = -x$, $0 \le x \le 1$. Thus, VI is the image of the transformation.

(b) Along S_1, $v = 0$, $0 \le u \le 1 \;\Rightarrow\; y = uv = 0$, $0 \le x \le 1$. Along S_2, $u = 1$, $0 \le v \le 1 \;\Rightarrow\; x = u - v = 1 - v$, $y = v \;\Rightarrow\; y = 1 - x$, $0 \le x \le 1$. Along S_3, $v = 1$, $0 \le u \le 1 \;\Rightarrow\; x = u - 1$, $y = u \;\Rightarrow\; y = x + 1$, $-1 \le x \le 0$. Finally, along S_4, $u = 0$, $0 \le v \le 1 \;\Rightarrow\; y = 0$, $-1 \le x \le 0$. Thus, I is the image of the transformation.

(c) Along S_1, $v = 0$, $0 \le u \le 1 \;\Rightarrow\; y = u\sin v = 0$ and $x = u\cos v = u\cos 0 = u \;\Rightarrow\; 0 \le x \le 1$. Along S_2, $u = 1$, $0 \le v \le 1 \;\Rightarrow\; x = \cos v$, $y = \sin v$, $0 \le v \le 1$, which are parametric equations for a circle of radius 1,

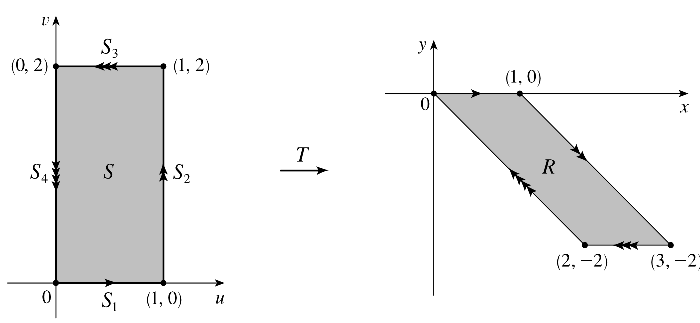
$1 \geq x \geq \cos 1$. Along S_3, $v = 1$, $0 \leq u \leq 1 \;\Rightarrow\; x = u\cos 1$, $y = u\sin 1 \;\Rightarrow\; y = \tan(1)x$, $0 \leq x \leq \cos 1$. Along S_4, $u = 0 \;\Rightarrow\; x = y = 0$. Thus, IV is the image of the transformation.

(d) Along S_1, $v = 0$, $0 \leq u \leq 1 \;\Rightarrow\; x = u - v = u$, $y = u + v^2 = u \;\Rightarrow\; y = x$, $0 \leq x \leq 1$. Along S_2, $u = 1$, $0 \leq v \leq 1 \;\Rightarrow\; x = 1 - v$, $y = 1 + v^2 \;\Rightarrow\; y = 1 + (1-x)^2$, $1 \geq x \geq 0$. Along S_3, $v = 1$, $0 \leq u \leq 1 \;\Rightarrow\; x = u - 1$, $y = u + 1 \;\Rightarrow\; y = x + 2$, $-1 \leq x \leq 0$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \;\Rightarrow\; x = -v$, $y = v^2 \;\Rightarrow\; y = x^2$, $-1 \leq x \leq 0$. Thus, V is the image of the transformation.

(e) Along S_1, $v = 0$, $0 \leq u \leq 1 \;\Rightarrow\; y = 2v = 0$, and since $x = u + v = u$, $0 \leq x \leq 1$. Along S_2, $u = 1$, $0 \leq v \leq 1 \;\Rightarrow\; x = 1 + v$, $y = 2v \;\Rightarrow\; y = 2x - 2$, $1 \leq x \leq 2$. Along S_3, $v = 1$, $0 \leq u \leq 1 \;\Rightarrow\; y = 2v = 2$, $1 \leq x \leq 2$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \;\Rightarrow\; x = v$, $y = 2v \;\Rightarrow\; y = 2x$, $0 \leq x \leq 1$. Thus, III is the image of the transformation.

(f) Along S_1, $v = 0$, $0 \leq u \leq 1 \;\Rightarrow\; x = uv = 0$, and since $y = u^3 - v^3 = u^3$, $0 \leq y \leq 1$. Along S_2, $u = 1$, $0 \leq u \leq 1 \;\Rightarrow\; x = v$, $y = 1 - v^3 \;\Rightarrow\; y = 1 - x^3$, $1 \geq x \geq 0$. Along S_3, $v = 1$, $0 \leq u \leq 1 \;\Rightarrow\; x = u$, $y = u^3 - 1 \;\Rightarrow\; y = x^3 - 1$, $0 \leq x \leq 1$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \;\Rightarrow\; x = 0$, $-1 \leq y \leq 0$. Thus, II is the image of the transformation.

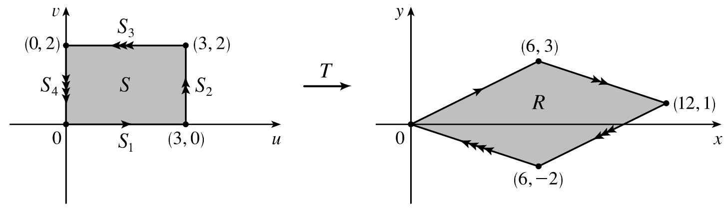
2. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$, $0 \leq u \leq 1$, so $x = u + v = u$ and $y = -v = 0$. Therefore, the image is the line segment $y = 0$, $0 \leq x \leq 1$. S_2 is the line segment $u = 1$, $0 \leq v \leq 2$, so $x = 1 + v$ and $y = -v$. Eliminating v, we have $y = 1 - x$, $1 \leq x \leq 3$. S_3 is the line segment $v = 2$, $0 \leq u \leq 1$, so $x = u + 2$ and $y = -2 \;\Rightarrow\;$ the image is the line segment $y = -2$, $2 \leq x \leq 3$. Finally, S_4 is the line segment $u = 0$, $0 \leq v \leq 2$, so $x = v$ and $y = -v \;\Rightarrow\;$ the image is the line segment $y = -x$, $0 \leq x \leq 2$. The image of the set S is the region R shown in the xy-plane, a parallelogram bounded by these four line segments.



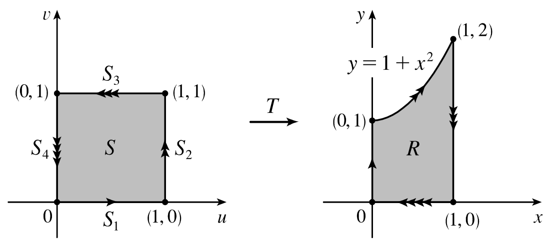
3. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$, $0 \leq u \leq 3$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \leq x \leq 6$. S_2 is the line segment $u = 3$, $0 \leq v \leq 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \;\Rightarrow\; x = 6 + 3(3 - y) = 15 - 3y$, $6 \leq x \leq 12$. S_3 is the line segment $v = 2$, $0 \leq u \leq 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \;\Rightarrow\; x = 2y + 10$, $6 \leq x \leq 12$. Finally, S_4 is the segment $u = 0$, $0 \leq v \leq 2$, so $x = 3v$ and $y = -v \;\Rightarrow\; x = -3y$, $0 \leq x \leq 6$.

[continued]

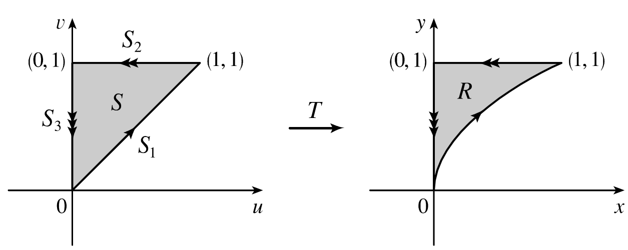
The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



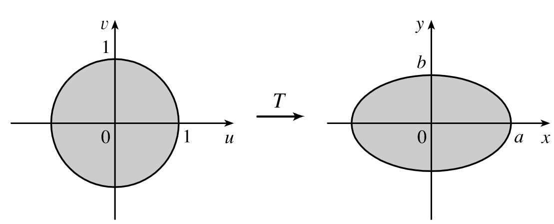
4. S_1 is the line segment $v = 0$, $0 \leq u \leq 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \leq u \leq 1$, the image is the line segment $x = 0$, $0 \leq y \leq 1$. S_2 is the segment $u = 1$, $0 \leq v \leq 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \leq x \leq 1$. S_3 is the segment $v = 1$, $0 \leq u \leq 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \leq y \leq 2$. S_4 is described by $u = 0$, $0 \leq v \leq 1$, so $0 \leq x = v \leq 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \leq x \leq 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



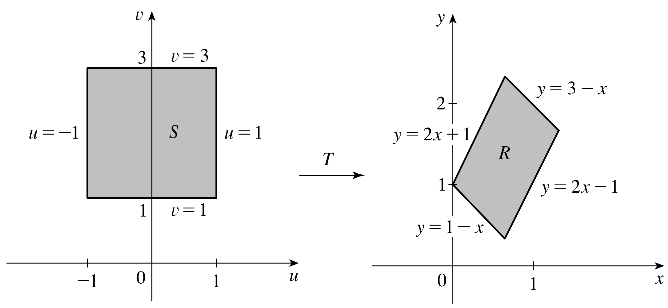
5. S_1 is the line segment $u = v$, $0 \leq u \leq 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \leq u \leq 1$, the image is the portion of the parabola $x = y^2$, $0 \leq y \leq 1$. S_2 is the segment $v = 1$, $0 \leq u \leq 1$, thus $y = v = 1$ and $x = u^2$, so $0 \leq x \leq 1$. The image is the line segment $y = 1$, $0 \leq x \leq 1$. S_3 is the segment $u = 0$, $0 \leq v \leq 1$, so $x = u^2 = 0$ and $y = v \quad \Rightarrow \quad 0 \leq y \leq 1$. The image is the segment $x = 0$, $0 \leq y \leq 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



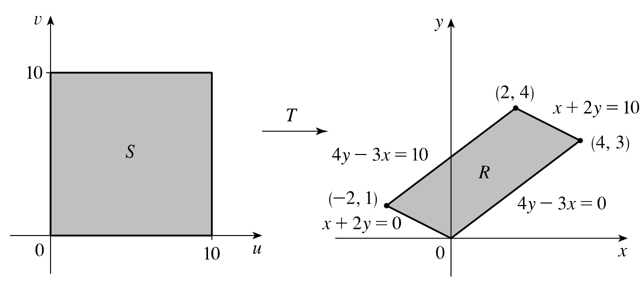
6. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \leq 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$, so the image of $u^2 + v^2 \leq 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.



7. R is a parallelogram enclosed by the parallel lines $y = 2x - 1$, $y = 2x + 1$ and the parallel lines $y = 1 - x$, $y = 3 - x$. The first pair of equations can be written as $y - 2x = -1$, $y - 2x = 1$. If we let $u = y - 2x$ then these lines are mapped to the vertical lines $u = -1$, $u = 1$ in the uv-plane. Similarly, the second pair of equations can be written as $x + y = 1$, $x + y = 3$, and setting $v = x + y$ maps these lines to the horizontal lines $v = 1$, $v = 3$ in the uv-plane. Boundary curves are mapped to boundary curves under a transformation, so here the equations $u = y - 2x$, $v = x + y$ define a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = -1$, $u = 1$, $v = 1$, $v = 3$ in the uv-plane. To find the transformation T that maps S to R we solve $u = y - 2x$, $v = x + y$ for x, y: Subtracting the first equation from the second gives $v - u = 3x \quad \Rightarrow \quad x = \frac{1}{3}(v - u)$ and adding twice the second equation to the first gives $u + 2v = 3y \quad \Rightarrow$ $y = \frac{1}{3}(u + 2v)$. Thus one possible transformation T (there are many) is given by $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$.

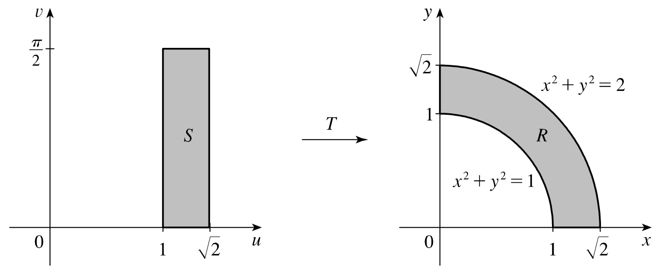


8. The boundaries of the parallelogram R are the lines $y = \frac{3}{4}x$ or $4y - 3x = 0$, $y = \frac{3}{4}x + \frac{5}{2}$ or $4y - 3x = 10$, $y = -\frac{1}{2}x$ or $x + 2y = 0$, $y = -\frac{1}{2}x + 5$ or $x + 2y = 10$. Setting $u = 4y - 3x$ and $v = x + 2y$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 0$, $u = 10$, $v = 0$, $v = 10$ in the uv-plane. Solving $u = 4y - 3x$, $v = x + 2y$ for x and y gives $2v - u = 5x \quad \Rightarrow \quad x = \frac{1}{5}(2v - u)$, $u + 3v = 10y \quad \Rightarrow \quad y = \frac{1}{10}(u + 3v)$. Thus one possible transformation T is given by $x = \frac{1}{5}(2v - u)$, $y = \frac{1}{10}(u + 3v)$.

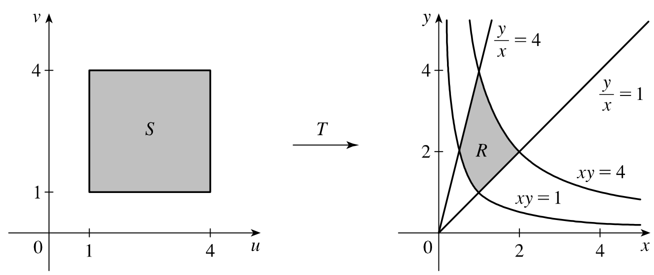


9. R is a portion of an annular region (see the figure) that is easily described in polar coordinates as $R = \left\{(r, \theta) \mid 1 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq \pi/2\right\}$. If we converted a double integral over R to polar coordinates the resulting region of integration is a rectangle (in the $r\theta$-plane), so we can create a transformation T here by letting u play the role of r and v the role of θ. Thus T is defined by $x = u\cos v$, $y = u\sin v$ and T maps the rectangle

$S = \{(u,v) \mid 1 \le u \le \sqrt{2},\ 0 \le v \le \pi/2\}$ in the uv-plane to R in the xy-plane.



10. The boundaries of the region R are the curves $y = 1/x$ or $xy = 1$, $y = 4/x$ or $xy = 4$, $y = x$ or $y/x = 1$, $y = 4x$ or $y/x = 4$. Setting $u = xy$ and $v = y/x$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 1$, $u = 4$, $v = 1$, $v = 4$ in the uv-plane. Solving $u = xy$, $v = y/x$ for x and y gives $x^2 = u/v \quad \Rightarrow$ $x = \sqrt{u/v}$ [since x, y, u, v are all positive], $y^2 = uv \quad \Rightarrow \quad y = \sqrt{uv}$. Thus one possible transformation T is given by $x = \sqrt{u/v}$, $y = \sqrt{uv}$.



11. $x = 2u + v,\ y = 4u - v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = (2)(-1) - (1)(4) = -6.$

12. $x = u^2 + uv,\ y = uv^2$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u+v & u \\ v^2 & 2uv \end{vmatrix} = (2u+v)(2uv) - u(v^2) = 4u^2v + 2uv^2 - uv^2 = 4u^2v + uv^2$$

13. $x = s\cos t,\ y = s\sin t$.

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = s\cos^2 t - (-s\sin^2 t) = s(\cos^2 t + \sin^2 t) = s$$

14. $x = pe^q,\ y = qe^p$.

$$\frac{\partial(x,y)}{\partial(p,q)} = \begin{vmatrix} \partial x/\partial p & \partial x/\partial q \\ \partial y/\partial p & \partial y/\partial q \end{vmatrix} = \begin{vmatrix} e^q & pe^q \\ qe^p & e^p \end{vmatrix} = e^q e^p - pe^q \cdot qe^p = e^{p+q} - pqe^{p+q} = (1-pq)e^{p+q}$$

15. $x = uv,\ y = vw,\ z = wu.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix} = v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix}$$

$$= v(uw - 0) - u(0 - vw) + 0 = uvw + uvw = 2uvw$$

16. $x = u + vw,\ y = v + wu,\ z = w + uv.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ u & 1 \end{vmatrix} - w\begin{vmatrix} w & u \\ v & 1 \end{vmatrix} + v\begin{vmatrix} w & 1 \\ v & u \end{vmatrix}$$

$$= 1(1 - u^2) - w\,(w - uv) + v\,(uw - v)$$

$$= 1 - u^2 - w^2 + uvw + uvw - v^2 = 1 + 2uvw - u^2 - v^2 - w^2$$

17. $x = 2u + v, y = u + 2v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$. The integrand $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \quad\Rightarrow\quad v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \quad\Rightarrow\quad u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \quad\Rightarrow\quad u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\,dA = \int_0^1 \int_0^{1-u} (-u - 5v)\,|3|\;dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u}\,du$$

$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right)\,du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

18. $x = \frac{1}{4}(u + v), y = \frac{1}{4}(u - 3u) \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$. The integrand $4x + 8y = 4\cdot\frac{1}{4}(u + v) + 8\cdot\frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\iint_R (4x + 8y)\,dA = \int_{-4}^4 \int_0^8 (3v - 5u)\left|\tfrac{1}{4}\right|\,dv\,du = \tfrac{1}{4}\int_{-4}^4 \left[\tfrac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8}\,du$$

$$= \tfrac{1}{4}\int_{-4}^4 (96 - 40u)\,du = \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^4 = 192$$

19. $x = 2u, y = 3v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$. The integrand $x^2 = 4u^2$. The planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus,

[continued]

$$\iint_R x^2\,dA = \iint\limits_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr$$
$$= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

20. $x = \sqrt{2}\,u - \sqrt{2/3}\,v$, $y = \sqrt{2}\,u + \sqrt{2/3}\,v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$. The integrand

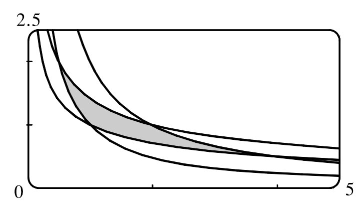
$x^2 - xy + y^2 = 2u^2 + 2v^2$. The planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus,

$$\iint_R (x^2 - xy + y^2)\,dA = \iint\limits_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

21. $x = u/v$, $y = v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$. The integrand $xy = u$. The line $y = x$ $[v = u/v]$ is the image of the parabola $v^2 = u$ and the line $y = 3x$ $[v = 3u/v]$ is the image of the parabola $v^2 = 3u$. The hyperbolas $xy = 1$ and $xy = 3$ are the images of the lines $u = 1$ and $u = 3$, respectively. Thus,

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right)dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right)du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

22. $u = xy$, $v = xy^2$. To solve for x and y in terms of u and v, try dividing.



$\dfrac{xy^2}{xy} = \dfrac{v}{u} \quad\Rightarrow\quad y = \dfrac{v}{u}$. Also, $\dfrac{(xy)^2}{xy^2} = \dfrac{u^2}{v} \quad\Rightarrow\quad x = \dfrac{u^2}{v}$. Then

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$. The integrand $y^2 = v^2/u^2$. R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. Thus,

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$

23. (a) $x = au$, $y = bv$, $z = cw \quad\Rightarrow\quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$. Since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$, the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. Thus,

$$\iiint_E dV = \iiint\limits_{u^2+v^2+w^2\,\le\,1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\text{ km}^3$.

(c) The moment of intertia about the z-axis is $I_z = \iiint_E \left(x^2 + y^2\right)\rho(x,y,z)\,dV$, where E is the solid enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. As in part (a), we use the transformation $x = au$, $y = bv$, $z = cw$, so $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ and

$$I_z = \iiint_E \left(x^2+y^2\right) k\,dV = \iiint\limits_{u^2+v^2+w^2\,\le\,1} k(a^2u^2+b^2v^2)(abc)\,du\,dv\,dw$$

$$= abck \int_0^{\pi}\int_0^{2\pi}\int_0^1 (a^2\rho^2\sin^2\phi\cos^2\theta + b^2\rho^2\sin^2\phi\sin^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= abck\left[a^2\int_0^{\pi}\int_0^{2\pi}\int_0^1(\rho^2\sin^2\phi\cos^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi + b^2\int_0^{\pi}\int_0^{2\pi}\int_0^1(\rho^2\sin^2\phi\sin^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\right]$$

$$= a^3bck\int_0^{\pi}\sin^3\phi\,d\phi\,\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1\rho^4\,d\rho + ab^3ck\int_0^{\pi}\sin^3\phi\,d\phi\,\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1\rho^4\,d\rho$$

$$= a^3bck\left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi}\,\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\,\left[\tfrac{1}{5}\rho^5\right]_0^1 + ab^3ck\left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi}\,\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\,\left[\tfrac{1}{5}\rho^5\right]_0^1$$

$$= a^3bck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) + ab^3ck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) = \tfrac{4}{15}\pi(a^2+b^2)abck$$

24. R is the region enclosed by the curves $xy=a$, $xy=b$, $xy^{1.4}=c$, and $xy^{1.4}=d$, so if we let $u=xy$ and $v=xy^{1.4}$ then R is the image of the rectangle enclosed by the lines $u=a$, $u=b$ $(a<b)$ and $v=c$, $v=d$ $(c<d)$. Now $x=u/y \quad\Rightarrow\quad v=(u/y)y^{1.4}=uy^{0.4} \quad\Rightarrow\quad y^{0.4}=u^{-1}v \quad\Rightarrow\quad y=(u^{-1}v)^{1/0.4}=u^{-2.5}v^{2.5}$ and $x=uy^{-1}=u(u^{-2.5}v^{2.5})^{-1}=u^{3.5}v^{-2.5}$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3.5u^{2.5}v^{-2.5} & -2.5u^{3.5}v^{-3.5} \\ -2.5u^{-3.5}v^{2.5} & 2.5u^{-2.5}v^{1.5}\end{vmatrix} = 8.75v^{-1} - 6.25v^{-1} = 2.5v^{-1}$$

Thus the area of R, and the work done by the engine, is

$$\iint_R dA = \int_a^b\int_c^d \left|2.5v^{-1}\right|\,dv\,du = 2.5\int_a^b du\,\int_c^d(1/v)\,dv = 2.5\big[u\big]_a^b\,\big[\ln|v|\,\big]_c^d = 2.5(b-a)(\ln d-\ln c) = 2.5(b-a)\ln\frac{d}{c}.$$

25. Letting $u=x-2y$ and $v=3x-y$, we have $x=\frac{1}{5}(2v-u)$ and $y=\frac{1}{5}(v-3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5\end{vmatrix} = \dfrac{1}{5}$.

R is the image of the rectangle enclosed by the lines $u=0$, $u=4$, $v=1$, and $v=8$. Thus,

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4\int_1^8 \frac{u}{v}\left|\frac{1}{5}\right|dv\;du = \frac{1}{5}\int_0^4 u\,du\int_1^8\frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4\left[\ln|v|\,\right]_1^8 = \tfrac{8}{5}\ln 8$$

26. Letting $u=x+y$ and $v=x-y$, we have $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2\end{vmatrix} = -\dfrac{1}{2}$.

R is the image of the rectangle enclosed by the lines $u=0$, $u=3$, $v=0$, and $v=2$. Thus,

$$\iint_R (x+y)\,e^{x^2-y^2}\,dA = \int_0^3\int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right|dv\,du = \tfrac{1}{2}\int_0^3\left[e^{uv}\right]_{v=0}^{v=2}du = \tfrac{1}{2}\int_0^3(e^{2u}-1)\,du$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u}-u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6-3-\tfrac{1}{2}\right) = \tfrac{1}{4}(e^6-7)$$

27. Letting $u=y-x$ and $v=y+x$, we have $x=\frac{1}{2}(v-u)$ and $y=\frac{1}{2}(u+v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2\end{vmatrix} = -\dfrac{1}{2}$.

R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus,

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)dA = \int_1^2\int_{-v}^{v}\cos\frac{u}{v}\left|-\frac{1}{2}\right|du\,dv = \frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v}dv$$

$$= \tfrac{1}{2}\int_1^2[v\sin(1)-v\sin(-1)]\,dv = \tfrac{1}{2}\int_1^2 2v\sin 1\,dv$$

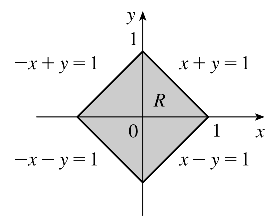
$$= \sin 1\left[\tfrac{1}{2}v^2\right]_1^2 = \tfrac{3}{2}\sin 1$$

28. Letting $u = 3x$ and $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = \dfrac{1}{6}$.

R is the image of the quarter-disk D given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus,

$$\iint_R \sin(9x^2 + 4y^2)\, dA = \iint_D \tfrac{1}{6} \sin(u^2 + v^2)\, du\, dv = \int_0^{\pi/2} \int_0^1 \tfrac{1}{6} \sin(r^2)\, r\, dr\, d\theta = \tfrac{\pi}{12} \left[-\tfrac{1}{2} \cos r^2 \right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

29. See the figure. Leting $u = x + y$ and $v = -x + y$, we have $x = \frac{1}{2}(u - v)$ and $y = \frac{1}{2}(u + v)$.



Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \le |x| + |y| \le 1 \quad \Rightarrow$

$-1 \le u \le 1$, and $|v| = |-x + y| \le |x| + |y| \le 1 \quad \Rightarrow \quad -1 \le v \le 1$.

R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

Thus, $\iint_R e^{x+y}\, dA = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 e^u\, du\, dv = \frac{1}{2} \left[e^u \right]_{-1}^1 \left[v \right]_{-1}^1 = e - e^{-1}$.

30. Letting $u = x + y$ and $v = y/x$, we have $x = \dfrac{u}{1+v}$ and $y = \dfrac{uv}{1+v}$. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/(1+v) & -u/(1+v)^2 \\ v/(1+v) & u/(1+v)^2 \end{vmatrix} = \frac{u + uv}{(1+v)^3} = \frac{u}{(1+v)^2}$$

R is the image of the rectangle enclosed by the lines $u = 1$, $u = 3$, $v = 2$, and $v = 1/2$. Thus,

$$\begin{aligned} \iint_R \frac{y}{x}\, dA &= \int_{1/2}^2 \int_1^3 v \left(\frac{u}{(1+v)^2} \right) du\, dv = \int_{1/2}^2 \frac{v}{(1+v)^2}\, dv \int_1^3 u\, du \\ &= \left[\ln|1+v| + \frac{1}{1+v} \right]_{1/2}^2 \left[\frac{u^2}{2} \right]_1^3 \qquad \left[\text{Use the substitution } w = 1 + v \text{ for the first integral} \right] \\ &= \left(\ln 3 + \tfrac{1}{3} - \ln \tfrac{3}{2} - \tfrac{2}{3} \right) \cdot 4 = 4 \ln 2 - \tfrac{4}{3} \end{aligned}$$

31. Letting $u = x + y$ and $v = y$, we have $x = u - v$ and $y = v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$.

R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus,

$$\iint_R f(x+y)\, dA = \int_0^1 \int_0^u (1)\, f(u)\, dv\, du = \int_0^1 f(u) \left[v \right]_{v=0}^{v=u} du = \int_0^1 u f(u)\, du, \text{ as desired.}$$

15 Review

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x + y^2}\, dy\, dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x + y^2}\, dx\, dy$.

3. True by Equation 15.1.11.

4. $\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^{1} e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function.

Therefore the statement is true.

5. True. By Equation 15.1.11 we can write $\int_0^1\int_0^1 f(x)\,f(y)\,dy\,dx = \int_0^1 f(x)\,dx\,\int_0^1 f(y)\,dy$. But $\int_0^1 f(y)\,dy = \int_0^1 f(x)\,dx$ so this becomes $\int_0^1 f(x)\,dx\,\int_0^1 f(x)\,dx = \left[\int_0^1 f(x)\,dx\right]^2$.

6. This statement is true because in the given region, $\left(x^2+\sqrt{y}\right)\sin(x^2y^2) \le (1+2)(1) = 3$, so

$\int_1^4\int_0^1 \left(x^2+\sqrt{y}\right)\sin(x^2y^2)\,dx\,dy \le \int_1^4\int_0^1 3\,dA = 3A(D) = 3(3) = 9.$

7. True.

$$\begin{aligned}\iint_D \sqrt{4-x^2-y^2}\,dA &= \text{the volume under the surface } x^2+y^2+z^2=4 \text{ and above the } xy\text{-plane}\\ &= \tfrac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \tfrac{1}{2}\cdot\tfrac{4}{3}\pi(2)^3 = \tfrac{16}{3}\pi\end{aligned}$$

8. True. The moment of inertia about the z-axis of a solid E with constant density k is

$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta.$

9. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates,

$V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A\\ &= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)\ +f(2,3)+f(3,1)+f(3,2)+f(3,3)]\end{aligned}$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m=n=3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A\\ &= \Delta A\,[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5)\\ &\qquad +f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)]\\ &\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$

$\qquad = 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1 \int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{in the first term}\end{array}\right]$

$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi \int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi \int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$

$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

8. $\int_0^1 \int_0^y \int_x^1 6xyz\,dz\,dx\,dy = \int_0^1 \int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1 \int_0^y (3xy - 3x^3y)\,dx\,dy$

$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus

$\iint_R f(x,y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid 0 \le y \le 4, y - 4 \le x \le 4 - y\}$ and $\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

11. $x = r\cos\theta = 2\sqrt{3}\cos\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{1}{2} = \sqrt{3}$, $y = r\sin\theta = 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 3$, $z = 2$, so in rectangular coordinates the point is $(\sqrt{3}, 3, 2)$. $\rho = \sqrt{r^2 + z^2} = \sqrt{12 + 4} = 4$, $\theta = \frac{\pi}{3}$, and $\cos\phi = z/\rho = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$ and spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

12. $r = \sqrt{4 + 4} = 2\sqrt{2}$; $z = -1$; $\tan\theta = \frac{2}{2} = 1$ and the point $(2,2)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4}$. Thus in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4 + 4 + 1} = 3$, $\cos\phi = z/\rho = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

13. $x = \rho\sin\phi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$, $y = \rho\sin\phi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{\pi}{4} = 2\sqrt{2}$, and $z = \rho\cos\phi = 8\cos\frac{\pi}{6} = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$. Thus rectangular coordinates for the point are $\left(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}\right)$.

$r^2 = x^2 + y^2 = 8 + 8 = 16 \quad\Rightarrow\quad r = 4$, $\theta = \frac{\pi}{4}$, and $z = 4\sqrt{3}$, so cylindrical coordinates are $\left(4, \frac{\pi}{4}, 4\sqrt{3}\right)$.

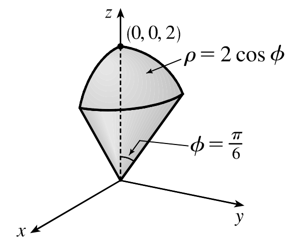
14. (a) $\theta = \frac{\pi}{4}$. In cylindrical coordinates (assuming that r can be negative), this is a vertical plane that includes the z-axis and intersects the xy-plane in the line $y = x$. In spherical coordinates, because $\rho \ge 0$ and $0 \le \phi \le \pi$, we get a vertical half-plane that includes the z-axis and intersects the xy-plane in the half-line $y = x$, $x \ge 0$.

(b) $\phi = \frac{\pi}{4}$. In spherical coordinates, this is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

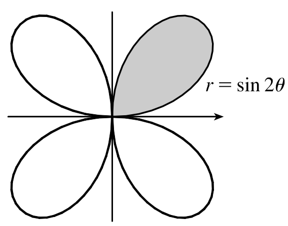
15. (a) $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.

(b) $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$ or $r = 2$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2 \cos^2\phi = 4$ or $\rho^2 \sin^2\phi = 4$ or $\rho \sin\phi = 2$.

16. $\rho = 2\cos\phi \;\Rightarrow\; \rho^2 = 2\rho\cos\phi \;\Rightarrow\; x^2 + y^2 + z^2 = 2z \;\Rightarrow$ $x^2 + y^2 + (z-1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \le \rho \le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.



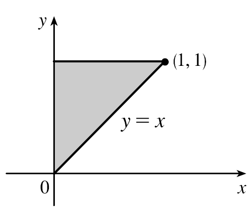
17.



The region whose area is given by $\int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r, \theta) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le \sin 2\theta\right\}$, which is the region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant.

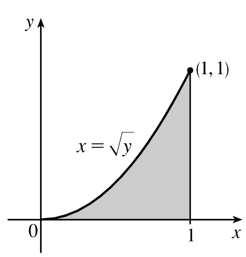
18. The solid is $\left\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\right\}$, which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

19.



$$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$$
$$= \int_0^1 \cos(y^2)\left[\,x\,\right]_{x=0}^{x=y} dy = \int_0^1 y\cos(y^2)\,dy$$
$$= \left[\tfrac{1}{2}\sin(y^2)\right]_0^1 = \tfrac{1}{2}\sin 1$$

20.

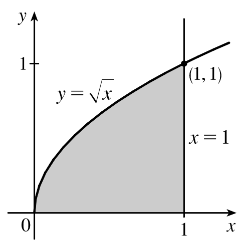


$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \frac{e^{x^2}}{x^3}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=x^2} dx$$
$$= \int_0^1 \tfrac{1}{2}xe^{x^2}\,dx = \tfrac{1}{4}e^{x^2}\Big]_0^1 = \tfrac{1}{4}(e-1)$$

21. $\iint_R ye^{xy}\,dA = \int_0^3 \int_0^2 ye^{xy}\,dx\,dy = \int_0^3 \left[e^{xy}\right]_{x=0}^{x=2} dy = \int_0^3 (e^{2y} - 1)\,dy = \left[\frac{1}{2}e^{2y} - y\right]_0^3 = \frac{1}{2}e^6 - 3 - \frac{1}{2} = \frac{1}{2}e^6 - \frac{7}{2}$

22. $\iint_D xy\,dA = \int_0^1 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^1 y\left[\frac{1}{2}x^2\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^1 y\left[(y+2)^2 - y^4\right] dy$

$$= \tfrac{1}{2}\int_0^1 (y^3 + 4y^2 + 4y - y^5)\,dy = \tfrac{1}{2}\left[\tfrac{1}{4}y^4 + \tfrac{4}{3}y^3 + 2y^2 - \tfrac{1}{6}y^6\right]_0^1 = \tfrac{41}{24}$$

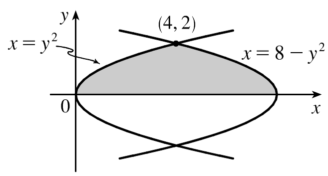
23. 

$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \tfrac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

24. $\displaystyle\iint_D \frac{1}{1+x^2}\,dA = \int_0^1 \int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\,\big[\,y\,\big]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1-x}{1+x^2}\,dx = \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right)dx$

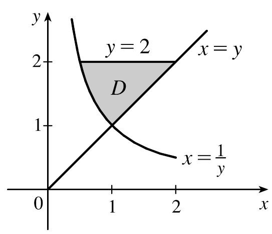
$$= \left[\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tan^{-1}1 - \tfrac{1}{2}\ln 2 - \left(\tan^{-1}0 - \tfrac{1}{2}\ln 1\right) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

25. 

$$\textstyle\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$

$$= \textstyle\int_0^2 y\big[x\big]_{x=y^2}^{x=8-y^2}\,dy = \int_0^2 y(8-y^2-y^2)\,dy$$

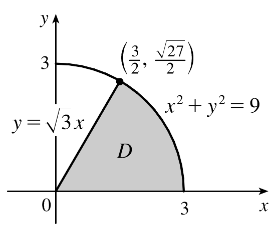
$$= \textstyle\int_0^2 (8y-2y^3)\,dy = \left[4y^2 - \tfrac{1}{2}y^4\right]_0^2 = 8$$

26. 

$$\iint_D y\,dA = \int_1^2 \int_{1/y}^{y} y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right)dy$$

$$= \textstyle\int_1^2 \left(y^2 - 1\right)dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$

$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

27. 

$$\iint_D \left(x^2+y^2\right)^{3/2}dA = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2} r\,dr\,d\theta$$

$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \big[\,\theta\,\big]_0^{\pi/3}\,\left[\tfrac{1}{5}r^5\right]_0^3$$

$$= \frac{\pi}{3}\,\frac{3^5}{5} = \frac{81\pi}{5}$$

28. $\iint_D x\,dA = \int_0^{\pi/2}\int_1^{\sqrt{2}} (r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\,d\theta\,\int_1^{\sqrt{2}} r^2\,dr = \big[\sin\theta\big]_0^{\pi/2}\,\left[\tfrac{1}{3}r^3\right]_1^{\sqrt{2}}$

$$= 1\cdot\tfrac{1}{3}(2^{3/2}-1) = \tfrac{1}{3}(2^{3/2}-1)$$

29. $\iiint_E xy\,dV = \int_0^3\int_0^x\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3\int_0^x xy\,\big[\,z\,\big]_{z=0}^{z=x+y}\,dy\,dx = \int_0^3\int_0^x xy(x+y)\,dy\,dx$

$$= \textstyle\int_0^3\int_0^x (x^2y+xy^2)\,dy\,dx = \int_0^3 \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{y=0}^{y=x}\,dx = \int_0^3 \left(\tfrac{1}{2}x^4 + \tfrac{1}{3}x^4\right)dx$$

$$= \textstyle\tfrac{5}{6}\int_0^3 x^4\,dx = \left[\tfrac{1}{6}x^5\right]_0^3 = \tfrac{81}{2} = 40.5$$

30. $\iiint_T xy\,dV = \int_0^{1/3}\int_0^{1-3x}\int_0^{1-3x-y} xy\,dz\,dy\,dx = \int_0^{1/3}\int_0^{1-3x} xy(1-3x-y)\,dy\,dx$

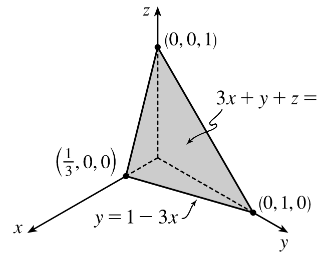
$$= \int_0^{1/3}\int_0^{1-3x}(xy - 3x^2y - xy^2)\,dy\,dx$$

$$= \int_0^{1/3}\left[\tfrac{1}{2}xy^2 - \tfrac{3}{2}x^2y^2 - \tfrac{1}{3}xy^3\right]_{y=0}^{y=1-3x} dx$$

$$= \int_0^{1/3}\left[\tfrac{1}{2}x(1-3x)^2 - \tfrac{3}{2}x^2(1-3x)^2 - \tfrac{1}{3}x(1-3x)^3\right] dx$$

$$= \int_0^{1/3}\left(\tfrac{1}{6}x - \tfrac{3}{2}x^2 + \tfrac{9}{2}x^3 - \tfrac{9}{2}x^4\right) dx$$

$$= \tfrac{1}{12}x^2 - \tfrac{1}{2}x^3 + \tfrac{9}{8}x^4 - \tfrac{9}{10}x^5\Big]_0^{1/3} = \tfrac{1}{1080}$$



31. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

$$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{2}\sin 2\theta\right)^2 d\theta\,\int_0^1 (r^5 - r^7)\,dr$$

$$= \int_0^{2\pi}\tfrac{1}{4}\left[\tfrac{1}{2}(1-\cos 4\theta)\right] d\theta\,\int_0^1 (r^5 - r^7)\,dr = \tfrac{1}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6 - \tfrac{1}{8}r^8\right]_0^1$$

$$= \tfrac{1}{8}(2\pi)\left(\tfrac{1}{6} - \tfrac{1}{8}\right) = \tfrac{\pi}{4}\cdot\tfrac{1}{24} = \tfrac{\pi}{96}$$

32. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)z\,dz\,dy = \int_0^1 \tfrac{1}{2}(2-y)(1-y^2)\,dy$

$$= \int_0^1 \tfrac{1}{2}(2 - y - 2y^2 + y^3)\,dy = \tfrac{13}{24}$$

33. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\left[\tfrac{1}{2}yz^2\right]_{z=0}^{z=y} dy\,dx = \tfrac{1}{2}\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y^3\,dy\,dx$

$$= \tfrac{1}{2}\int_0^{\pi}\int_0^2 (r\sin\theta)^3\,r\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi}\sin^3\theta\,d\theta\,\int_0^2 r^4\,dr = \tfrac{1}{2}\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta\,\int_0^2 r^4\,dr$$

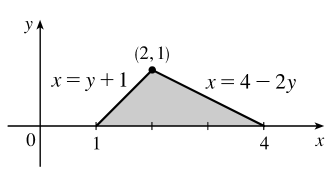
$$= \tfrac{1}{2}\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi}\left[\tfrac{1}{5}r^5\right]_0^2 = \tfrac{1}{2}\left(\tfrac{2}{3} + \tfrac{2}{3}\right)\left(\tfrac{32}{5}\right) = \tfrac{64}{15}$$

34. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$$= \int_0^{2\pi} d\theta\,\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,\int_0^1 \rho^6\,d\rho = 2\pi\left[-\tfrac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\tfrac{1}{7}\right) = \tfrac{\pi}{14}$$

35. $V = \int_0^2\int_1^4 (x^2 + 4y^2)\,dy\,dx = \int_0^2\left[x^2y + \tfrac{4}{3}y^3\right]_{y=1}^{y=4} dx = \int_0^2 (3x^2 + 84)\,dx = x^3 + 84x\Big]_0^2 = 176$

36.

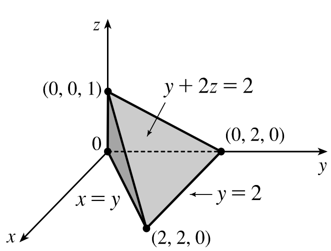


$$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{3}\left[(4-2y)^3y - (y+1)^3y\right] dy$$

$$= \int_0^1 3(-y^4 + 5y^3 - 11y^2 + 7y)\,dy = 3\left(-\tfrac{1}{5} + \tfrac{5}{4} - \tfrac{11}{3} + \tfrac{7}{2}\right) = \tfrac{53}{20}$$

37.



$$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2\int_0^y \left(1 - \tfrac{1}{2}y\right) dx\,dy$$

$$= \int_0^2 \left(y - \tfrac{1}{2}y^2\right) dy = \tfrac{1}{2}y^2 - \tfrac{1}{6}y^3\Big]_0^2 = \tfrac{2}{3}$$

38. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \Big[rz\Big]_0^{3-r\sin\theta} dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{3}{2}r^2 - \frac{1}{3}r^3\sin\theta\right]_0^2 d\theta = \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right]d\theta = \left[6\theta + \frac{8}{3}\cos\theta\right]_0^{2\pi} = 12\pi$

39. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\frac{1}{2}m(a^2 - 9y^2)\,dy = m\left[a^2y - 3y^3\right]_0^{a/3} = m\left(\frac{1}{3}a^3 - \frac{1}{9}a^3\right) = \frac{2}{9}ma^3.$

40. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2+y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$V = \iint\limits_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{3} - \frac{1}{4}\right)d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}.$

41. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\Big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x},\overline{y}) = \left(\frac{1}{3},\frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\Big]_0^1 = \frac{1}{24}$,

$\overline{\overline{y}}^2 = I_x/m = \frac{1/12}{1/4} = \frac{1}{3} \;\Rightarrow\; \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = I_y/m = \frac{1/24}{1/4} = \frac{1}{6} \;\Rightarrow\; \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

42. (a) In polar coordinates, the lamina occupies the region $D = \{(r,\theta) \mid 0 \le r \le a,\, 0 \le \theta \le \pi/2\}$. Assuming constant density K, then $m = K\,A(D) = K\cdot\frac{1}{4}\pi a^2 = \frac{1}{4}\pi Ka^2$,

$M_y = \iint_D Kx\,dA = K\int_0^{\pi/2}\int_0^a (r\cos\theta)\,r\,dr\,d\theta = K\int_0^{\pi/2}\cos\theta\,d\theta\int_0^a r^2\,dr = K\,[\sin\theta]_0^{\pi/2}\left[\frac{1}{3}r^3\right]_0^a = \frac{1}{3}Ka^3$, and

$M_x = \iint_D Ky\,dA = K\int_0^{\pi/2}\sin\theta\,d\theta\int_0^a r^2\,dr = K\,[-\cos\theta]_0^{\pi/2}\left[\frac{1}{3}r^3\right]_0^a = \frac{1}{3}Ka^3$ [by symmetry $M_y = M_x$].

Thus the centroid is $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \iint_D \rho(x,y)\,dA = \iint_D xy^2\,dA = \int_0^{\pi/2}\int_0^a (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\sin^2\theta\,\cos\theta\,d\theta\int_0^a r^4\,dr = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left[\frac{1}{5}r^5\right]_0^a = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left[\frac{1}{6}r^6\right]_0^a = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left[\frac{1}{6}r^6\right]_0^a = \frac{1}{24}a^6$. Hence $(\overline{x},\overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

43. (a) A right circular cone with axis the z-axis and vertex at the origin has equation $z^2 = c^2(x^2 + y^2)$. Here we have the bottom frustum, shifted upward h units, and with $c^2 = h^2/a^2$ so that the cone includes the point $(a,0,0)$. Thus an equation of the cone in rectangular coordinates is $z = h - \frac{h}{a}\sqrt{x^2+y^2}$, $0 \le z \le h$. In cylindrical coordinates, the cone is described by $E = \left\{(r,\theta,z) \mid 0 \le r \le a,\; 0 \le \theta \le 2\pi,\; 0 \le z \le h\left(1 - \frac{1}{a}r\right)\right\}$, and its volume is $V = \frac{1}{3}\pi a^2 h$. By symmetry

$M_{yz} = M_{xz} = 0$, and

$$M_{xy} = \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)} z \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \left[\frac{1}{2}rz^2\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^a rh^2\left(1-\frac{r}{a}\right)^2 dr\,d\theta = \frac{1}{2}h^2\int_0^{2\pi} d\theta \int_0^a \left(r - \frac{2}{a}r^2 + \frac{1}{a^2}r^3\right) dr$$
$$= \tfrac{1}{2}h^2\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{2}r^2 - \tfrac{2}{3a}r^3 + \tfrac{1}{4a^2}r^4\right]_0^a = \tfrac{1}{2}h^2\,(2\pi)\left(\tfrac{1}{2}a^2 - \tfrac{2}{3}a^2 + \tfrac{1}{4}a^2\right)$$
$$= \pi h^2\left(\tfrac{1}{12}a^2\right) = \tfrac{1}{12}\pi a^2h^2$$

Hence the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, [\pi a^2h^2/12]/[\pi a^2h/3]\right) = \left(0, 0, \frac{1}{4}h\right)$.

(b) The density function is $\rho = \sqrt{x^2+y^2} = \sqrt{r^2} = r$, so the moment of inertia about the cone's axis (the z-axis) is

$$I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)}(r^2)(r)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^a \left[r^4z\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta = \int_0^{2\pi}\int_0^a r^4h\left(1-\tfrac{1}{a}r\right) dr\,d\theta$$
$$= h\int_0^{2\pi} d\theta \int_0^a \left(r^4 - \tfrac{1}{a}r^5\right) dr \;= h\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{5}r^5 - \tfrac{1}{6a}r^6\right]_0^a$$
$$= h\,(2\pi)\left(\tfrac{1}{5}a^5 - \tfrac{1}{6}a^5\right) = \tfrac{1}{15}\pi a^5h$$

44. $1 \le z^2 \le 4 \quad\Rightarrow\quad 1/a^2 \le x^2+y^2 \le 4/a^2$. Let $D = \left\{(x,y) \mid 1/a^2 \le x^2+y^2 \le 4/a^2\right\}$. $z = f(x,y) = a\sqrt{x^2+y^2}$, so $f_x(x,y) = ax(x^2+y^2)^{-1/2}$, $f_y(x,y) = ay(x^2+y^2)^{-1/2}$, and

$$A(S) = \iint_D \sqrt{\frac{a^2x^2+a^2y^2}{x^2+y^2}+1}\,dA = \iint_D \sqrt{a^2+1}\,dA = \sqrt{a^2+1}\,A(D)$$
$$= \sqrt{a^2+1}\left[\pi\left(\frac{2}{a}\right)^2 - \pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2+1}$$

45. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2-2x$, or equivalently $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2-2x\}$. We want to find the surface area of the part of the graph of $z = x^2+y$ that lies over D, so using Formula 15.5.3 we have

$$A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D \sqrt{1+(2x)^2+(1)^2}\,dA = \int_0^1\int_0^{2-2x}\sqrt{2+4x^2}\,dy\,dx$$
$$= \int_0^1 \sqrt{2+4x^2}\left[\,y\,\right]_{y=0}^{y=2-2x} dx = \int_0^1 (2-2x)\sqrt{2+4x^2}\,dx = \int_0^1 2\sqrt{2+4x^2}\,dx - \int_0^1 2x\sqrt{2+4x^2}\,dx$$

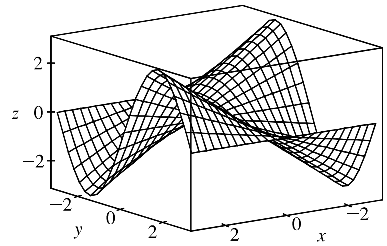
Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have

$\int 2\sqrt{2+4x^2}\,dx = x\sqrt{2+4x^2} + \ln\left(2x+\sqrt{2+4x^2}\right)$. If we substitute $u = 2+4x^2$ in the second integral, then $du = 8x\,dx$ and $\int 2x\sqrt{2+4x^2}\,dx = \frac{1}{4}\int\sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}(2+4x^2)^{3/2}$. Thus

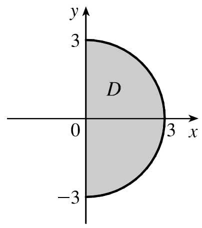
$$A(S) = \left[x\sqrt{2+4x^2} + \ln\left(2x+\sqrt{2+4x^2}\right) - \tfrac{1}{6}(2+4x^2)^{3/2}\right]_0^1$$
$$= \sqrt{6} + \ln\left(2+\sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} = \ln\tfrac{2+\sqrt{6}}{\sqrt{2}} + \tfrac{\sqrt{2}}{3}$$
$$= \ln\left(\sqrt{2}+\sqrt{3}\right) + \tfrac{\sqrt{2}}{3} \approx 1.6176$$

46. Using Formula 15.5.3 with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get

$S = \int_{-\pi}^{\pi}\int_{-3}^{3}\sqrt{1+\sin^2 y + x^2\cos^2 y}\,dx\,dy \approx 62.9714.$



47.



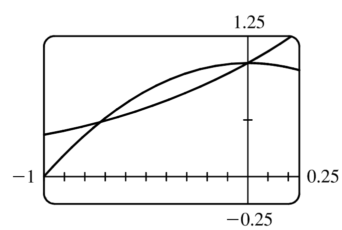
$$\begin{aligned}\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x^3+xy^2)\,dy\,dx &= \int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}x(x^2+y^2)\,dy\,dx\\ &= \int_{-\pi/2}^{\pi/2}\int_0^3(r\cos\theta)(r^2)\,r\,dr\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\int_0^3 r^4\,dr\\ &= \left[\sin\theta\right]_{-\pi/2}^{\pi/2}\left[\tfrac15 r^5\right]_0^3 = 2\cdot\tfrac15(243) = \tfrac{486}{5} = 97.2\end{aligned}$$

48. The region of integration is the solid hemisphere $x^2+y^2+z^2\le 4$, $x\ge 0$.

$$\begin{aligned}&\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy\\ &\quad= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2(\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho\\ &\quad= \left[\tfrac12\theta-\tfrac14\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\cos\phi+\tfrac13\cos^3\phi\right]_0^{\pi}\left[\tfrac16\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac23+\tfrac23\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi\end{aligned}$$

49. From the graph, it appears that $1-x^2=e^x$ at $x\approx -0.71$ and at $x=0$, with $1-x^2>e^x$ on $(-0.71,0)$. So the desired integral is

$$\begin{aligned}\iint_D y^2\,dA &\approx \int_{-0.71}^{0}\int_{e^x}^{1-x^2}y^2\,dy\,dx\\ &= \tfrac13\int_{-0.71}^{0}\left[(1-x^2)^3-e^{3x}\right]dx\\ &= \tfrac13\left[x-x^3+\tfrac35x^5-\tfrac17x^7-\tfrac13e^{3x}\right]_{-0.71}^{0}\approx 0.0512\end{aligned}$$



50. Let the tetrahedron be called T. The front face of T is given by the plane $x+\frac12y+\frac13z=1$, or $z=3-3x-\frac32y$, which intersects the xy-plane in the line $y=2-2x$. So the total mass is

$m=\iiint_T\rho(x,y,z)\,dV=\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}(x^2+y^2+z^2)\,dz\,dy\,dx=\frac75$. The center of mass is

$(\overline{x},\overline{y},\overline{z})=\left(m^{-1}\iiint_T x\rho(x,y,z)\,dV, m^{-1}\iiint_T y\rho(x,y,z)\,dV, m^{-1}\iiint_T z\rho(x,y,z)\,dV\right)=\left(\frac{4}{21},\frac{11}{21},\frac87\right)$.

51. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2}f(x,y)\,dA=1$. Since $f(x,y)=0$ outside the rectangle $[0,3]\times[0,2]$, we can say

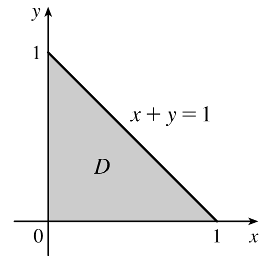
$$\begin{aligned}\iint_{\mathbb{R}^2}f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx = \int_0^3\int_0^2 C(x+y)\,dy\,dx\\ &= C\int_0^3\left[xy+\tfrac12y^2\right]_{y=0}^{y=2}dx = C\int_0^3(2x+2)\,dx = C\left[x^2+2x\right]_0^3 = 15C\end{aligned}$$

Then $15C=1 \quad\Rightarrow\quad C=\frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2} \int_{1}^{\infty} f(x,y)\,dy\,dx = \int_0^2 \int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15}\int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$$= \frac{1}{15}\int_0^2 \left(x + \frac{3}{2}\right) dx = \frac{1}{15}\left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_0^2 = \frac{1}{3}$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y \le 1) = \iint_D f(x,y)\,dA = \int_0^1 \int_0^{1-x} \frac{1}{15}(x+y)\,dy\,dx$$
$$= \frac{1}{15}\int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \frac{1}{15}\int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right] dx$$
$$= \frac{1}{30}\int_0^1 (1-x^2)\,dx = \frac{1}{30}\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{1}{45}$$

52. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

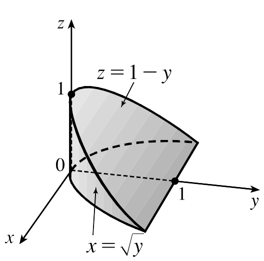
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

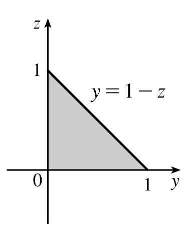
$$f(x,y,z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x \ge 0,\, y \ge 0,\, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X+Y+Z \le 1000)$, or equivalently $P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z = 1000$. The plane $x+y+z = 1000$ meets the xy-plane in the line $x+y = 1000$, so we have

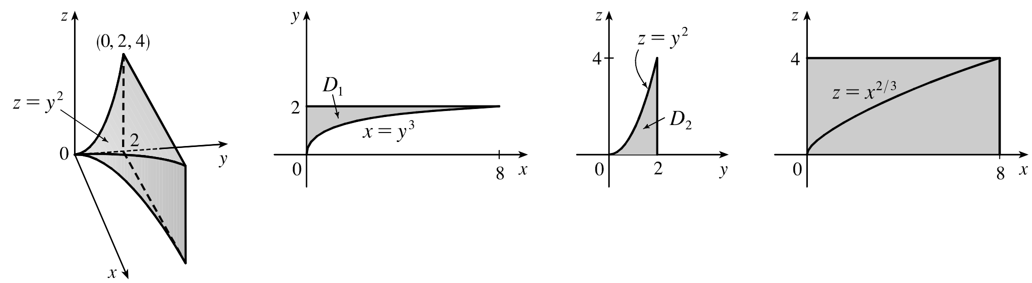
$$P(X+Y+Z \le 1000) = \iiint_E f(x,y,z)\,dV = \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \frac{1}{800^3}e^{-(x+y+z)/800}\,dz\,dy\,dx$$
$$= \frac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y} dy\,dx$$
$$= \frac{-1}{800^2}\int_0^{1000}\int_0^{1000-x} \left[e^{-5/4} - e^{-(x+y)/800}\right] dy\,dx$$
$$= \frac{-1}{800^2}\int_0^{1000} \left[e^{-5/4}y + 800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x} dx$$
$$= \frac{-1}{800^2}\int_0^{1000} \left[e^{-5/4}(1800-x) - 800e^{-x/800}\right] dx$$
$$= \frac{-1}{800^2}\left[-\frac{1}{2}e^{-5/4}(1800-x)^2 + 800^2e^{-x/800}\right]_0^{1000}$$
$$= \frac{-1}{800^2}\left[-\frac{1}{2}e^{-5/4}(800)^2 + 800^2e^{-5/4} + \frac{1}{2}e^{-5/4}(1800)^2 - 800^2\right]$$
$$= 1 - \frac{97}{32}e^{-5/4} \approx 0.1315$$

53.





$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

54. 

$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E=\left\{(x,y,z)\mid 0\le y\le 2, 0\le x\le y^3, 0\le z\le y^2\right\}$.

If D_1, D_2, and D_3 are the projections of E onto the xy-, yz-, and xz-planes, then

$$D_1=\left\{(x,y)\mid 0\le y\le 2, 0\le x\le y^3\right\}=\{(x,y)\mid 0\le x\le 8, \sqrt[3]{x}\le y\le 2\},$$

$$D_2=\{(y,z)\mid 0\le z\le 4, \sqrt{z}\le y\le 2\}=\left\{(y,z)\mid 0\le y\le 2, 0\le z\le y^2\right\},$$

$$D_3=\{(x,z)\mid 0\le x\le 8, 0\le z\le 4\}.$$

Therefore we have

$$\begin{aligned}\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^8\int_{\sqrt[3]{x}}^2\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx\\ &= \int_0^4\int_{\sqrt{z}}^2\int_0^{y^3} f(x,y,z)\,dx\,dy\,dz\\ &= \int_0^2\int_0^{y^2}\int_0^{y^3} f(x,y,z)\,dx\,dz\,dy\\ &= \int_0^8\int_0^{x^{2/3}}\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8\int_{x^{2/3}}^4\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^4\int_0^{z^{3/2}}\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4\int_{z^{3/2}}^8\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

55. Since $u=x-y$ and $v=x+y$, $x=\frac12(u+v)$ and $y=\frac12(v-u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{vmatrix}=\dfrac12.$

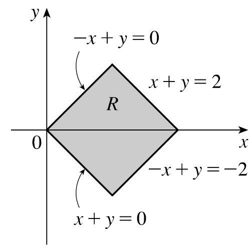
R is the image under this transformation of the square with vertices $(u,v)=(-2,2)$, $(0,2)$, $(0,4)$, and $(-2,4)$. Thus,

$$\begin{aligned}\iint_R \frac{x-y}{x+y}\,dA &= \int_2^4\int_{-2}^0 \frac{u}{v}\left(\frac12\right)du\,dv = \frac12\int_2^4\left[\frac{u^2}{2v}\right]_{u=-2}^{u=0} dv = \frac12\int_2^4\left(-\frac{2}{v}\right)dv\\ &= \Big[-\ln v\Big]_2^4 = -\ln 4+\ln 2 = -2\ln 2+\ln 2 = -\ln 2\end{aligned}$$

56. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}2u & 0 & 0\\ 0 & 2v & 0\\ 0 & 0 & 2w\end{vmatrix}=8uvw$, so

$$\begin{aligned}V &= \iiint_E dV = \int_0^1\int_0^{1-u}\int_0^{1-u-v} 8uvw\,dw\,dv\,du = \int_0^1\int_0^{1-u} 4uv(1-u-v)^2\,du\\ &= \int_0^1\int_0^{1-u}\left[4u(1-u)^2v-8u(1-u)v^2+4uv^3\right]dv\,du\\ &= \int_0^1\left[2u(1-u)^4-\tfrac83u(1-u)^4+u(1-u)^4\right]du = \int_0^1\tfrac13u(1-u)^4du\\ &= \int_0^1\tfrac13\left[(1-u)^4-(1-u)^5\right]du = \tfrac13\left[-\tfrac15(1-u)^5+\tfrac16(1-u)^6\right]_0^1 = \tfrac13\left(-\tfrac16+\tfrac15\right) = \tfrac{1}{90}\end{aligned}$$

57. See the figure. Letting $u = -x + y$ and $v = x + y$, we have $x = \frac{1}{2}(v - u)$ and $y = \frac{1}{2}(v + u)$. Then



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = \left|-\tfrac{1}{2}\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

R is the image under this transformation of the square region with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. Thus,

$$\iint_R xy\,dA = \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4}\left(\frac{1}{2}\right) du\,dv = \frac{1}{8}\int_0^2 \left[v^2 u - \frac{1}{3}u^3\right]_{u=-2}^{u=0} dv$$
$$= \frac{1}{8}\int_0^2 \left(2v^2 - \frac{8}{3}\right) dv = \frac{1}{8}\left[\frac{2}{3}v^3 - \frac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

58. (a) $\displaystyle\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi \int_r^R t^{1-n}\,dt$

$$= \begin{cases} \left.\dfrac{2\pi}{2-n}t^{2-n}\right]_r^R = \dfrac{2\pi}{2-n}\left(R^{2-n} - r^{2-n}\right) & \text{if } n \neq 2 \\ 2\pi \ln(R/r) & \text{if } n = 2 \end{cases}$$

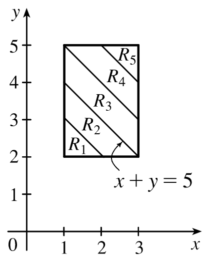
(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2 - n > 0 \iff n < 2$.

(c) $\displaystyle\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R \int_0^\pi \int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2 \sin\phi\,d\theta\,d\phi\,d\rho = 2\pi \int_r^R \int_0^\pi \rho^{2-n}\sin\phi\,d\phi\,d\rho$

$$= \begin{cases} \left.\dfrac{4\pi}{3-n}\,\rho^{3-n}\right]_r^R = \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{if } n \neq 3 \\ 4\pi \ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3 - n > 0 \iff n < 3$.

PROBLEMS PLUS

1. 

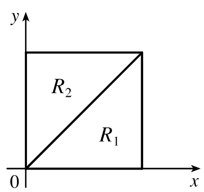
Let $R = \bigcup_{i=1}^{5} R_i$, where

$R_i = \{(x, y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x + y]\!]\, dA = \sum\limits_{i=1}^{5} \iint_{R_i} [\![x + y]\!]\, dA = \sum\limits_{i=1}^{5} [\![x + y]\!] \iint_{R_i} dA$, since

$[\![x + y]\!] = \text{constant} = i + 2$ for $(x, y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x + y]\!]\, dA &= \textstyle\sum_{i=1}^{5} (i + 2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

2. 

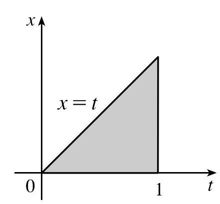
Let $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \geq y$, and $\max\{x^2, y^2\} = y^2$ if $x \leq y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\{x^2, y^2\} = x^2$ for $(x, y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x, y) \in R_2 \quad \Rightarrow$

$$\begin{aligned}\int_0^1\!\int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx &= \iint_R e^{\max\{x^2, y^2\}}\, dA = \iint_{R_1} e^{\max\{x^2, y^2\}}\, dA + \iint_{R_2} e^{\max\{x^2, y^2\}}\, dA \\ &= \int_0^1\!\int_0^x e^{x^2}\, dy\, dx + \int_0^1\!\int_0^y e^{y^2}\, dx\, dy = \int_0^1 x e^{x^2}\, dx + \int_0^1 y e^{y^2}\, dy = e^{x^2}\Big]_0^1 = e - 1\end{aligned}$$

3. $$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\, dx = \frac{1}{1-0}\int_0^1 \left[\int_x^1 \cos(t^2)\, dt\right] dx$$

$$= \int_0^1\!\int_x^1 \cos(t^2)\, dt\, dx = \int_0^1\!\int_0^t \cos(t^2)\, dx\, dt \quad [\text{changing the order of integration}]$$

$$= \int_0^1 t\cos(t^2)\, dt = \tfrac{1}{2}\sin(t^2)\Big]_0^1 = \tfrac{1}{2}\sin 1$$



4. To show that $\int_0^2\!\int_0^x 2e^{x^2 - y^2}\, dy\, dx = \int_0^2\!\int_0^x 2e^{(x+y)(x-y)}\, dy\, dx$ is equal to $\int_0^2\!\int_y^{4-y} e^{xy}\, dx\, dy$, we use a change of variables on the left-hand side. Leting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

R is bounded by $y = x$, $x = 2$, and $y = 0$.

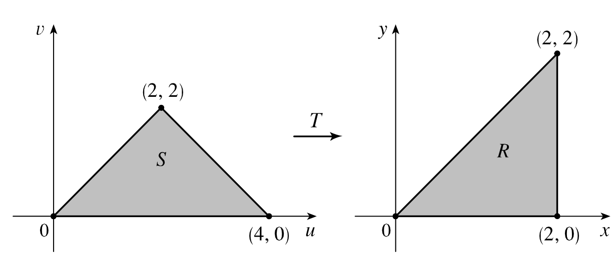
$y = x \quad \Rightarrow \quad \frac{1}{2}(u - v) = \frac{1}{2}(u + v) \quad \Rightarrow \quad v = 0$;

$x = 2 \quad \Rightarrow \quad \frac{1}{2}(u + v) = 2 \quad \Rightarrow$

$\qquad v = 4 - u$, or $u = 4 - v$;

$y = 0 \quad \Rightarrow \quad \frac{1}{2}(u - v) = 0 \quad \Rightarrow \quad u = v$.

So R is the image of S. Thus,



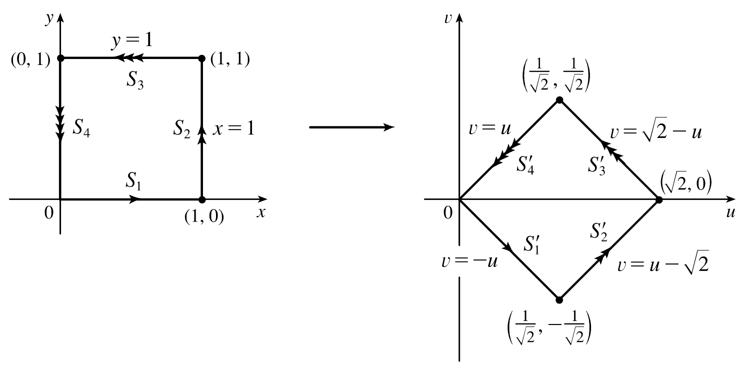
$$\int_0^2 \int_0^x 2e^{x^2-y^2}\,dy\,dx = \int_0^2 \int_0^x 2e^{(x+y)(x-y)}\,dy\,dx = \int_0^2 \int_v^{4-v} 2e^{uv}\left|-\frac{1}{2}\right|\,du\,dv$$
$$= \int_0^2 \int_v^{4-v} e^{uv}\,du\,dv,$$

which is the same as $\int_0^2 \int_y^{4-y} e^{xy}\,dx\,dy$.

5. Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \sum\limits_{n=0}^{\infty} (xy)^n$, so

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n\,dx\,dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n\,dx\,dy$$
$$= \sum_{n=0}^{\infty} \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\cdot\frac{1}{n+1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Let $x = \dfrac{u-v}{\sqrt{2}}$ and $y = \dfrac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of Section 15.9. $x + y = \dfrac{u-v}{\sqrt{2}} + \dfrac{u+v}{\sqrt{2}} = \sqrt{2}\,u$, so $u = \dfrac{x+y}{\sqrt{2}}$, and similarly $v = \dfrac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \le x \le 1$, so from the equations derived above, the image of S_1 is S_1': $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \le x \le 1$, that is, $v = -u$, $0 \le u \le \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S_2': $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, the image of S_3 is S_3': $v = \sqrt{2} - u$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, and the image of S_4 is S_4': $v - u$, $0 \le u \le \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$$

From the diagram, we see that we must evaluate two integrals: one over the region $\left\{(u,v) \mid 0 \le u \le \frac{1}{\sqrt{2}},\ -u \le v \le u\right\}$ and the other over $\left\{(u,v) \mid \frac{1}{\sqrt{2}} \le u \le \sqrt{2},\ -\sqrt{2} + u \le v \le \sqrt{2} - u\right\}$. So

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$

$$= \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{2\,dv\,du}{2-u^2+v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2\,dv\,du}{2-u^2+v^2}$$

$$= 2\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u} du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du\right]$$

$$= 4\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,du\right]$$

Now let $u = \sqrt{2}\sin\theta$, so $du = \sqrt{2}\cos\theta\,d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right) + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right)\right]$$

$$= 4\left[\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\,(1-\sin\theta)}{\sqrt{2}\cos\theta}\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta\right]$$

But (following the hint)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1-\left[1-2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} \qquad \text{[half-angle formulas]}$$

$$= \frac{2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} = \tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)$$

Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \theta\,d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right]d\theta\right] = 4\left(\left[\frac{\theta^2}{2}\right]_0^{\pi/6} + \left[\frac{\pi\theta}{4}-\frac{\theta^2}{4}\right]_{\pi/6}^{\pi/2}\right) = 4\left(\frac{3\pi^2}{72}\right) = \frac{\pi^2}{6}$$

7. (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty}(xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(b) Since $|-xyz| < 1$, except at $(1, 1, 1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1 + xyz} = \sum\limits_{n=0}^{\infty}(-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 11.5, we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8. $$\int_0^{\infty}\frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^{\infty}\left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi}dx = \int_0^{\infty}\int_1^{\pi}\frac{1}{1+y^2x^2}\,dy\,dx$$

$$= \int_1^{\pi}\int_0^{\infty}\frac{1}{1+y^2x^2}\,dx\,dy = \int_1^{\pi}\lim_{t\to\infty}\left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t}dy$$

$$= \int_1^{\pi}\frac{\pi}{2y}\,dy = \frac{\pi}{2}\big[\ln y\big]_1^{\pi} = \frac{\pi}{2}\ln\pi$$

9. (a) $x = r\cos\theta,\ y = r\sin\theta,\ z = z.$ Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right]$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}r\sin\theta + \dfrac{\partial u}{\partial y}r\cos\theta$ and

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\,\frac{\partial^2 u}{\partial y\,\partial x}r^2\sin\theta\,\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta.$$ So

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r}$$

$$+ \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\,\cos\theta$$

$$-\frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(b) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Then

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = \frac{\partial u}{\partial x}\sin\phi\,\cos\theta + \frac{\partial u}{\partial y}\sin\phi\,\sin\theta + \frac{\partial u}{\partial z}\cos\phi,$$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial\rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\rho}+\frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial\rho}+\frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial\rho}\right]\\ &\quad+\sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\rho}+\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial\rho}+\frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial\rho}\right]\\ &\quad+\cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial\rho}+\frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial\rho}+\frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial\rho}\right]\\ &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\,\sin\theta\,\cos\theta+2\,\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\,\cos\phi\,\cos\theta+2\,\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\,\cos\phi\,\sin\theta\\ &\quad+\frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta+\frac{\partial^2 u}{\partial z^2}\cos^2\phi\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial\phi}=\dfrac{\partial u}{\partial x}\,\rho\cos\phi\cos\theta+\dfrac{\partial u}{\partial y}\,\rho\cos\phi\sin\theta-\dfrac{\partial u}{\partial z}\,\rho\sin\phi$, and

$$\begin{aligned}\frac{\partial^2 u}{\partial\phi^2} &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\cos^2\phi\,\sin\theta\,\cos\theta-2\,\frac{\partial^2 u}{\partial x\,\partial z}\,\rho^2\sin\phi\,\cos\phi\,\cos\theta\\ &\quad-2\,\frac{\partial^2 u}{\partial y\,\partial z}\,\rho^2\sin\phi\,\cos\phi\,\sin\theta+\frac{\partial^2 u}{\partial x^2}\,\rho^2\cos^2\phi\,\cos^2\theta+\frac{\partial^2 u}{\partial y^2}\,\rho^2\cos^2\phi\,\sin^2\theta\\ &\quad+\frac{\partial^2 u}{\partial z^2}\,\rho^2\sin^2\phi-\frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta-\frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta-\frac{\partial u}{\partial z}\,\rho\cos\phi\end{aligned}$$

And $\dfrac{\partial u}{\partial\theta}=-\dfrac{\partial u}{\partial x}\,\rho\sin\phi\sin\theta+\dfrac{\partial u}{\partial y}\,\rho\sin\phi\cos\theta$, while

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta^2} &= -2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\sin^2\phi\,\cos\theta\,\sin\theta+\frac{\partial^2 u}{\partial x^2}\,\rho^2\sin^2\phi\,\sin^2\theta\\ &\quad+\frac{\partial^2 u}{\partial y^2}\,\rho^2\sin^2\phi\cos^2\theta-\frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta-\frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\partial^2 u}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial\rho}+\frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial\phi}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\\ &\qquad=\frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\,\cos^2\theta)+(\cos^2\phi\,\cos^2\theta)+\sin^2\theta\right]\\ &\qquad\quad+\frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\,\sin^2\theta)+(\cos^2\phi\,\sin^2\theta)+\cos^2\theta\right]+\frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi+\sin^2\phi\right]\\ &\qquad\quad+\frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\,\cos\theta+\cos^2\phi\,\cos\theta-\sin^2\phi\,\cos\theta-\cos\theta}{\rho\sin\phi}\right]\\ &\qquad\quad+\frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\,\sin\theta+\cos^2\phi\,\sin\theta-\sin^2\phi\,\sin\theta-\sin\theta}{\rho\sin\phi}\right]\end{aligned}$$

But

$$2\sin^2\phi\cos\theta+\cos^2\phi\cos\theta-\sin^2\phi\cos\theta-\cos\theta=\left(\sin^2\phi+\cos^2\phi-1\right)\,\cos\theta=0$$

and similarly the coefficient of $\partial u/\partial y$ is 0. Also

$$\sin^2\phi\cos^2\theta+\cos^2\phi\cos^2\theta+\sin^2\theta=\cos^2\theta\,(\sin^2\phi+\cos^2\phi)+\sin^2\theta=1$$

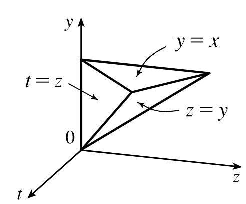
and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

10. (a) Consider a polar division of the disk, similar to that in Figure 15.3.4, where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d(r_i^*)\,\Delta r\,\Delta\theta}{\left[(r_i^*)^2 + d^2\right]^{3/2}}$ which becomes $\displaystyle\int_0^R\int_0^{2\pi}\frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$ as $m \to \infty$ and $n \to \infty$. Therefore,

$$F = 2\pi Gm\rho d\int_0^R \frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R \to \infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}} \to 0$, and we are left with $F = 2\pi Gm\rho d\left(\dfrac{1}{d} - 0\right) = 2\pi Gm\rho$.

11. $\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where $E = \{(t,z,y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$. If we let D be the projection of E on the yt-plane then $D = \{(y,t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram that $E = \{(t,z,y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So



$$\begin{aligned}\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy &= \int_0^x\int_t^x\int_t^y f(t)\,dz\,dy\,dt = \int_0^x\left[\int_t^x (y-t)\,f(t)\,dy\right]dt \\ &= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right)f(t)\right]_{y=t}^{y=x}\,dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right]f(t)\,dt \\ &= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right]f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right)f(t)\,dt \\ &= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt\end{aligned}$$

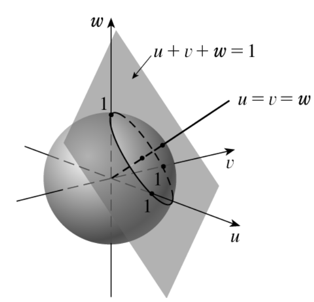
12. $n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\sqrt{n^2+ni+j}} = \sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\frac{1}{n}\sqrt{n^2+ni+j}}\cdot\dfrac{1}{n^3} = \sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\sqrt{1+\frac{i}{n}+\frac{j}{n^2}}}\cdot\dfrac{1}{n^3}$ can be considered a double Riemann sum of the function $f(x,y)=\dfrac{1}{\sqrt{1+x+y}}$ where the square region $R=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$ is divided into subrectangles by dividing the interval $[0,1]$ on the x-axis into n subintervals, each of width $\frac{1}{n}$, and $[0,1]$ on the y-axis is divided into n^2 subintervals, each of width $\frac{1}{n^2}$. Then the area of each subrectangle is $\Delta A=\frac{1}{n^3}$, and if we take the upper right corners of the subrectangles as sample points, we have $(x_{ij}^*, y_{ij}^*)=\left(\frac{i}{n},\frac{j}{n^2}\right)$. Finally, note that $n^2\to\infty$ as $n\to\infty$, so

$$\lim_{n\to\infty} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2+ni+j}} = \lim_{n,n^2\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{1+\frac{i}{n}+\frac{j}{n^2}}}\cdot\frac{1}{n^3} = \lim_{n,n^2\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n^2} f(x_{ij}^*,y_{ij}^*)\,\Delta A$$

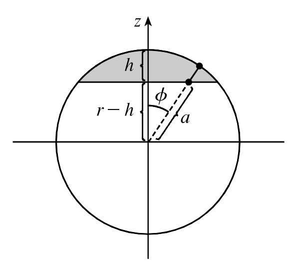
But by Definition 15.1.5 this is equal to $\iint_R f(x,y)\,dA$, so

$$\begin{aligned}\lim_{n\to\infty} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2+ni+j}} &= \iint_R f(x,y)\,dA = \int_0^1\int_0^1 \frac{1}{\sqrt{1+x+y}}\,dy\,dx\\ &= \int_0^1\left[2(1+x+y)^{1/2}\right]_{y=0}^{y=1}dx = 2\int_0^1\left(\sqrt{2+x}-\sqrt{1+x}\right)dx\\ &= 2\left[\tfrac{2}{3}(2+x)^{3/2}-\tfrac{2}{3}(1+x)^{3/2}\right]_0^1 = \tfrac{4}{3}(3^{3/2}-2^{3/2}-2^{3/2}+1)\\ &= \tfrac{4}{3}(3\sqrt{3}-4\sqrt{2}+1) = 4\sqrt{3}-\tfrac{16}{3}\sqrt{2}+\tfrac{4}{3}\end{aligned}$$

13. The volume is $V=\iiint_R dV$ where R is the solid region given. From Exercise 15.9.23(a), the transformation $x=au$, $y=bv$, $z=cw$ maps the unit ball $u^2+v^2+w^2\le 1$ to the solid ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1$ with $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=abc$. The same transformation maps the plane $u+v+w=1$ to $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$. Thus the region R in xyz-space corresponds to the region S in uvw-space consisting of the smaller piece of the unit ball cut off by the plane $u+v+w=1$, a "cap of a sphere" (see the figure).



We will need to compute the volume of S, but first consider the general case where a horizontal plane slices the upper portion of a sphere of radius r to produce a cap of height h. We use spherical coordinates. From the figure, a line through the origin at angle ϕ from the z-axis intersects the plane when $\cos\phi=(r-h)/a \Rightarrow a=(r-h)/\cos\phi$, and the line passes through the outer rim of the cap when $a=r \Rightarrow \cos\phi=(r-h)/r \Rightarrow \phi=\cos^{-1}((r-h)/r)$. Thus the cap

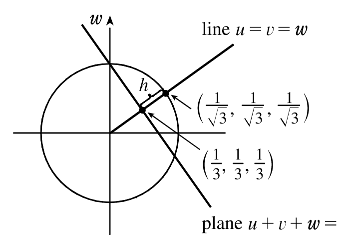


is described by $\{(\rho,\theta,\phi) \mid (r-h)/\cos\phi \le \rho \le r, 0 \le \theta \le 2\pi, 0 \le \phi \le \cos^{-1}((r-h)/r)\}$ and its volume is

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\cos^{-1}((r-h)/r)}\int_{(r-h)/\cos\phi}^{r} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\cos^{-1}((r-h)/r)} \left[\tfrac{1}{3}\rho^3\sin\phi\right]_{\rho=(r-h)/\cos\phi}^{\rho=r} d\phi\,d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\cos^{-1}((r-h)/r)} \left[r^3\sin\phi - \frac{(r-h)^3}{\cos^3\phi}\sin\phi\right] d\phi\,d\theta \\
&= \tfrac{1}{3}\int_0^{2\pi} \left[-r^3\cos\phi - \tfrac{1}{2}(r-h)^3\cos^{-2}\phi\right]_{\phi=0}^{\phi=\cos^{-1}((r-h)/r)} d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \left[-r^3\left(\frac{r-h}{r}\right) - \frac{1}{2}(r-h)^3\left(\frac{r-h}{r}\right)^{-2} + r^3 + \frac{1}{2}(r-h)^3\right] d\theta \\
&= \tfrac{1}{3}\int_0^{2\pi}\left(\tfrac{3}{2}rh^2 - \tfrac{1}{2}h^3\right) d\theta = \tfrac{1}{3}\left(\tfrac{3}{2}rh^2 - \tfrac{1}{2}h^3\right)(2\pi) = \pi h^2\left(r - \tfrac{1}{3}h\right)
\end{aligned}$$

(This volume can also be computed by treating the cap as a solid of revolution and using the single variable disk method; see Exercise 6.2.61.)

To determine the height h of the cap cut from the unit ball by the plane $u+v+w=1$, note that the line $u=v=w$ passes through the origin with direction vector $\langle 1,1,1\rangle$ which is perpendicular to the plane. Therefore this line coincides with a radius of the sphere that passes through the center of the cap and h is measured along this line. The line intersects the plane at $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and the sphere at $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$. (See the figure.)



The distance between these points is $h = \sqrt{3\left(\frac{1}{\sqrt{3}}-\frac{1}{3}\right)^2} = \sqrt{3}\left(\frac{1}{\sqrt{3}}-\frac{1}{3}\right) = 1 - \frac{1}{\sqrt{3}}$. Thus the volume of R is

$$\begin{aligned}
V &= \iiint_R dV = \iiint_S \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV = abc\iiint_S dV = abc\,V(S) \\
&= abc\cdot\pi h^2\left(r-\tfrac{1}{3}h\right) = abc\cdot\pi\left(1-\tfrac{1}{\sqrt{3}}\right)^2\left[1-\tfrac{1}{3}\left(1-\tfrac{1}{\sqrt{3}}\right)\right] \\
&= abc\pi\left(\tfrac{4}{3}-\tfrac{2}{\sqrt{3}}\right)\left(\tfrac{2}{3}+\tfrac{1}{3\sqrt{3}}\right) = abc\pi\left(\tfrac{2}{3}-\tfrac{8}{9\sqrt{3}}\right) \approx 0.482abc
\end{aligned}$$

16 □ VECTOR CALCULUS

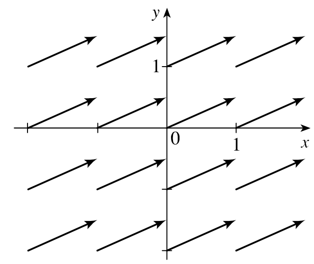
16.1 Vector Fields

1. $\mathbf{F}(x,y) = \mathbf{i} + \frac{1}{2}\,\mathbf{j}$

All vectors in this field are identical with length

$\sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ and parallel to $\left\langle 1, \frac{1}{2} \right\rangle$, or,

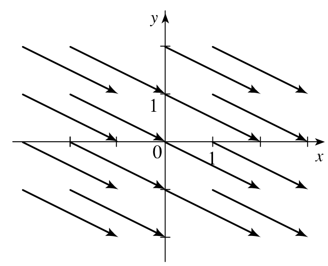
equivalently, $\langle 2, 1 \rangle$.



2. $\mathbf{F}(x,y) = 2\,\mathbf{i} - \mathbf{j}$

All vectors in this field are identical with length

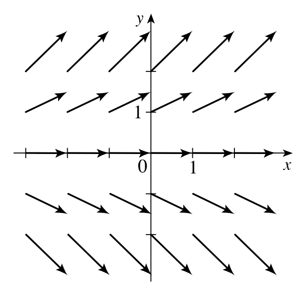
$\sqrt{2^2 + (-1)^2} = \sqrt{5}$ and parallel to $\langle 2, -1 \rangle$.



3. $\mathbf{F}(x,y) = \mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

The length of the vector $\mathbf{i} + \frac{1}{2}y\,\mathbf{j}$ is $\sqrt{1 + \frac{1}{4}y^2}$. Vectors

along the line $y = 0$ are horizontal with length 1.

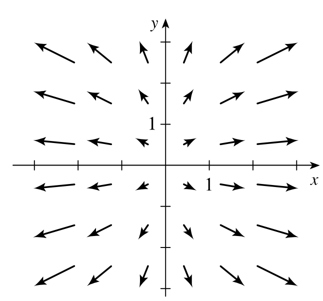


4. $\mathbf{F}(x,y) = x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

The length of the vector $x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j}$ is $\sqrt{x^2 + \frac{1}{4}y^2}$.

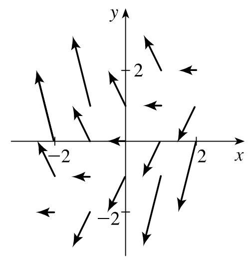
Vectors point roughly away from the origin and vectors

farther from the origin are longer.



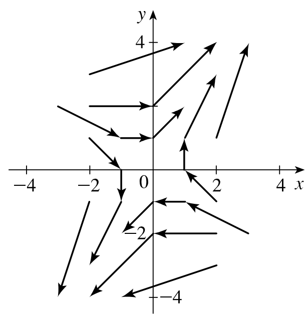
5. $\mathbf{F}(x, y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

The length of the vector $-\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$ is $\sqrt{\frac{1}{4} + (y - x)^2}$. Vectors along the line $y = x$ are horizontal with length $\frac{1}{2}$.



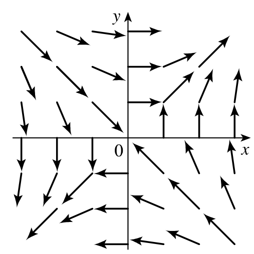
6. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

The length of the vector $y\,\mathbf{i} + (x + y)\,\mathbf{j}$ is $\sqrt{y^2 + (x + y)^2}$. Vectors along the x-axis are vertical, and vectors along the line $y = -x$ are horizontal with length $|y|$.



7. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

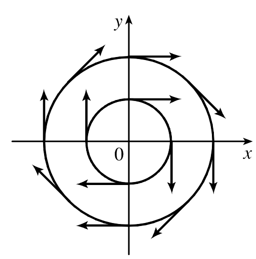
The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ is $\sqrt{\dfrac{y^2}{x^2 + y^2} + \dfrac{x^2}{x^2 + y^2}} = 1$.



Vectors along the x-axis are vertical, and vectors along the y-axis are horizontal. In general, vectors in Q1 and QIII point away from the origin, whereas vectors in QII and QIV point toward the origin.

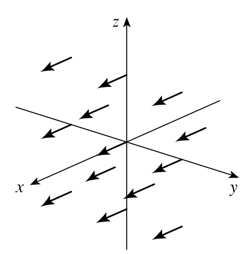
8. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

All the vectors $\mathbf{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.

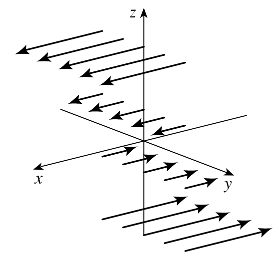


9. $\mathbf{F}(x, y, z) = \mathbf{i}$

All vectors in this field are identical, with length 1 and pointing in the direction of the positive x-axis.

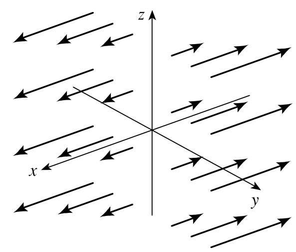


10. $\mathbf{F}(x, y, z) = z\,\mathbf{i}$



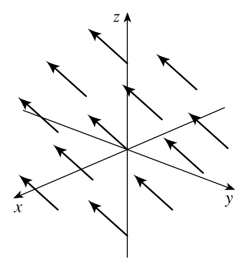
At each point (x, y, z), $\mathbf{F}(x, y, z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive x-axis, while for $z < 0$, all are in the direction of the negative x-axis. In each plane $z = k$, all the vectors are identical.

11. $\mathbf{F}(x, y, z) = -y\,\mathbf{i}$



At each point (x, y, z), $\mathbf{F}(x, y, z)$ is a vector of length $|y|$. For $y > 0$, all point in the direction of the negative x-axis, while for $y < 0$, all are in the direction of the positive x-axis. In each plane $y = k$, all the vectors are identical.

12. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$



All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xz-plane.

13. $\mathbf{F}(x, y) = \langle x, -y \rangle$ corresponds to graph IV. In the first quadrant all the vectors have positive x-components and negative y-components, in the second quadrant all vectors have negative x- and y-components, in the third quadrant all vectors have negative x-components and positive y-components, and in the fourth quadrant all vectors have positive x- and y-components. In addition, the vectors get shorter as we approach the origin.

14. $\mathbf{F}(x, y) = \langle y, x - y \rangle$ corresponds to graph V. All vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. In addition, vectors along the line $y = x$ are horizontal, and vectors get shorter as we approach the origin.

15. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$ corresponds to graph I. As in Exercise 14, all vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. Vectors along the line $y = -2$ are horizontal, and the vectors are independent of x (vectors along horizontal lines are identical).

16. $\mathbf{F}(x, y) = \langle y, 2x \rangle$ corresponds to graph VI. In the first quadrant all the vectors have positive x- and y-components. In the second quadrant all vectors have positive x-components and negative y-components. In the third quadrant all vectors have negative x- and y-components. In the fourth quadrant all vectors have negative x-components and positive y-components.

17. $\mathbf{F}(x, y) = \langle \sin y, \cos x \rangle$ corresponds to graph III. Both the x- and y-components oscillate in all four quadrants.

18. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$ corresponds to graph II. All vectors in quadrants I and IV have positive y-components while all vectors in quadrants II and III have negative y-components. Also, the y-components of vectors along any vertical line remain constant while the x-component oscillates.

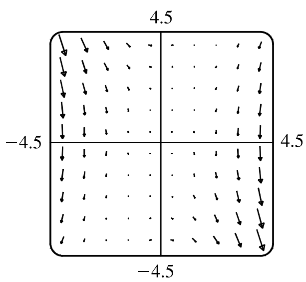
19. $\mathbf{F}(x,y,z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

20. $\mathbf{F}(x,y,z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

21. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

22. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x,y,z)$ has the same length and direction as the position vector of the point (x,y,z), and therefore the vectors all point directly away from the origin.

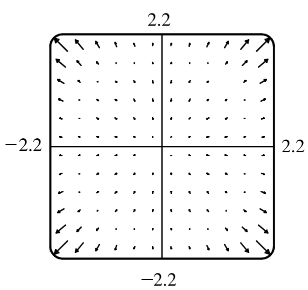
23.



$\mathbf{F}(x,y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$.

The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x,y) = \langle 0, 0\rangle$, we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x,y) = \mathbf{0}$] along the line $y = 2x$.

24.



$\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\,\mathbf{x}$, where $\mathbf{x} = \langle x, y\rangle$ and $r = |\mathbf{x}|$.

From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\mathbf{x}| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = \mathbf{0} \;\Leftrightarrow\; r(r-2) = 0 \;\Leftrightarrow\; r = 0$ or 2, so as we suspected, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for $|\mathbf{x}| = 2$ and for $|\mathbf{x}| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

25. $f(x,y) = y\sin(xy) \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y\,(x,y)\;\mathbf{j} = (y\cos(xy)\cdot y)\,\mathbf{i} + [y\cdot x\cos(xy) + \sin(xy)\cdot 1]\,\mathbf{j}\\ &= y^2\cos(xy)\,\mathbf{i} + [xy\cos(xy) + \sin(xy)]\,\mathbf{j}\end{aligned}$$

26. $f(s,t) = \sqrt{2s+3t} \quad \Rightarrow$

$$\nabla f(s,t) = f_s(s,t)\,\mathbf{i} + f_t(s,t)\,\mathbf{j} = \left[\tfrac{1}{2}(2s+3t)^{-1/2}\cdot 2\right]\mathbf{i} + \left[\tfrac{1}{2}(2s+3t)^{-1/2}\cdot 3\right]\mathbf{j} = \frac{1}{\sqrt{2s+3t}}\,\mathbf{i} + \frac{3}{2\sqrt{2s+3t}}\,\mathbf{j}$$

27. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}\\ &= \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)\,\mathbf{i} + \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)\,\mathbf{j} + \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)\,\mathbf{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}\end{aligned}$$

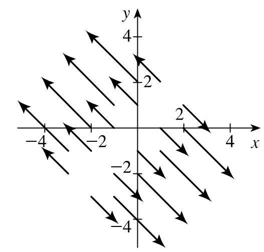
28. $f(x,y,z) = x^2 y e^{y/z} \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}\\ &= 2xye^{y/z}\,\mathbf{i} + x^2\left[y\cdot e^{yz}(1/z) + e^{y/z}\cdot 1\right]\mathbf{j} + \left[x^2ye^{y/z}(-y/z^2)\right]\mathbf{k}\\ &= 2xye^{y/z}\,\mathbf{i} + x^2e^{y/z}\left(\frac{y}{z}+1\right)\mathbf{j} - \frac{x^2y^2}{z^2}e^{y/z}\,\mathbf{k}\end{aligned}$$

29. $f(x,y) = \frac{1}{2}(x-y)^2 \quad \Rightarrow$

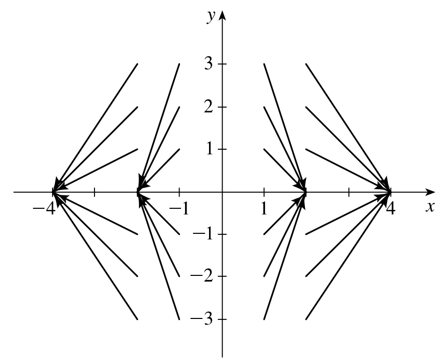
$\nabla f(x,y) = (x-y)(1)\,\mathbf{i} + (x-y)(-1)\,\mathbf{j} = (x-y)\,\mathbf{i} + (y-x)\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{(x-y)^2+(y-x)^2} = \sqrt{2}\,|x-y|$. The vectors are $\mathbf{0}$ along the line $y=x$. Elsewhere the vectors point away from the line $y=x$ with length that increases as the distance from the line increases.



30. $f(x,y) = \frac{1}{2}(x^2-y^2) \quad \Rightarrow \quad \nabla f(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{x^2+y^2}$. The lengths of the vectors increase as the distance from the origin increases, and the terminal point of each vector lies on the x-axis.



31. $f(x,y) = x^2+y^2 \quad \Rightarrow \quad \nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph III.

32. $f(x,y) = x(x+y) = x^2+xy \quad \Rightarrow \quad \nabla f(x,y) = (2x+y)\,\mathbf{i} + x\,\mathbf{j}$. The y-component of each vector is x, so the vectors point upward in quadrants I and IV and downward in quadrants II and III. Also, the x-component of each vector is 0 along the line $y=-2x$ so the vectors are vertical there. Thus, ∇f is graph IV.

33. $f(x,y) = (x+y)^2 \quad \Rightarrow \quad \nabla f(x,y) = 2(x+y)\,\mathbf{i} + 2(x+y)\,\mathbf{j}$. The x- and y-components of each vector are equal, so all vectors are parallel to the line $y=x$. The vectors are $\mathbf{0}$ along the line $y=-x$ and their length increases as the distance from this line increases. Thus, ∇f is graph II.

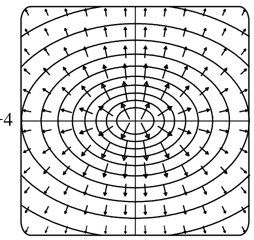
34. $f(x,y) = \sin\sqrt{x^2+y^2} \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y) &= \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2x)\right]\mathbf{i} + \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2y)\right]\mathbf{j}\\ &= \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,x\,\mathbf{i} + \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,y\,\mathbf{j} \;\text{ or }\; \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,(x\,\mathbf{i}+y\,\mathbf{j})\end{aligned}$$

Thus each vector is a scalar multiple of its position vector, so the vectors point toward or away from the origin with length that changes in a periodic fashion as we move away from the origin. ∇f is graph I.

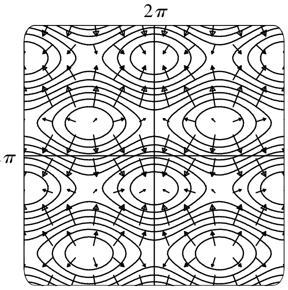
35. $f(x,y) = \ln(1 + x^2 + 2y^2)$. We graph

$\nabla f(x,y) = \dfrac{2x}{1 + x^2 + 2y^2}\,\mathbf{i} + \dfrac{4y}{1 + x^2 + 2y^2}\,\mathbf{j}$ along with a contour map of f. The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.



36. $f(x,y) = \cos x - 2\sin y$.

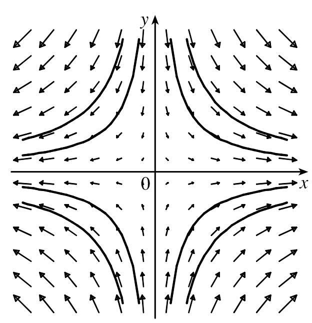
We graph $\nabla f(x,y) = -\sin x\,\mathbf{i} - 2\cos y\,\mathbf{j}$ along with a contour map of f. The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.



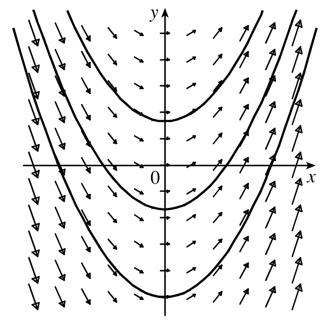
37. $\mathbf{V}(x,y) = \langle x^2, x + y^2\rangle$. At $t = 3$ the particle is at $(2,1)$ so its velocity is $\mathbf{V}(2,1) = \langle 4,3\rangle$. After 0.01 units of time, the particle's change in location should be approximately $0.01\,\mathbf{V}(2,1) = 0.01\,\langle 4,3\rangle = \langle 0.04, 0.03\rangle$, so the particle should be approximately at the point $(2.04, 1.03)$.

38. $\mathbf{F}(x,y) = \langle xy - 2, y^2 - 10\rangle$. At $t = 1$ the particle is at $(1,3)$ so its velocity is $\mathbf{F}(1,3) = \langle 1, -1\rangle$. After 0.05 units of time, the particle's change in location should be approximately $0.05\,\mathbf{F}(1,3) = 0.05\,\langle 1,-1\rangle = \langle 0.05, -0.05\rangle$, so the particle should be approximately at the point $(1.05, 2.95)$.

39. (a) We sketch the vector field $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \;\Rightarrow\; dx/dt = x,\, dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \;\Rightarrow\; dx/x = dt \;\Rightarrow\; \ln|x| = t + C \;\Rightarrow\; x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \;\Rightarrow\; dy/y = -dt \;\Rightarrow\; \ln|y| = -t + K \;\Rightarrow\; y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^tBe^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)\,(1) =$ constant $= 1 \;\Rightarrow\; xy = 1 \;\Rightarrow\; y = 1/x,\, x > 0$.



40. (a) We sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.

(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\,\mathbf{j} \quad\Rightarrow\quad \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = x$. Thus, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{1} = x$.

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0, 0)$ is on the curve, so $0 = 0 + c \quad\Rightarrow\quad c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

16.2 Line Integrals

1. $x = t^2$ and $y = 2t$, $0 \le t \le 3$, so by Formula 3

$$\int_C y\,ds = \int_0^3 2t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 2t\,\sqrt{(2t)^2 + (2)^2}\,dt = \int_0^3 2t\,\sqrt{4t^2 + 4}\,dt$$
$$= \int_0^3 4t\,\sqrt{t^2 + 1}\,dt = \left[2 \cdot \tfrac{2}{3}(t^2 + 1)^{3/2}\right]_0^3 = \tfrac{4}{3}(10^{3/2} - 1)$$

2. $x = t^3$ and $y = t^4$, $1 \le t \le 2$, so by Formula 3

$$\int_C (x/y)\,ds = \int_1^2 (t^3/t^4)\sqrt{(3t^2)^2 + (4t^3)^2}\,dt = \int_1^2 (1/t) \cdot t^2\sqrt{9 + 16t^2}\,dt = \int_1^2 t\sqrt{9 + 16t^2}\,dt$$
$$= \left[\tfrac{1}{32} \cdot \tfrac{2}{3}(9 + 16t^2)^{3/2}\right]_1^2 = \tfrac{1}{48}(73^{3/2} - 25^{3/2}) \ \text{ or } \ \tfrac{1}{48}(73\sqrt{73} - 125)$$

3. Parametric equations for C are $x = 4\cos t,\ \ y = 4\sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^5\cos t\,\sin^4 t\,\sqrt{16(\sin^2 t + \cos^2 t)}\,dt$$
$$= 4^5\int_{-\pi/2}^{\pi/2} (\sin^4 t\cos t)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = 4^6 \cdot \tfrac{2}{5} = 1638.4$$

4. Parametric equations for C are $x = 2 + 3t,\ \ y = 4t,\ \ 0 \le t \le 1$. Then

$$\int_C xe^y\,ds = \int_0^1 (2 + 3t)\,e^{4t}\,\sqrt{3^2 + 4^2}\,dt = 5\int_0^1 (2 + 3t)\,e^{4t}\,dt$$

Integrating by parts with $u = 2 + 3t \quad\Rightarrow\quad du = 3\,dt,\ dv = e^{4t}\,dt \quad\Rightarrow\quad v = \frac{1}{4}e^{4t}$ gives

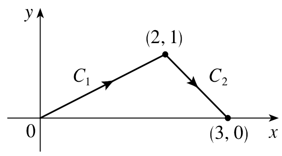
$$\int_C xe^y\,ds = 5\left[\tfrac{1}{4}(2 + 3t)e^{4t} - \tfrac{3}{16}e^{4t}\right]_0^1 = 5\left[\tfrac{5}{4}e^4 - \tfrac{3}{16}e^4 - \tfrac{1}{2} + \tfrac{3}{16}\right] = \tfrac{85}{16}e^4 - \tfrac{25}{16}$$

5. If we choose x as the parameter, parametric equations for C are $x = x,\ \ y = x^2$ for $0 \le x \le \pi$ and by Equations 7

$$\int_C \left(x^2y + \sin x\right)dy = \int_0^\pi \left[x^2(x^2) + \sin x\right] \cdot 2x\,dx = 2\int_0^\pi \left(x^5 + x\sin x\right)dx$$
$$= 2\left[\tfrac{1}{6}x^6 - x\cos x + \sin x\right]_0^\pi \quad \left[\begin{array}{c}\text{where we integrated by parts}\\ \text{in the second term}\end{array}\right]$$
$$= 2\left[\tfrac{1}{6}\pi^6 + \pi + 0 - 0\right] = \tfrac{1}{3}\pi^6 + 2\pi$$

6. Choosing y as the parameter, we have $x = y^3$, $y = y$, $-1 \le y \le 1$. Then

$$\int_C e^x\,dx = \int_{-1}^{1} e^{y^3} \cdot 3y^2\,dy = e^{y^3}\Big]_{-1}^{1} = e^1 - e^{-1} = e - \tfrac{1}{e}.$$

7. 

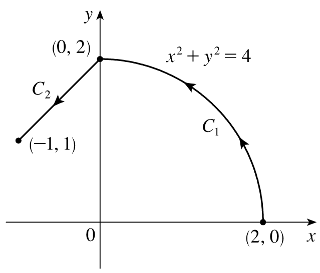
$C = C_1 + C_2$

On C_1: $x = x, y = \frac{1}{2}x \;\Rightarrow\; dy = \frac{1}{2}\,dx,\; 0 \le x \le 2.$

On C_2: $x = x, y = 3 - x \;\Rightarrow\; dy = -dx,\; 2 \le x \le 3.$

Then

$$\begin{aligned}\int_C (x+2y)\,dx + x^2\,dy &= \int_{C_1} (x+2y)\,dx + x^2\,dy + \int_{C_2} (x+2y)\,dx + x^2\,dy \\ &= \int_0^2 \left[x + 2\left(\tfrac{1}{2}x\right) + x^2\left(\tfrac{1}{2}\right)\right] dx + \int_2^3 \left[x + 2(3-x) + x^2(-1)\right] dx \\ &= \int_0^2 \left(2x + \tfrac{1}{2}x^2\right) dx + \int_2^3 \left(6 - x - x^2\right) dx \\ &= \left[x^2 + \tfrac{1}{6}x^3\right]_0^2 + \left[6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_2^3 = \tfrac{16}{3} - 0 + \tfrac{9}{2} - \tfrac{22}{3} = \tfrac{5}{2}\end{aligned}$$

8. 

$C = C_1 + C_2$

On C_1: $x = 2\cos t \;\Rightarrow\; dx = -2\sin t\,dt,$

$y = 2\sin t \;\Rightarrow\; dy = 2\cos t\,dt, 0 \le t \le \frac{\pi}{2}.$

On C_2: $x = -t \;\Rightarrow\; dx = -dt,$

$y = 2 - t \;\Rightarrow\; dy = -dt, 0 \le t \le 1.$

Then

$$\begin{aligned}\int_C x^2\,dx + y^2\,dy &= \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy \\ &= \int_0^{\pi/2} (2\cos t)^2(-2\sin t\,dt) + (2\sin t)^2(2\cos t\,dt) + \int_0^1 (-t)^2(-dt) + (2-t)^2(-dt) \\ &= \int_0^{\pi/2} (-8\cos^2 t\sin t + 8\sin^2 t\cos t)\,dt - 2\int_0^1 (t^2 - 2t + 2)\,dt \\ &= 8\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} - 2\left[\tfrac{1}{3}t^3 - t^2 + 2t\right]_0^1 = 8\left(\tfrac{1}{3} - \tfrac{1}{3}\right) - 2\left(\tfrac{1}{3} - 1 + 2\right) = -\tfrac{8}{3}\end{aligned}$$

9. $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le \pi/2$. Then by Formula 9,

$$\begin{aligned}\int_C x^2y\,ds &= \int_0^{\pi/2} (\cos t)^2(\sin t)\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt \\ &= \int_0^{\pi/2} \cos^2 t\sin t\,\sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\,dt = \int_0^{\pi/2} \cos^2 t\sin t\,\sqrt{\sin^2 t + \cos^2 t + 1}\,dt \\ &= \sqrt{2}\int_0^{\pi/2} \cos^2 t\sin t\,dt = \sqrt{2}\left[-\tfrac{1}{3}\cos^3 t\right]_0^{\pi/2} = \sqrt{2}\left(0 + \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{3}\end{aligned}$$

10. Parametric equations for the line segment C from $(3, 1, 2)$ to $(1, 2, 5)$ are $x = 3 - 2t$, $y = 1 + t$, $z = 2 + 3t$, $0 \le t \le 1$.

Then by Formula 9,

$$\begin{aligned}\int_C y^2z\,ds &= \int_0^1 (1+t)^2(2+3t)\sqrt{(-2)^2 + 1^2 + 3^2}\,dt = \sqrt{14}\int_0^1 (3t^3 + 8t^2 + 7t + 2)\,dt \\ &= \sqrt{14}\left[\tfrac{3}{4}t^4 + \tfrac{8}{3}t^3 + \tfrac{7}{2}t^2 + 2t\right]_0^1 = \sqrt{14}\left(\tfrac{3}{4} + \tfrac{8}{3} + \tfrac{7}{2} + 2\right) = \tfrac{107}{12}\sqrt{14}\end{aligned}$$

11. Parametric equations for the line segment C from $(0,0,0)$ to $(1,2,3)$ are $x=t,\ y=2t,\ z=3t,\ 0\le t\le 1$. Then by Formula 9, $\int_C xe^{yz}\,ds=\int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt=\sqrt{14}\int_0^1 te^{6t^2}dt=\sqrt{14}\left[\frac{1}{12}e^{6t^2}\right]_0^1=\frac{\sqrt{14}}{12}(e^6-1)$.

12. C: $x=t$, $y=\cos 2t$, $z=\sin 2t$, $0\le t\le 2\pi$.

$\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}=\sqrt{1^2+(-2\sin 2t)^2+(2\cos 2t)^2}=\sqrt{1+4(\sin^2 2t+\cos^2 2t)}=\sqrt{5}$.

Then by Formula 9,

$$\begin{aligned}\int_C(x^2+y^2+z^2)\,ds&=\int_0^{2\pi}(t^2+\cos^2 2t+\sin^2 2t)\sqrt{5}\,dt=\sqrt{5}\int_0^{2\pi}(t^2+1)\,dt\\&=\sqrt{5}\left[\tfrac13 t^3+t\right]_0^{2\pi}=\sqrt{5}\left[\tfrac13(8\pi^3)+2\pi\right]=\sqrt{5}\left(\tfrac83\pi^3+2\pi\right)\end{aligned}$$

13. C: $x=t$, $y=t^2$, $z=t^3$, $0\le t\le 1$.

$$\int_C xye^{yz}\,dy=\int_0^1(t)(t^2)e^{(t^2)(t^3)}\cdot 2t\,dt=\int_0^1 2t^4e^{t^5}\,dt=\left[\tfrac25 e^{t^5}\right]_0^1=\tfrac25(e^1-e^0)=\tfrac25(e-1)$$

14. C: $x=e^t$, $y=2t$, $z=\ln t$, $1\le t\le 2$.

$$\begin{aligned}\int_C ye^z\,dz+x\ln x\,dy-y\,dx&=\int_1^2 2te^{\ln t}\,\tfrac1t\,dt+e^t\ln e^t\cdot 2\,dt-2te^t\,dt=\int_1^2\left(2t+2te^t-2te^t\right)\,dt\\&=\int_1^2 2t\,dt=\left[t^2\right]_1^2=4-1=3\end{aligned}$$

15. C: $x=\sin t$, $y=\cos t$, $z=\tan t$, $-\pi/4\le t\le\pi/4$.

$$\begin{aligned}\int_C z\,dx+xy\,dy+y^2\,dz&=\int_{-\pi/4}^{\pi/4}(\tan t)(\cos t\,dt)+(\sin t)(\cos t)(-\sin t\,dt)+(\cos^2 t)(\sec^2 t\,dt)\\&=\int_{-\pi/4}^{\pi/4}(\sin t-\sin^2 t\cos t+1)\,dt=\left[-\cos t-\tfrac13\sin^3 t+t\right]_{-\pi/4}^{\pi/4}\\&=\left[-\frac{\sqrt2}{2}-\frac13\left(\frac{\sqrt2}{2}\right)^3+\frac{\pi}{4}\right]-\left[-\frac{\sqrt2}{2}-\frac13\left(-\frac{\sqrt2}{2}\right)^3-\frac{\pi}{4}\right]=\frac{\pi}{2}-\frac{\sqrt2}{6}\end{aligned}$$

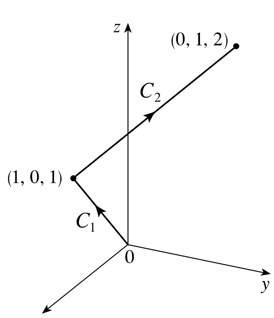
16. C: $x=\sqrt{t}$, $y=t$, $z=t^2$, $1\le t\le 4$.

$$\begin{aligned}\int_C y\,dx+z\,dy+x\,dz&=\int_1^4 t\cdot\tfrac12 t^{-1/2}\,dt+t^2\cdot dt+\sqrt{t}\cdot 2t\,dt=\int_1^4\left(\tfrac12 t^{1/2}+t^2+2t^{3/2}\right)dt\\&=\left[\tfrac13 t^{3/2}+\tfrac13 t^3+\tfrac45 t^{5/2}\right]_1^4=\tfrac83+\tfrac{64}{3}+\tfrac{128}{5}-\tfrac13-\tfrac13-\tfrac45=\tfrac{722}{15}\end{aligned}$$

17. Parametric equations for the line segment C from $(1,0,0)$ to $(4,1,2)$ are $x=1+3t,\ \ y=t,\ \ z=2t,\ 0\le t\le 1$. Then

$$\begin{aligned}\int_C z^2\,dx+x^2\,dy+y^2\,dz&=\int_0^1(2t)^2\cdot 3\,dt+(1+3t)^2\,dt+t^2\cdot 2\,dt=\int_0^1\left(23t^2+6t+1\right)dt\\&=\left[\tfrac{23}{3}t^3+3t^2+t\right]_0^1=\tfrac{23}{3}+3+1=\tfrac{35}{3}\end{aligned}$$

18.



$C=C_1+C_2$

On C_1 from $(0,0,0)$ to $(1,0,1)$: $x=t\ \Rightarrow\ dx=dt$, $y=0\ \Rightarrow\ dy=0\,dt$, $z=t\ \Rightarrow\ dz=dt$, $0\le t\le 1$.

On C_2 from $(1,0,1)$ to $(0,1,2)$: $x=1-t\ \Rightarrow\ dx=-dt$, $y=t\ \Rightarrow\ dy=dt$, $z=1+t\ \Rightarrow\ dz=dt$, $0\le t\le 1$. [continued]

Then

$$\begin{aligned}
&\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz \\
&\quad = \int_{C_1} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz + \int_{C_2} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz \\
&\quad = \int_0^1 (0+t)\,dt + (t+t)\cdot 0\,dt + (t+0)\,dt + \int_0^1 (t+1+t)(-dt) + (1-t+1+t)\,dt + (1-t+t)\,dt \\
&\quad = \int_0^1 2t\,dt + \int_0^1 (-2t+2)\,dt = \left[t^2\right]_0^1 + \left[-t^2+2t\right]_0^1 = 1+1 = 2
\end{aligned}$$

19. (a) Along the line $x=-3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F}\cdot\mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F}\cdot\mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

20. Vectors starting on C_1 point in roughly the same direction as C_1, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ to be negative.

21. $\mathbf{F}(x,y) = xy^2\,\mathbf{i} - x^2\,\mathbf{j}$ and $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$, $0 \le t \le 1 \quad\Rightarrow$

$\mathbf{F}(\mathbf{r}(t)) = (t^3)(t^2)^2\,\mathbf{i} - (t^3)^2\,\mathbf{j} = t^7\,\mathbf{i} - t^6\,\mathbf{j}$ and $\mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$. Then

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (t^7\cdot 3t^2 - t^6\cdot 2t)\,dt = \int_0^1 (3t^9 - 2t^7)\,dt = \left[\frac{3}{10}t^{10} - \frac{1}{4}t^8\right]_0^1 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

22. $\mathbf{F}(x,y,z) = (x+y^2)\,\mathbf{i} + xz\,\mathbf{j} + (y+z)\,\mathbf{k}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k}$, $0 \le t \le 2 \quad\Rightarrow$

$\mathbf{F}(\mathbf{r}(t)) = \left(t^2 + (t^3)^2\right)\mathbf{i} + (t^2)(-2t)\,\mathbf{j} + (t^3-2t)\,\mathbf{k} = (t^2+t^6)\,\mathbf{i} - 2t^3\,\mathbf{j} + (t^3-2t)\,\mathbf{k}$ and $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - 2\,\mathbf{k}$.

Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^2 (2t^3 + 2t^7 - 6t^5 - 2t^3 + 4t)\,dt = \int_0^2 (2t^7 - 6t^5 + 4t)\,dt \\
&= \left[\tfrac{1}{4}t^8 - t^6 + 2t^2\right]_0^2 = 64 - 64 + 8 = 8
\end{aligned}$$

23. $\mathbf{F}(x,y,z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$ and $\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 1 \quad\Rightarrow$

$\mathbf{F}(\mathbf{r}(t)) = \sin t^3\,\mathbf{i} + \cos(-t^2)\,\mathbf{j} + t^3\cdot t\,\mathbf{k}$ and $\mathbf{r}'(t) = 3t^2\,\mathbf{i} - 2t\,\mathbf{j} + 1\,\mathbf{k}$. Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (3t^2\sin t^3 - 2t\cos t^2 + t^4)\,dt \\
&= \left[-\cos t^3 - \sin t^2 + \tfrac{1}{5}t^5\right]_0^1 = \tfrac{6}{5} - \cos 1 - \sin 1
\end{aligned}$$

24. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + z^3\,\mathbf{j} + y\,\mathbf{k}$ and $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $-1 \le t \le 1 \quad\Rightarrow$

$\mathbf{F}(\mathbf{r}(t)) = e^t e^{-t}\,\mathbf{i} + (e^{-t})^3\,\mathbf{j} + e^{2t}\,\mathbf{k} = \mathbf{i} + e^{-3t}\,\mathbf{j} + e^{2t}\,\mathbf{k}$ and $\mathbf{r}'(t) = e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j} - e^{-t}\,\mathbf{k}$. Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{-1}^1 \left(1\cdot e^t + e^{-3t}\cdot 2e^{2t} - e^{2t}e^{-t}\right)\,dt \\
&= \int_{-1}^1 2e^{-t}\,dt = -2\left[e^{-t}\right]_{-1}^1 = -2(e^{-1} - e)
\end{aligned}$$

25. $\mathbf{F}(\mathbf{r}(t)) = \sqrt{\sin^2 t + \sin t \cos t}\ \mathbf{i} + \left[(\sin t \cos t)/\sin^2 t\right]\mathbf{j} = \sqrt{\sin^2 t + \sin t \cos t}\ \mathbf{i} + \cot t\,\mathbf{j}$,

$\mathbf{r}'(t) = 2\sin t \cos t\,\mathbf{i} + (\cos^2 t - \sin^2 t)\,\mathbf{j}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{\pi/6}^{\pi/3} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_{\pi/6}^{\pi/3} \left[2\sin t\cos t\sqrt{\sin^2 t + \sin t\cos t} + (\cot t)(\cos^2 t - \sin^2 t)\right] dt \approx 0.5424\end{aligned}$$

26. $$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= (\cos t \tan t)e^{\sin t}\,\mathbf{i} + (\tan t \sin t)e^{\cos t}\,\mathbf{j} + (\sin t \cos t)e^{\tan t}\,\mathbf{k} \\ &= (\sin t)e^{\sin t}\,\mathbf{i} + (\tan t \sin t)e^{\cos t}\,\mathbf{j} + (\sin t \cos t)e^{\tan t}\,\mathbf{k},\end{aligned}$$

$\mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/4} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_0^{\pi/4} \left[(\sin t\cos t)e^{\sin t} - (\tan t \sin^2 t)e^{\cos t} + (\tan t)e^{\tan t}\right] dt \approx 0.8527\end{aligned}$$

27. $x = t^2,\ \ y = t^3,\ \ z = \sqrt{t}$ so by Formula 9,

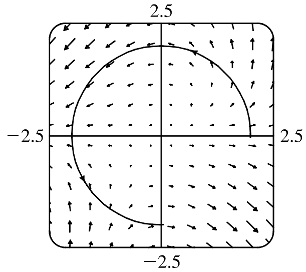
$$\begin{aligned}\int_C xy \arctan z\,ds &= \int_1^2 (t^2)(t^3)\arctan\sqrt{t}\cdot\sqrt{(2t)^2 + (3t^2)^2 + \left[1/(2\sqrt{t}\,)\right]^2}\,dt \\ &= \int_1^2 t^5\sqrt{4t^2 + 9t^4 + 1/(4t)}\ \arctan\sqrt{t}\,dt \approx 94.8231\end{aligned}$$

28. $x = 1 + 3t,\ \ y = 2 + t^2,\ \ z = t^4$ so by Formula 9,

$$\begin{aligned}\int_C z\ln(x+y)\,ds &= \int_{-1}^1 t^4 \ln(1 + 3t + 2 + t^2)\cdot\sqrt{(3)^2 + (2t)^2 + (4t^3)^2}\,dt \\ &= \int_{-1}^1 t^4\sqrt{9 + 4t^2 + 16t^6}\ \ln(3 + 3t + t^2)\,dt \approx 1.7260\end{aligned}$$

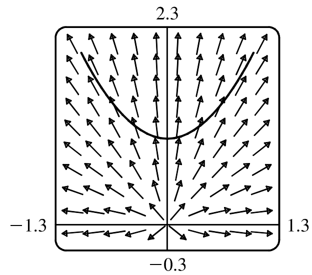
29. We graph $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ \ 0 \le t \le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t)\,\mathbf{i} + 4\cos t\sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Then



$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_0^{3\pi/2} [-2\sin t(2\cos t - 2\sin t) + 2\cos t(4\cos t\sin t)]\,dt \\ &= 4\int_0^{3\pi/2} (\sin^2 t - \sin t\cos t + 2\sin t\cos^2 t)\,dt \\ &= 3\pi + \tfrac{2}{3} \qquad \text{[using a CAS]}\end{aligned}$$

30. We graph $\mathbf{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\mathbf{F}\cdot\mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is zero. To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by



$\mathbf{r}(t) = t\,\mathbf{i} + (1+t^2)\,\mathbf{j},\ -1 \le t \le 1$, so $\mathbf{F}(\mathbf{r}(t)) = \dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{i} + \dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{-1}^{1}\left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}}\right)dt$$

$$= \int_{-1}^{1} \frac{t(3+2t^2)}{\sqrt{t^4+3t^2+1}}\,dt = 0 \qquad \text{[since the integrand is an odd function]}$$

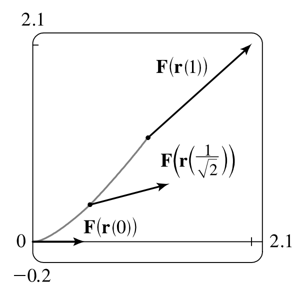
31. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, (t^2)(t^3)\right\rangle \cdot \langle 2t, 3t^2\rangle\,dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0},\ \ \mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0\rangle;$

$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle,\ \ \mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle;$

$\mathbf{r}(1) = \langle 1,1\rangle,\ \ \mathbf{F}(\mathbf{r}(1)) = \langle 1,1\rangle.$

In order to generate the graph with Maple, we use the `line` command in the `plottools` package to define each of the vectors. For example,

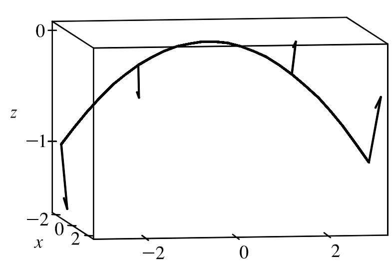


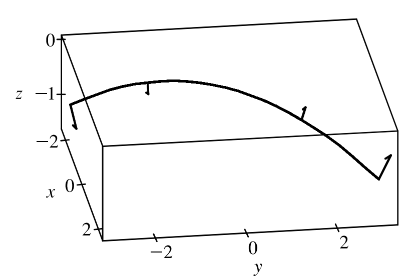

```
v1:=line([0,0],[exp(-1),0]):
```

generates the vector from the vector field at the point $(0,0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

32. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \langle 2t, t^2, 3t\rangle \cdot \langle 2, 3, -2t\rangle\,dt = \int_{-1}^{1} (4t + 3t^2 - 6t^2)\,dt = \left[2t^2 - t^3\right]_{-1}^{1} = -2$

(b) Now $\mathbf{F}(\mathbf{r}(t)) = \langle 2t, t^2, 3t\rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3\rangle$, $\mathbf{F}\left(\mathbf{r}\left(-\frac{1}{2}\right)\right) = \left\langle -1, \frac{1}{4}, -\frac{3}{2}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{2}\right)\right) = \left\langle 1, \frac{1}{4}, \frac{3}{2}\right\rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2, 1, 3\rangle$.





33. $x = e^{-t}\cos 4t,\quad y = e^{-t}\sin 4t,\quad z = e^{-t},\quad 0 \le t \le 2\pi$.

Then $\dfrac{dx}{dt} = e^{-t}(-\sin 4t)(4) - e^{-t}\cos 4t = -e^{-t}(4\sin 4t + \cos 4t)$,

$\dfrac{dy}{dt} = e^{-t}(\cos 4t)(4) - e^{-t}\sin 4t = -e^{-t}(-4\cos 4t + \sin 4t)$, and $\dfrac{dz}{dt} = -e^{-t}$, so

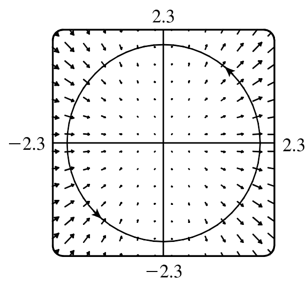
$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} &= \sqrt{(-e^{-t})^2[(4\sin 4t + \cos 4t)^2 + (-4\cos 4t + \sin 4t)^2 + 1]} \\ &= e^{-t}\sqrt{16(\sin^2 4t + \cos^2 4t) + \sin^2 4t + \cos^2 4t + 1} = 3\sqrt{2}\,e^{-t}\end{aligned}$$

So by Formula 9,
$$\begin{aligned}\int_C x^3y^2z\,ds &= \int_0^{2\pi}(e^{-t}\cos 4t)^3(e^{-t}\sin 4t)^2(e^{-t})\,(3\sqrt{2}\,e^{-t})\,dt \\ &= \int_0^{2\pi} 3\sqrt{2}\,e^{-7t}\cos^3 4t\sin^2 4t\,dt = \tfrac{172{,}704}{5{,}632{,}705}\sqrt{2}\,(1 - e^{-14\pi})\end{aligned}$$

34. (a) We parametrize the circle C as $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\quad 0 \le t \le 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \langle 4\cos^2 t, 4\cos t\sin t\rangle$,

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, and $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-8\cos^2 t\sin t + 8\cos^2 t\sin t)\,dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C, $\mathbf{F}\cdot\mathbf{T} = 0$, and so certainly $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$.

35. We use the parametrization $x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = 2\,dt$, so $m = \int_C k\,ds = 2k\int_{-\pi/2}^{\pi/2} dt = 2k(\pi)$,

$\overline{x} = \frac{1}{2\pi k}\int_C xk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt = \frac{1}{2\pi}\big[4\sin t\big]_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$, $\overline{y} = \frac{1}{2\pi k}\int_C yk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt = 0$.

Hence $(\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, 0\right)$.

36. We use the parametrization $x = a\cos t,\quad y = a\sin t,\quad 0 \le t \le \frac{\pi}{2}$. Then

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$, so

$m = \int_C \rho(x,y)\,ds = \int_C kxy\,ds = \int_0^{\pi/2} k(a\cos t)(a\sin t)\,a\,dt = ka^3\int_0^{\pi/2}\cos t\sin t\,dt = ka^3\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2} = \frac{1}{2}ka^3$,

$$\begin{aligned}\overline{x} &= \frac{1}{ka^3/2}\int_C x(kxy)\,ds = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)^2(a\sin t)a\,dt = \frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\cos^2 t\,\sin t\,dt \\ &= 2a\left[-\tfrac{1}{3}\cos^3 t\right]_0^{\pi/2} = 2a\left(0 + \tfrac{1}{3}\right) = \tfrac{2}{3}a, \text{ and}\end{aligned}$$

$$\begin{aligned}\overline{y} &= \frac{1}{ka^3/2}\int_C y(kxy)\,ds = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)(a\sin t)^2a\,dt = \frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\sin^2 t\,\cos t\,dt \\ &= 2a\left[\tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = 2a\left(\tfrac{1}{3} - 0\right) = \tfrac{2}{3}a.\end{aligned}$$

Therefore the mass is $\frac{1}{2}ka^3$ and the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{2}{3}a, \frac{2}{3}a\right)$.

37. (a) $\overline{x} = \dfrac{1}{m}\displaystyle\int_C x\rho(x,y,z)\,ds$, $\overline{y} = \dfrac{1}{m}\displaystyle\int_C y\rho(x,y,z)\,ds$, $\overline{z} = \dfrac{1}{m}\displaystyle\int_C z\rho(x,y,z)\,ds$, where $m = \int_C \rho(x,y,z)\,ds$.

(b) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2}\,dt = \sqrt{4(\cos^2 t + \sin^2 t) + 9}\,dt = \sqrt{13}\,dt$.

$$m = \int_C k\,ds = k\int_0^{2\pi}\sqrt{13}\,dt = k\sqrt{13}\int_0^{2\pi}dt = 2\pi k\sqrt{13},$$

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,ds = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} 2k\sqrt{13}\,\sin t\,dt = 0,$$

$$\overline{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,ds = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} 2k\sqrt{13}\,\cos t\,dt = 0,$$

$$\overline{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,ds = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}\left(k\sqrt{13}\right)(3t)\,dt = \frac{3}{2\pi}\left(2\pi^2\right) = 3\pi. \text{ Hence, } (\overline{x},\overline{y},\overline{z}) = (0,0,3\pi).$$

38. $m = \int_C (x^2+y^2+z^2)\,ds = \int_0^{2\pi}(t^2+\cos^2 t+\sin^2 t)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\,dt$

$= \int_0^{2\pi}(t^2+1)\sqrt{2}\,dt = \sqrt{2}\left[\frac{1}{3}t^3+t\right]_0^{2\pi} = \sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)$,

$$\overline{x} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3+t)\,dt = \frac{1}{\frac{8}{3}\pi^3+2\pi}\left[\tfrac{1}{4}t^4+\tfrac{1}{2}t^2\right]_0^{2\pi} = \frac{4\pi^4+2\pi^2}{\frac{8}{3}\pi^3+2\pi}\cdot\frac{3/(2\pi)}{3/(2\pi)}$$

$$= \frac{3\pi\left(2\pi^2+1\right)}{4\pi^2+3},$$

$$\overline{y} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\left(\sqrt{2}\cos t\right)(t^2+1)\,dt = 0, \text{ and}$$

$$\overline{z} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\left(\sqrt{2}\sin t\right)(t^2+1)\,dt = 0. \text{ Hence, } (\overline{x},\overline{y},\overline{z}) = \left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3},0,0\right).$$

39. From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \Rightarrow$

$I_x = \int_C y^2\rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi(\sin^2 t - \sin^3 t)\,dt$

$= \frac{1}{2}k\int_0^\pi(1-\cos 2t)\,dt - k\int_0^\pi(1-\cos^2 t)\sin t\,dt$ $\quad$ [Let $u = \cos t$, $du = -\sin t\,dt$ in the second integral]

$= k\left[\frac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\frac{\pi}{2}-\frac{4}{3}\right)$

$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^\pi \cos^2 t\,(1-\sin t)\,dt = \frac{k}{2}\int_0^\pi(1+\cos 2t)\,dt - k\int_0^\pi \cos^2 t\sin t\,dt$

$= k\left(\frac{\pi}{2}-\frac{2}{3}\right)$, using the same substitution as above.

40. The wire is given as $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$ with $\rho(x,y,z) = k$. Then

$ds = \sqrt{(2\cos t)^2+(-2\sin t)^2+3^2}\,dt = \sqrt{4(\cos^2 t+\sin^2 t)+9}\,dt = \sqrt{13}\,dt$ and

$I_x = \int_C(y^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\cos^2 t+9t^2)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$

$= \sqrt{13}\,k(4\pi+24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$

[continued]

$I_y = \int_C (x^2 + z^2)\rho(x,y,z)\,ds = \int_0^{2\pi} \left(4\sin^2 t + 9t^2\right)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$

$= \sqrt{13}\,k(4\pi + 24\pi^3) = 4\sqrt{13}\,\pi k(1 + 6\pi^2)$

$I_z = \int_C (x^2 + y^2)\rho(x,y,z)\,ds = \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)(k)\sqrt{13}\,dt = 4\sqrt{13}\,k\int_0^{2\pi} dt = 8\pi\sqrt{13}\,k$

41. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle t - \sin t, (1 - \cos t) + 2\rangle \cdot \langle 1 - \cos t, \sin t\rangle\,dt$

$= \int_0^{2\pi} (t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\,dt$

$= \int_0^{2\pi} (t - t\cos t + 2\sin t)\,dt = \left[\frac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi}$ $\quad\left[\begin{array}{l}\text{integrate by parts}\\ \text{in the second term}\end{array}\right]$

$= 2\pi^2$

42. Choosing y as the parameter, the curve C is parametrized by $x = y^2 + 1$, $y = y$, $0 \le y \le 1$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle \left(y^2+1\right)^2, ye^{y^2+1}\right\rangle \cdot \langle 2y, 1\rangle\,dy = \int_0^1 \left[2y\left(y^2+1\right)^2 + ye^{y^2+1}\right]dy$$
$$= \left[\tfrac{1}{3}\left(y^2+1\right)^3 + \tfrac{1}{2}e^{y^2+1}\right]_0^1 = \tfrac{8}{3} + \tfrac{1}{2}e^2 - \tfrac{1}{3} - \tfrac{1}{2}e = \tfrac{1}{2}e^2 - \tfrac{1}{2}e + \tfrac{7}{3}$$

43. $\mathbf{r}(t) = \langle 2t, t, 1 - t\rangle,\ \ 0 \le t \le 1.$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle 2t - t^2, t - (1-t)^2, 1 - t - (2t)^2\right\rangle \cdot \langle 2, 1, -1\rangle\,dt$$
$$= \int_0^1 (4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2)\,dt = \int_0^1 (t^2 + 8t - 2)\,dt = \left[\tfrac{1}{3}t^3 + 4t^2 - 2t\right]_0^1 = \tfrac{7}{3}$$

44. $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + 5t\,\mathbf{k},\ \ 0 \le t \le 1$. Therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \frac{K\langle 2, t, 5t\rangle}{(4 + 26t^2)^{3/2}} \cdot \langle 0, 1, 5\rangle\,dt = K\int_0^1 \frac{26t}{(4+26t^2)^{3/2}}\,dt = K\left[-(4+26t^2)^{-1/2}\right]_0^1 = K\left(\tfrac{1}{2} - \tfrac{1}{\sqrt{30}}\right).$$

45. (a) $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = 2at\,\mathbf{i} + 3bt^2\,\mathbf{j} \quad\Rightarrow\quad \mathbf{a}(t) = \mathbf{v}'(t) = 2a\,\mathbf{i} + 6bt\,\mathbf{j}$, and force is mass times acceleration: $\mathbf{F}(t) = m\,\mathbf{a}(t) = 2ma\,\mathbf{i} + 6mbt\,\mathbf{j}$, $0 \le t \le 1$.

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2ma\,\mathbf{i} + 6mbt\,\mathbf{j}) \cdot (2at\,\mathbf{i} + 3bt^2\,\mathbf{j})\,dt = \int_0^1 (4ma^2 t + 18mb^2 t^3)\,dt$

$= \left[2ma^2t^2 + \frac{9}{2}mb^2t^4\right]_0^1 = 2ma^2 + \frac{9}{2}mb^2$

46. $\mathbf{r}(t) = a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + ct\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k} \quad\Rightarrow\quad \mathbf{a}(t) = \mathbf{v}'(t) = -a\sin t\,\mathbf{i} - b\cos t\,\mathbf{j}$ and $\mathbf{F}(t) = m\,\mathbf{a}(t) = -ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}$. Thus

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}) \cdot (a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k})\,dt$$
$$= \int_0^{\pi/2} (-ma^2\sin t\cos t + mb^2\sin t\cos t)\,dt = m(b^2 - a^2)\left[\tfrac{1}{2}\sin^2 t\right]_0^{\pi/2} = \tfrac{1}{2}m(b^2 - a^2)$$

47. The combined weight of the man and the paint is 185 lb, so the force exerted (equal and opposite to that exerted by gravity) is $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t,\ \ y = 20\sin t,\ \ z = \frac{90}{6\pi}t = \frac{15}{\pi}t,\ \ 0 \le t \le 6\pi$. Then the work done is

$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185\rangle \cdot \left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = (185)\frac{15}{\pi}\int_0^{6\pi} dt = (185)\left(\frac{15}{\pi}\right)(6\pi) \approx 1.67 \times 10^4$ ft-lb

48. This time m is a function of t: $m = 185 - \frac{9}{6\pi}t = 185 - \frac{3}{2\pi}t$. So let $\mathbf{F} = \left(185 - \frac{3}{2\pi}t\right)\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t,\;\; y = 20\sin t,\;\; z = \frac{90}{6\pi}t = \frac{15}{\pi}t,\;\; 0 \le t \le 6\pi$. Therefore

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{6\pi} \left\langle 0, 0, 185 - \tfrac{3}{2\pi}t\right\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi}\right\rangle dt = \tfrac{15}{\pi}\int_0^{6\pi}\left(185 - \tfrac{3}{2\pi}t\right)dt \\ &= \tfrac{15}{\pi}\left[185t - \tfrac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185 - \tfrac{9}{2}\right) \approx 1.62\times 10^4 \text{ ft-lb} \end{aligned}$$

49. (a) $\mathbf{r}(t) = \langle \cos t, \sin t\rangle,\;\; 0 \le t \le 2\pi$, and let $\mathbf{F} = \langle a, b\rangle$. Then

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle a, b\rangle \cdot \langle -\sin t, \cos t\rangle\; dt = \int_0^{2\pi} (-a\sin t + b\cos t)\,dt = \big[a\cos t + b\sin t\big]_0^{2\pi} \\ &= a + 0 - a + 0 = 0 \end{aligned}$$

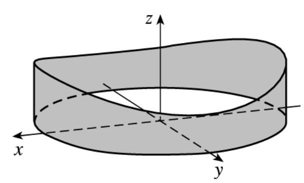
(b) Yes. $\mathbf{F}\,(x,y) = k\,\mathbf{x} = \langle kx, ky\rangle$ and

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle k\cos t, k\sin t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi}(-k\sin t\;\cos t + k\sin t\;\cos t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

50. Consider the base of the fence in the xy-plane, centered at the origin, with the height given by $z = h(x,y) = 4 + 0.01(x^2 - y^2)$. To graph the fence, observe that the fence is highest when $y = 0$ (where the height is 5 m) and lowest when $x = 0$ (a height of 3 m). When $y = \pm x$, the height is 4 m.

Also, the fence can be graphed using parametric equations (see Section 16.6): $x = 10\cos u,\;\; y = 10\sin u$,

$$\begin{aligned} z &= v\big[4 + 0.01((10\cos u)^2 - (10\sin u)^2)\big] = v(4 + \cos^2 u - \sin^2 u) \\ &= v(4 + \cos 2u),\;\; 0 \le u \le 2\pi,\;\; 0 \le v \le 1. \end{aligned}$$



The surface area of one side of the fence is $\int_C h(x,y)\,ds$, where the base C of the fence is given by $x = 10\cos t,\;\; y = 10\sin t,\;\; 0 \le t \le 2\pi$. Then

$$\begin{aligned} \int_C h(x,y)\,ds &= \int_0^{2\pi} \left[4 + 0.01((10\cos t)^2 - (10\sin t)^2)\right]\sqrt{(-10\sin t)^2 + (10\cos t)^2}\,dt \\ &= \int_0^{2\pi} (4 + \cos 2t)\sqrt{100}\,dt = 10\big[4t + \tfrac{1}{2}\sin 2t\big]_0^{2\pi} = 10(8\pi) = 80\pi \text{ m}^2 \end{aligned}$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2, and since 1 L of paint covers 100 m^2, we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03$ L of paint.

51. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned} \int_C \mathbf{v}\cdot d\mathbf{r} &= \int_a^b \langle v_1, v_2, v_3\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b [v_1\,x'(t) + v_2\,y'(t) + v_3\,z'(t)]\,dt \\ &= \big[v_1\,x(t) + v_2\,y(t) + v_3\,z(t)\big]_a^b = [v_1\,x(b) + v_2\,y(b) + v_3\,z(b)] - [v_1\,x(a) + v_2\,y(a) + v_3\,z(a)] \\ &= v_1\,[x(b) - x(a)] + v_2\,[y(b) - y(a)] + v_3\,[z(b) - z(a)] \\ &= \langle v_1, v_2, v_3\rangle \cdot \langle x(b) - x(a), y(b) - y(a), z(b) - z(a)\rangle \\ &= \langle v_1, v_2, v_3\rangle \cdot [\langle x(b), y(b), z(b)\rangle - \langle x(a), y(a), z(a)\rangle] = \mathbf{v}\cdot[\mathbf{r}(b) - \mathbf{r}(a)] \end{aligned}$$

52. If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ then

$$\begin{aligned}\int_C \mathbf{r} \cdot d\mathbf{r} &= \int_a^b \langle x(t), y(t), z(t) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt = \int_a^b [x(t)\, x'(t) + y(t)\, y'(t) + z(t)\, z'(t)] \, dt \\ &= \left[\tfrac{1}{2}[x(t)]^2 + \tfrac{1}{2}[y(t)]^2 + \tfrac{1}{2}[z(t)]^2\right]_a^b \\ &= \tfrac{1}{2}\left\{\left([x(b)]^2 + [y(b)]^2 + [z(b)]^2\right) - \left([x(a)]^2 + [y(a)]^2 + [z(a)]^2\right)\right\} \\ &= \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]\end{aligned}$$

53. The work done in moving the object is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F} \cdot \mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F} \cdot \mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds \approx \sum_{i=1}^{7} \left[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)\right] \Delta s = [2 + 2 + 2 + 2 + 1 + 1 + 1](2) = 22$$

Thus, we estimate the work done to be approximately 22 J.

54. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}| \, \mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta$, $y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}| \langle -\sin\theta, \cos\theta \rangle$. Then $\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}| \langle -\sin\theta, \cos\theta \rangle \cdot \langle -r\sin\theta, r\cos\theta \rangle \, d\theta = \int_0^{2\pi} |\mathbf{B}| \, r \, d\theta = 2\pi r \, |\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

16.3 The Fundamental Theorem for Line Integrals

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^3 + t)\,\mathbf{j}$, $0 \le t \le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2 + 1)\,\mathbf{j}$. Since $3t^2 + 1 \neq 0$, we have $\mathbf{r}'(t) \neq \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6$.

3. Let $P(x, y) = xy + y^2$ and $Q(x, y) = x^2 + 2xy$. Then $\partial P/\partial y = x + 2y$ and $\partial Q/\partial x = 2x + 2y$. Since $\partial P/\partial y \neq \partial Q/\partial x$, $\mathbf{F}(x, y) = P\,\mathbf{i} + Q\,\mathbf{j}$ is not conservative by Theorem 5.

4. $\partial(y^2 - 2x)/\partial y = 2y = \partial(2xy)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative by Theorem 6. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x, y) = y^2 - 2x$ and $f_y(x, y) = 2xy$. But $f_x(x, y) = y^2 - 2x$ implies $f(x, y) = xy^2 - x^2 + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y) = 2xy + g'(y)$. Thus $2xy = 2xy + g'(y)$ so $g'(y) = 0$ and $g(y) = K$ where K is a constant. Hence $f(x, y) = xy^2 - x^2 + K$ is a potential function for $\mathbf{F}$.

5. $\dfrac{\partial}{\partial y}\left(y^2 e^{xy}\right) = y^2 \cdot xe^{xy} + 2ye^{xy} = (xy^2 + 2y)e^{xy}$,

$\dfrac{\partial}{\partial x}\left[(1+xy)e^{xy}\right] = (1+xy) \cdot ye^{xy} + ye^{xy} = ye^{xy} + xy^2e^{xy} + ye^{xy} = (xy^2+2y)e^{xy}$.

Since these partial derivatives are equal and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, $\mathbf{F}$ is conservative by Theorem 6. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = y^2e^{xy}$ and $f_y(x,y) = (1+xy)e^{xy}$. But $f_x(x,y) = y^2e^{xy}$ implies $f(x,y) = ye^{xy} + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = (1+xy)e^{xy} + g'(y)$. Thus $(1+xy)e^{xy} = (1+xy)e^{xy} + g'(y)$ so $g'(y) = 0$ and $g(y) = K$ where K is a constant. Hence $f(x,y) = ye^{xy} + K$ is a potential function for $\mathbf{F}$.

6. $\partial(ye^x)/\partial y = e^x = \partial(e^x + e^y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative. Hence there exists a function f such that $\nabla f = \mathbf{F}$. Here $f_x(x,y) = ye^x$ implies $f(x,y) = ye^x + g(y)$ and then $f_y(x,y) = e^x + g'(y)$. But $f_y(x,y) = e^x + e^y$ so $g'(y) = e^y \quad\Rightarrow\quad g(y) = e^y + K$ and $f(x,y) = ye^x + e^y + K$ is a potential function for $\mathbf{F}$.

7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

8. $\partial(2xy + y^{-2})/\partial y = 2x - 2y^{-3} = \partial(x^2 - 2xy^{-3})/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid y > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 2xy + y^{-2}$ implies $f(x,y) = x^2y + xy^{-2} + g(y)$ and $f_y(x,y) = x^2 - 2xy^{-3} + g'(y)$. But $f_y(x,y) = x^2 - 2xy^{-3}$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. Then $f(x,y) = x^2y + xy^{-2} + K$ is a potential function for $\mathbf{F}$.

9. $\partial(y^2\cos x + \cos y)/\partial y = 2y\cos x - \sin y = \partial(2y\sin x - x\sin y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = y^2\cos x + \cos y$ implies $f(x,y) = y^2\sin x + x\cos y + g(y)$ and $f_y(x,y) = 2y\sin x - x\sin y + g'(y)$. But $f_y(x,y) = 2y\sin x - x\sin y$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$ and $f(x,y) = y^2\sin x + x\cos y + K$ is a potential function for $\mathbf{F}$.

10. $\partial(\ln y + y/x)/\partial y = 1/y + 1/x = \partial(\ln x + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid x > 0,\ y > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = \ln y + y/x$ implies $f(x,y) = x\ln y + y\ln x + g(y)$ and $f_y(x,y) = x/y + \ln x + g'(y)$. But $f_y(x,y) = \ln x + x/y$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$ and $f(x,y) = x\ln y + y\ln x + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}(2xy) = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path;

in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Integrating $f_x(x, y)$ with respect to x, we have $f(x, y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x, y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = K$, a constant.
Thus $f(x, y) = x^2y + K$, and we can take $K = 0$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(3, 2) - f(1, 2) = 3^2(2) - 1^2(2) = 16$ for each curve.

12. (a) $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 + \sin y)\,\mathbf{j}$.

Solution 1: $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial(2xy)}{\partial y} = 2x = \dfrac{\partial(x^2 + \sin y)}{\partial x}$ on $\mathbb{R}^2$, which is open and simply-connected. Therefore, the vector field is conservative and there exists a function $f(x, y)$ such that $\nabla f = \mathbf{F}$. Here, $f_x(x, y) = 2xy$ implies $f(x, y) = x^2y + g(y)$ and $f_y(x, y) = x^2 + g'(y)$, but $f_y(x, y) = x^2 + \sin y$, which implies $g'(y) = \sin y \quad \Rightarrow \quad g(y) = -\cos y + K$ and $f(x, y) = x^2y - \cos y + K$ is a potential function for $\mathbf{F}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f\left(2, \tfrac{\pi}{2}\right) - f(0, 0) = 2^2\left(\tfrac{\pi}{2}\right) - \cos\left(\tfrac{\pi}{2}\right) + K - 0 + \cos 0 - K = 2\pi + 1$$

Solution 2: As in Example 4, since $\mathbf{F}$ is conservative, we know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, so we replace the curve C by the simpler curve C_1 consisting of the line segment connecting the two endpoints of C. Thus, C_1 can be represented by $\mathbf{r}(t) = 2t\,\mathbf{i} + \frac{\pi}{2}t\,\mathbf{j}$, $0 \le t \le 1$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^1 \left\{2(2t)\left(\tfrac{\pi}{2}t\right)(2) + \left[(2t)^2 + \sin\left(\tfrac{\pi}{2}t\right)\right]\left(\tfrac{\pi}{2}\right)\right\} dt = \left[\int_0^1 6\pi t^2 + \tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}t\right)\right] dt \\
&= \left[2\pi t^3 - \cos\left(\tfrac{\pi}{2}t\right)\right]_0^1 = (2\pi - 0) - (0 - 1) = 2\pi + 1
\end{aligned}$$

(b) From part (a), we know that $\mathbf{F}$ is conservative and therefore independent of path. Thus, since C is a closed path, by Theorem 3, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

13. (a) $\mathbf{F}(x, y) = (3x^2 + y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $\pi \le t \le 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = [3(2\cos t)^2 + (2\sin t)^2]\,\mathbf{i} + 2(2\cos t)(2\sin t)\,\mathbf{j} = (12\cos^2 t + 4\sin^2 t)\,\mathbf{i} + 8\cos t \sin t\,\mathbf{j}$$

and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_\pi^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_\pi^{2\pi} \left[-2\sin t(12\cos^2 t + 4\sin^2 t) + 2\cos t(8\cos t\sin t)\right]\,dt \\
&= \int_\pi^{2\pi} (-8\sin t\cos^2 t - 8\sin^3 t)\,dt = -8\int_\pi^{2\pi} \sin t\,dt = 8\Big[\cos t\Big]_\pi^{2\pi} = 8[1 - (-1)] = 16
\end{aligned}$$

(b) $\dfrac{\partial(3x^2 + y^2)}{\partial y} = 2y = \dfrac{\partial(2xy)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative and there exists a function $f(x, y)$ such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 3x^2 + y^2$ implies $f(x, y) = x^3 + xy^2 + g(y)$.

[continued]

Differentiating both sides with respect to y gives $f_y(x,y) = 2xy + g'(y)$, so we must have $2xy + g'(y) = 2xy \Rightarrow$ $g'(y) = 0 \Rightarrow g(y) = K$, a constant. Thus, $f(x,y) = x^3 + xy^2 + K$ is a potential function for $\mathbf{F}$.

(c) From part (b), $f(x,y) = x^3 + xy^2 + K$. Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(2,0) - f(-2,0) = 2^3 + 0 + K - (-2)^3 - 0 - K = 16$$

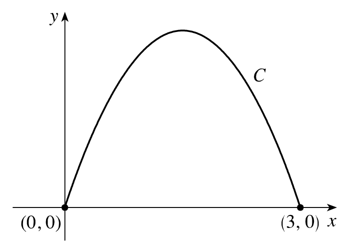
(d) We replace C with the line segment from $(-2,0)$ to $(2,0)$ so $\mathbf{r}(t) = \langle t, 0\rangle$, $-2 \le t \le 2$, which implies $\mathbf{r}'(t) = \langle 1, 0\rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \langle 3t^2, 0\rangle$. Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_{-2}^{2} 3t^2\,dt = \left[t^3\right]_{-2}^{2} = 2^3 - (-2)^3 = 16.$

14. (a) $\mathbf{F}(x,y) = \langle \sin y + e^x, x\cos y\rangle$ and C: $x = t$, $y = t(3-t)$, $0 \le t \le 3$.

$\dfrac{\partial(\sin y + e^x)}{\partial y} = \cos y = \dfrac{\partial(x\cos y)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative and there exists a function $f(x,y)$ such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = \sin y + e^x$ implies $f(x,y) = x\sin y + e^x + g(y)$, so $f_y(x,y) = x\cos y + g'(y)$, which implies $g'(y) = 0 \Rightarrow g(y) = K$. Therefore, $f(x,y) = x\sin y + e^x + K$ is a potential function for $\mathbf{F}$.

(b) The endpoints of C are $(0,0)$ and $(3,0)$. Thus, by Theorem 2,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= f(3,0) - f(0,0) \\ &= 3\cdot\sin 0 + e^3 + K - (0\cdot\sin 0 + e^0 + K) \\ &= e^3 - 1\end{aligned}$$



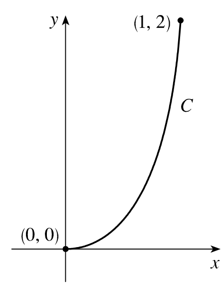
(c) We replace C with the line segment that connects the endpoints of C along the x-axis. So $x = t$, $y = 0$, $0 \le t \le 3$. Then $\mathbf{r}(t) = \langle t, 0\rangle$, $\mathbf{r}'(t) = \langle 1, 0\rangle$, and $\mathbf{F}(\mathbf{r}(t)) = \langle \sin 0 + e^t, t\cos 0\rangle = \langle e^t, t\rangle$. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^3 e^t\,dt = \left[e^t\right]_0^3 = e^3 - 1.$

15. (a) $\mathbf{F}(x,y) = \langle ye^{xy}, xe^{xy}\rangle$ and C: $x = \sin\frac{\pi}{2}t$, $y = e^{t-1}(1 - \cos\pi t)$, $0 \le t \le 1$.

$\dfrac{\partial(ye^{xy})}{\partial y} = e^{xy} + xye^{xy} = \dfrac{\partial(xe^{xy})}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is open and simply connected. Thus, $\mathbf{F}$ is conservative and there exists a function $f(x,y)$ such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^{xy}$ implies $f(x,y) = e^{xy} + g(y)$, so $f_y(x,y) = xe^{xy} + g'(y)$, which implies $g'(y) = 0 \Rightarrow g(y) = K$. Therefore, $f(x,y) = e^{xy} + K$ is a potential function for $\mathbf{F}$.

(b) The endpoints of C are $(0,0)$ and $(1,2)$. Thus, by Theorem 2,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= f(1,2) - f(0,0) \\ &= e^2 + K - e^0 - K = e^2 - 1\end{aligned}$$



(c) We replace C with the line segment that connects the endpoints of C. So $x = t$ and $y = 2t$, $0 \le t \le 1$. Then $\mathbf{r}(t) = \langle t, 2t\rangle$, $\mathbf{r}'(t) = \langle 1, 2\rangle$, and

$\mathbf{F}(\mathbf{r}(t)) = \left\langle 2te^{2t^2}, te^{2t^2} \right\rangle$. Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 4te^{2t^2}\, dt \\ &= \int_0^2 e^u\, du \qquad [u = 2t^2,\, du = 4t\, dt] \\ &= \left[e^u\right]_0^2 = e^2 - 1 \end{aligned}$$

16. $f(x, y, z) = xy^2z + x^2$ and C: $x = t^2$, $y = e^{t^2-1}$, $z = t^2 + t$, $-1 \le t \le 1$. ∇f is the gradient vector field of f and therefore conservative. Thus, by Theorem 2, $\int_C \nabla f \cdot d\mathbf{r} = f(1, 1, 2) - f(1, 1, 0) = [(1)(1^2)(2) + 1^2] - (0 + 1^2) = 2$.

17. (a) $\mathbf{F}(x, y) = \langle 2x, 4y \rangle$. If $\mathbf{F} = \nabla f$, then $f_x(x, y) = 2x$ and $f_y(x, y) = 4y$. $f_x(x, y) = 2x$ implies that $f(x, y) = x^2 + g(y)$ and $f_y(x, y) = g'(y) = 4y$, so $g(y) = 2y^2 + K$. We can take $K = 0$, so $f(x, y) = x^2 + 2y^2$.

(b) C is a smooth curve with initial point $(4, -2)$ and terminal point $(1, 1)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 1) - f(4, -2) = (1 + 2) - (16 + 8) = -21$.

18. (a) $\mathbf{F}(x, y) = (3 + 2xy^2)\, \mathbf{i} + 2x^2y\, \mathbf{j}$. If $\mathbf{F} = \nabla f$, then $f_x(x, y) = 3 + 2xy^2$ and $f_y(x, y) = 2x^2y$.

$f_x(x, y) = 3 + 2xy^2$ implies $f(x, y) = 3x + x^2y^2 + g(y)$ and $f_y(x, y) = 2x^2y + g'(y)$. But $f_y(x, y) = 2x^2y$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. We can take $K = 0$, so $f(x, y) = 3x + x^2y^2$.

(b) C is a smooth curve with initial point $(1, 1)$ and terminal point $\left(4, \frac{1}{4}\right)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\left(4, \frac{1}{4}\right) - f(1, 1) = (12 + 1) - (3 + 1) = 9$.

19. (a) $\mathbf{F}(x, y) = x^2y^3\, \mathbf{i} + x^3y^2\, \mathbf{j}$. If $\mathbf{F} = \nabla f$, then $f_x(x, y) = x^2y^3$ and $f_y(x, y) = x^3y^2$.

$f_x(x, y) = x^2y^3$ implies $f(x, y) = \frac{1}{3}x^3y^3 + g(y)$ and $f_y(x, y) = x^3y^2 + g'(y)$. But $f_y(x, y) = x^3y^2$, so $g'(y) = 0 \quad \Rightarrow$ $g(y) = K$, a constant. We can take $K = 0$, so $f(x, y) = \frac{1}{3}x^3y^3$.

(b) C is a smooth curve with initial point $\mathbf{r}(0) = (0, 0)$ and terminal point $\mathbf{r}(1) = (-1, 3)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(-1, 3) - f(0, 0) = -9 - 0 = -9$.

20. (a) $\mathbf{F}(x, y) = (1 + xy)e^{xy}\, \mathbf{i} + x^2e^{xy}\, \mathbf{j}$. $f_y(x, y) = x^2e^{xy}$ implies $f(x, y) = xe^{xy} + g(x) \quad \Rightarrow$ $f_x(x, y) = xye^{xy} + e^{xy} + g'(x) = (1 + xy)e^{xy} + g'(x)$. But $f_x(x, y) = (1 + xy)e^{xy}$ so $g'(x) = 0 \quad \Rightarrow \quad g(x) = K$. We can take $K = 0$, so $f(x, y) = xe^{xy}$.

(b) The initial point of C is $\mathbf{r}(0) = (1, 0)$ and the terminal point is $\mathbf{r}(\pi/2) = (0, 2)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2) - f(1, 0) = 0 - e^0 = -1$.

21. (a) $\mathbf{F}(x, y, z) = 2xy\, \mathbf{i} + (x^2 + 2yz)\, \mathbf{j} + y^2\, \mathbf{k}$. $f_x(x, y, z) = 2xy$ implies that $f(x, y, z) = x^2y + g(y, z)$ and so $f_y(x, y, z) = x^2 + g_y(y, z)$. But $f_y(x, y, z) = x^2 + 2yz$, which implies $g_y(y, z) = 2yz + h_y(z) \quad \Rightarrow$ $g(y, z) = y^2z + h(z)$. So $f(x, y, z) = x^2y + y^2z + h(z)$ and $f_z(x, y, z) = y^2 + h'(z)$. But $f_z(x, y, z) = y^2 \quad \Rightarrow$ $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. We can take $K = 0$, so $f(x, y, z) = x^2y + y^2z$.

(b) C is a smooth curve with initial point $(2, -3, 1)$ and terminal point $(-5, 1, 2)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(-5, 1, 2) - f(2, -3, 1) = (25 + 2) - (-12 + 9) = 30.$

22. (a) $\mathbf{F}(x, y, z) = (y^2 z + 2xz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (xy^2 + 2x^2 z)\,\mathbf{k}$. $f_x(x, y, z) = y^2 z + 2xz^2$ implies $f(x, y, z) = xy^2 z + x^2 z^2 + g(y, z)$ and so $f_y(x, y, z) = 2xyz + g_y(y, z)$. But $f_y(x, y, z) = 2xyz$ so $g_y(y, z) = 0 \quad \Rightarrow \quad g(y, z) = h(z)$. Thus, $f(x, y, z) = xy^2 z + x^2 z^2 + h(z)$ and $f_z(x, y, z) = xy^2 + 2x^2 z + h'(z)$. But $f_z(x, y, z) = xy^2 + 2x^2 z$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence, $f(x, y, z) = xy^2 z + x^2 z^2$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0, 1, 0)$ and $t = 1$ corresponds to $(1, 2, 1)$, so by Theorem 2,

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, 0) = 5 - 0 = 5.$

23. (a) $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$. $f_x(x, y, z) = yze^{xz}$ implies $f(x, y, z) = ye^{xz} + g(y, z)$ and so $f_y(x, y, z) = e^{xz} + g_y(y, z)$. But $f_y(x, y, z) = e^{xz}$ so $g_y(y, z) = 0 \quad \Rightarrow \quad g(y, z) = h(z)$. Thus, $f(x, y, z) = ye^{xz} + h(z)$ and $f_z(x, y, z) = xye^{xz} + h'(z)$. But $f_z(x, y, z) = xye^{xz}$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence $f(x, y, z) = ye^{xz}$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 1, -1, 0 \rangle$, $\mathbf{r}(2) = \langle 5, 3, 0 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(5, 3, 0) - f(1, -1, 0) = 3e^0 + e^0 = 4.$

24. (a) $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$. $f_x(x, y, z) = \sin y$ implies $f(x, y, z) = x\sin y + g(y, z)$ and so $f_y(x, y, z) = x\cos y + g_y(y, z)$. But $f_y(x, y, z) = x\cos y + \cos z$ so $g_y(y, z) = \cos z \quad \Rightarrow \quad g(y, z) = y\cos z + h(z)$. Thus, $f(x, y, z) = x\sin y + y\cos z + h(z)$ and $f_z(x, y, z) = -y\sin z + h'(z)$. But $f_z(x, y, z) = -y\sin z$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence, $f(x, y, z) = x\sin y + y\cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(\pi/2) = \langle 1, \pi/2, \pi \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, \pi/2, \pi) - f(0, 0, 0) = 1 - \frac{\pi}{2} - 0 = 1 - \frac{\pi}{2}.$

25. The functions $2xe^{-y}$ and $2y - x^2 e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and

$\dfrac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y} = \dfrac{\partial}{\partial x}\left(2y - x^2 e^{-y}\right)$, so $\mathbf{F}(x, y) = 2xe^{-y}\,\mathbf{i} + \left(2y - x^2 e^{-y}\right)\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = 2xe^{-y}$ implies $f(x, y) = x^2 e^{-y} + g(y)$ and $f_y(x, y) = -x^2 e^{-y} + g'(y)$. But $f_y(x, y) = 2y - x^2 e^{-y}$ so $g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + K$. We can take $K = 0$, so $f(x, y) = x^2 e^{-y} + y^2$. Then

$\int_C 2xe^{-y}\,dx + (2y - x^2 e^{-y})\,dy = f(2, 1) - f(1, 0) = 4e^{-1} + 1 - 1 = 4/e.$

26. The functions $\sin y$ and $x\cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and

$\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x\cos y - \sin y)$, so $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (x\cos y - \sin y)\,\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = \sin y$ implies $f(x, y) = x\sin y + g(y)$ and $f_y(x, y) = x\cos y + g'(y)$. But $f_y(x, y) = x\cos y - \sin y$, so $g'(y) = -\sin y \quad \Rightarrow$ $g(y) = \cos y + K$. We can take $K = 0$, so $f(x, y) = x\sin y + \cos y$. Then

$\int_C \sin y\,dx + (x\cos y - \sin y)\,dy = f(1, \pi) - f(2, 0) = -1 - 1 = -2.$

27. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. This means that the work done along all piecewise-smooth curves that have the described initial and terminal points is the same. Your reply: It doesn't matter which curve is chosen.

28. The curves C_1 and C_2 connect the same two points but $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus $\mathbf{F}$ is not independent of path, and therefore is not conservative.

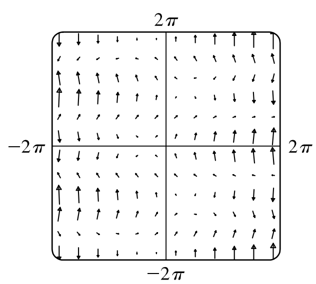
29. $\mathbf{F}(x,y) = x^3\,\mathbf{i} + y^3\,\mathbf{j},\ W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(x^3)/\partial y = 0 = \partial(y^3)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x,y) = x^3 \quad\Rightarrow\quad f(x,y) = \frac{1}{4}x^4 + g(y) \quad\Rightarrow\quad f_y(x,y) = 0 + g'(y)$. But $f_y(x,y) = y^3$ so $g'(y) = y^3 \quad\Rightarrow\quad g(y) = \frac{1}{4}y^4 + K$. We can take $K = 0$, so $f(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,2) - f(1,0) = (4+4) - \left(\frac{1}{4} + 0\right) = \frac{31}{4}$.

30. $\mathbf{F}(x,y) = (2x+y)\,\mathbf{i} + x\,\mathbf{j},\ W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(2x+y)/\partial y = 1 = \partial(x)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x,y) = 2x + y \quad\Rightarrow\quad f(x,y) = x^2 + xy + g(y) \quad\Rightarrow\quad f_y(x,y) = x + g'(y)$. But $f_y(x,y) = x$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. We can take $K = 0$, so $f(x,y) = x^2 + xy$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(4,3) - f(1,1) = (16+12) - (1+1) = 26$.

31. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on C are roughly in the direction of motion along C, so the integral around C will be positive. Therefore the field is not conservative.

32. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. For any closed path we draw in the field, it appears that some vectors on the curve point in approximately the same direction as the curve and a similar number point in roughly the opposite direction. (Some appear perpendicular to the curve as well.) Therefore it is plausible that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C which means $\mathbf{F}$ is conservative.

33.



From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(1 + x\cos y)$. Thus, $\mathbf{F}$ is conservative, by Theorem 6.

34. $f(x,y) = \sin(x - 2y) \quad\Rightarrow\quad \mathbf{F} = \nabla f(x,y) = \cos(x-2y)\,\mathbf{i} - 2\cos(x-2y)\,\mathbf{j}$

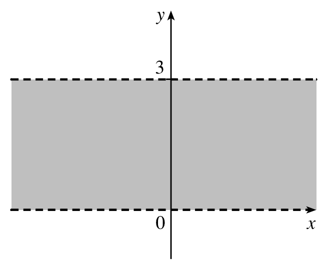
(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j},\ 0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

35. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q, and R have continuous first-order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

36. Here $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 35, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

37. $D = \{(x, y) \mid 0 < y < 3\}$ consists of those points between, but not on, the horizontal lines $y = 0$ and $y = 3$.

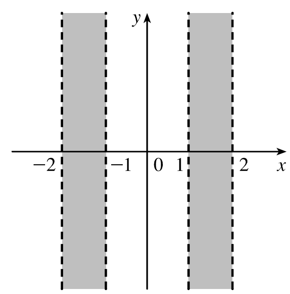


(a) Since D does not include any of its boundary points, it is open. More formally, at any point in D there is a disk centered at that point that lies entirely in D.

(b) Any two points chosen in D can always be joined by a path that lies entirely in D, so D is connected. (D consists of just one "piece.")

(c) D is connected and it has no holes, so it's simply-connected. (Every simple closed curve in D encloses only points that are in D.)

38. $D = \{(x, y) \mid 1 < |x| < 2\}$ consists of those points between, but not on, the vertical lines $x = 1$ and $x = 2$, together with the points between the vertical lines $x = -1$ and $x = -2$.

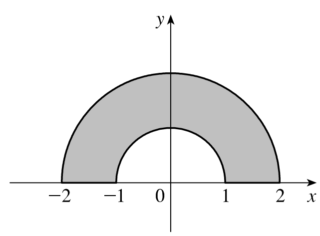


(a) The region does not include any of its boundary points, so it is open.

(b) D consists of two separate pieces, so it is not connected. [For instance, both the points $(-1.5, 0)$ and $(1.5, 0)$ lie in D but they cannot be joined by a path that lies entirely in D.]

(c) Because D is not connected, it's not simply-connected.

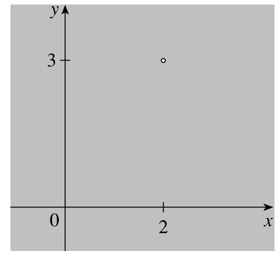
39. $D = \left\{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ y \ge 0\right\}$ is the semiannular region in the upper half-plane between circles centered at the origin of radii 1 and 2 (including all boundary points).



(a) D includes boundary points, so it is not open. [Note that at any boundary point, $(1, 0)$ for instance, any disk centered there cannot lie entirely in D.]

(b) The region consists of one piece, so it's connected.

(c) D is connected and has no holes, so it's simply-connected.



40. $D = \{(x,y) \mid (x,y) \neq (2,3)\}$ consists of all points in the xy-plane except for $(2,3)$.

(a) D has only one boundary point, namely $(2,3)$, which is not included, so the region is open.

(b) D is connected, as it consists of only one piece.

(c) D is not simply-connected, as it has a hole at $(2,3)$. Thus any simple closed curve that encloses $(2,3)$ lies in D but includes a point that is not in D.

41. (a) $P = -\dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ and $Q = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\,dt = \int_0^{\pi} dt = \pi$ and $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi$ Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

42. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 16.1.) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\dfrac{c}{(x_2^2+y_2^2+z_2^2)^{1/2}} + \dfrac{c}{(x_1^2+y_1^2+z_1^2)^{1/2}} = c\left(\dfrac{1}{d_1} - \dfrac{1}{d_2}\right)$.

(b) In this case, $c = -(mMG) \quad \Rightarrow$

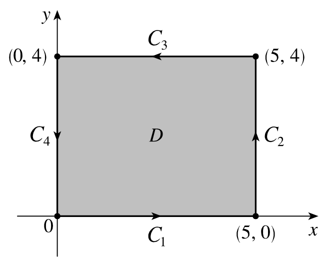
$$\begin{aligned} W &= -mMG\left(\frac{1}{1.52 \times 10^{11}} - \frac{1}{1.47 \times 10^{11}}\right) \\ &= -(5.97 \times 10^{24})(1.99 \times 10^{30})(6.67 \times 10^{-11})(-2.2377 \times 10^{-13}) \\ &\approx 1.77 \times 10^{32}\ \text{J} \end{aligned}$$

(c) In this case, $c = \epsilon q Q \quad \Rightarrow$

$W = \epsilon q Q\left(\dfrac{1}{10^{-12}} - \dfrac{1}{5 \times 10^{-13}}\right) = (8.985 \times 10^9)(1)(-1.6 \times 10^{-19})(-10^{12}) \approx 1400$ J.

16.4 Green's Theorem

1. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 0 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 5.$

C_2: $x = 5 \;\Rightarrow\; dx = 0\,dt,\; y = t \;\Rightarrow\; dy = dt,\; 0 \le t \le 4.$

C_3: $x = 5 - t \;\Rightarrow\; dx = -dt,\; y = 4 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 5.$

C_4: $x = 0 \;\Rightarrow\; dx = 0\,dt,\; y = 4 - t \;\Rightarrow\; dy = -dt,\; 0 \le t \le 4$

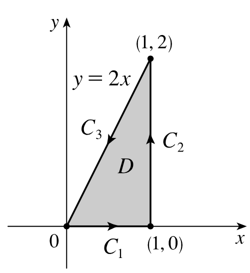
Thus
$$\oint_C y^2\,dx + x^2y\,dy = \oint_{C_1+C_2+C_3+C_4} y^2\,dx + x^2y\,dy = \int_0^5 0\,dt + \int_0^4 25t\,dt + \int_0^5 (-16+0)\,dt + \int_0^4 0\,dt$$
$$= 0 + \left[\tfrac{25}{2}t^2\right]_0^4 + [-16t]_0^5 + 0 = 200 + (-80) = 120$$

(b) Note that C as given in part (a) is a positively oriented, piecewise-smooth, simple closed curve. Then by Green's Theorem,
$$\oint_C y^2\,dx + x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(y^2)\right] dA = \int_0^5\int_0^4 (2xy - 2y)\,dy\,dx = \int_0^5 \left[xy^2 - y^2\right]_{y=0}^{y=4} dx$$
$$= \int_0^5 (16x - 16)\,dx = \left[8x^2 - 16x\right]_0^5 = 200 - 80 = 120$$

2. (a) Parametric equations for C are $x = 4\cos t,\; y = 4\sin t,\; 0 \le t \le 2\pi$. Then $dx = -4\sin t\,dt,\; dy = 4\cos t\,dt$ and
$$\oint_C y\,dx - x\,dy = \int_0^{2\pi} [(4\sin t)(-4\sin t) - (4\cos t)(4\cos t)]\,dt$$
$$= -16\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = -16\int_0^{2\pi} 1\,dt = -16(2\pi) = -32\pi$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,
$$\oint_C y\,dx - x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] dA = \iint_D (-1-1)\,dA = -2\iint_D dA$$
$$= -2(\text{area of } D) = -2\cdot\pi(4)^2 = -32\pi$$

3. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 0 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 1.$

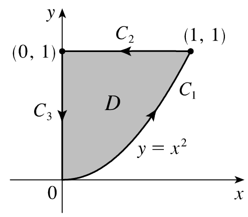
C_2: $x = 1 \;\Rightarrow\; dx = 0\,dt,\; y = t \;\Rightarrow\; dy = dt,\; 0 \le t \le 2.$

C_3: $x = 1 - t \;\Rightarrow\; dx = -dt,\; y = 2 - 2t \;\Rightarrow\; dy = -2\,dt,\; 0 \le t \le 1.$

Thus
$$\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$$
$$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$
$$= 0 + \left[\tfrac14 t^4\right]_0^2 + \int_0^1 \left[-2(1-t)^2 - 16(1-t)^5\right] dt$$
$$= 4 + \left[\tfrac23(1-t)^3 + \tfrac83(1-t)^6\right]_0^1 = 4 + 0 - \tfrac{10}{3} = \tfrac23$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1\int_0^{2x} (2xy^3 - x)\,dy\,dx$
$$= \int_0^1 \left[\tfrac12 xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac43 - \tfrac23 = \tfrac23$$

4. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\ y = t^2 \;\Rightarrow\; dy = 2t\,dt,\ \ 0 \le t \le 1$

C_2: $x = 1 - t \;\Rightarrow\; dx = -dt,\ y = 1 \;\Rightarrow\; dy = 0\,dt,\ \ 0 \le t \le 1$

C_3: $x = 0 \;\Rightarrow\; dx = 0\,dt,\ y = 1 - t \;\Rightarrow\; dy = -dt,\ \ 0 \le t \le 1$

Thus

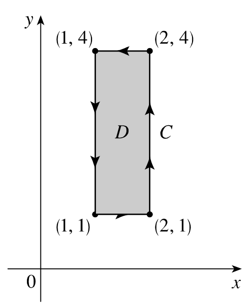
$$\begin{aligned}
\oint_C x^2y^2\,dx + xy\,dy &= \oint_{C_1+C_2+C_3} x^2y^2\,dx + xy\,dy \\
&= \int_0^1 \left[t^2(t^2)^2\,dt + t(t^2)(2t\,dt)\right] + \int_0^1 \left[(1-t)^2(1)^2(-dt) + (1-t)(1)(0\,dt)\right] \\
&\quad + \int_0^1 \left[(0)^2(1-t)^2(0\,dt) + (0)(1-t)(-dt)\right] \\
&= \int_0^1 \left(t^6 + 2t^4\right)dt + \int_0^1 \left(-1 + 2t - t^2\right)dt + \int_0^1 0\,dt \\
&= \left[\tfrac{1}{7}t^7 + \tfrac{2}{5}t^5\right]_0^1 + \left[-t + t^2 - \tfrac{1}{3}t^3\right]_0^1 + 0 = \left(\tfrac{1}{7} + \tfrac{2}{5}\right) + \left(-1 + 1 - \tfrac{1}{3}\right) = \tfrac{22}{105}
\end{aligned}$$

(b)
$$\begin{aligned}
\oint_C x^2y^2\,dx + xy\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(xy) - \tfrac{\partial}{\partial y}(x^2y^2)\right]dA = \int_0^1 \int_{x^2}^1 (y - 2x^2y)\,dy\,dx \\
&= \int_0^1 \left[\tfrac{1}{2}y^2 - x^2y^2\right]_{y=x^2}^{y=1} dx = \int_0^1 \left(\tfrac{1}{2} - x^2 - \tfrac{1}{2}x^4 + x^6\right)dx \\
&= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 - \tfrac{1}{10}x^5 + \tfrac{1}{7}x^7\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{10} + \tfrac{1}{7} = \tfrac{22}{105}
\end{aligned}$$

5. The region D enclosed by C is $[0,3] \times [0,4]$, so

$$\begin{aligned}
\int_C ye^x\,dx + 2e^x\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(2e^x) - \tfrac{\partial}{\partial y}(ye^x)\right]dA = \int_0^3 \int_0^4 (2e^x - e^x)\,dy\,dx \\
&= \int_0^3 e^x\,dx \int_0^4 dy = \left[e^x\right]_0^3 \left[y\right]_0^4 = (e^3 - e^0)(4 - 0) = 4(e^3 - 1)
\end{aligned}$$

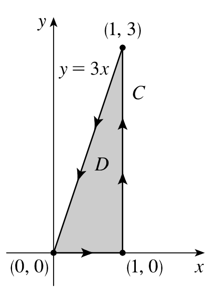
6.



The region D enclosed by C can be given by $\{(x,y) \mid 1 \le x \le 2, 1 \le y \le 4\}$, so

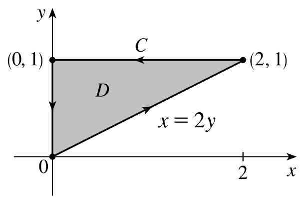
$$\begin{aligned}
\int_C \ln(xy)\,dx + \frac{y}{x}\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{y}{x}\right) - \frac{\partial}{\partial y}\ln(xy)\right]dA \\
&= \int_1^4 \int_1^2 \left(-\frac{y}{x^2} - \frac{1}{y}\right)dx\,dy = \int_1^4 \left[\frac{y}{x} - \frac{x}{y}\right]_{x=1}^{x=2} dy \\
&= \int_1^4 \left(\frac{y}{2} - \frac{2}{y} - y + \frac{1}{y}\right)dy = \int_1^4 \left(-\frac{y}{2} - \frac{1}{y}\right)dy \\
&= \left[-\frac{y^2}{4} - \ln y\right]_1^4 = -\frac{16}{4} - \ln 4 + \frac{1}{4} + 0 = -\frac{15}{4} - \ln 4
\end{aligned}$$

7.



The region D enclosed by C can be given by $\{(x,y) | 0 \le y \le 3x, 0 \le x \le 1\}$, so

$$\begin{aligned}
\int_C x^2y^2\,dx + y\tan^{-1}y\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(y\tan^{-1}y\right) - \frac{\partial}{\partial y}\left(x^2y^2\right)\right]dA \\
&= \int_0^1 \int_0^{3x} (-2x^2y)\,dy\,dx = -\int_0^1 \left[x^2y^2\right]_{y=0}^{y=3x} dx \\
&= -9\int_0^1 x^4\,dx = -\tfrac{9}{5}\left[x^5\right]_0^1 = -\tfrac{9}{5}
\end{aligned}$$

8. 

The region D enclosed by C is given by $\{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$, so

$$\begin{aligned}\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(x^2-y^2) - \tfrac{\partial}{\partial y}(x^2+y^2)\right] dA \\ &= \int_0^1 \int_0^{2y} (2x-2y)\,dx\,dy \\ &= \int_0^1 \left[x^2-2xy\right]_{x=0}^{x=2y} dy \\ &= \int_0^1 (4y^2-4y^2)\,dy = \int_0^1 0\,dy = 0\end{aligned}$$

9. $$\begin{aligned}\int_C \left(y+e^{\sqrt{x}}\right) dx + (2x+\cos y^2)\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(2x+\cos y^2) - \tfrac{\partial}{\partial y}\left(y+e^{\sqrt{x}}\right)\right] dA \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)\,dy\,dx = \int_0^1 (\sqrt{x}-x^2)\,dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\end{aligned}$$

10. $$\begin{aligned}\int_C y^4\,dx + 2xy^3\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(2xy^3) - \tfrac{\partial}{\partial y}(y^4)\right] dA = \iint_D (2y^3-4y^3)\,dA \\ &= -2\iint_D y^3\,dA = 0\end{aligned}$$

because $f(x,y) = y^3$ is an odd function with respect to y and D is symmetric about the x-axis.

11. $$\begin{aligned}\int_C y^3\,dx - x^3\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(-x^3) - \tfrac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2-3y^2)\,dA = \int_0^{2\pi}\int_0^2 (-3r^2)\,r\,dr\,d\theta \\ &= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3\big[\theta\big]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^2 = -3(2\pi)(4) = -24\pi\end{aligned}$$

12. $$\begin{aligned}\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(x^3+e^{y^2}) - \tfrac{\partial}{\partial y}(1-y^3)\right] dA = \iint_D (3x^2+3y^2)\,dA \\ &= \int_0^{2\pi}\int_2^3 (3r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_2^3 r^3\,dr \\ &= 3\big[\theta\big]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_2^3 = 3(2\pi)\cdot\tfrac{1}{4}(81-16) = \tfrac{195}{2}\pi\end{aligned}$$

13. The region D enclosed by C is given by $\{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi/2\}$ (in polar coordinates),which is traversed counterclockwise, so C has positive orientation. Thus,

$$\begin{aligned}\int_C \left(3+e^{x^2}\right) dx + \left(\tan^{-1}y+3x^2\right) dy &= \iint_D \left[\tfrac{\partial}{\partial x}(\tan^{-1}y+3x^2) - \tfrac{\partial}{\partial y}(3+e^{x^2})\right] dA \\ &= \iint_D 6x\,dA = 6\int_0^{\pi/2}\int_1^2 r\cos\theta\,r\,dr\,d\theta \qquad \text{[Switching to polar coordinates]} \\ &= 6\int_0^{\pi/2}\cos\theta\,d\theta \int_1^2 r^2\,dr = 6\left[\sin\theta\right]_{\theta=0}^{\theta=\pi/2}\left[\tfrac{r^3}{3}\right]_{r=1}^{r=2} \\ &= 6(1-0)\left(\tfrac{8}{3}-\tfrac{1}{3}\right) = 14\end{aligned}$$

14. The region D enclosed by C is given by $\{(x,y) \mid y^2 \le x \le 4, 0 \le y \le 2\}$. C is traversed clockwise, so $-C$ gives the positive orientation. Then

$$\begin{aligned}-\int_{-C} (x^{2/3}+y^2)\,dx + (y^{4/3}-x^2)\,dy &= -\iint_D \left[\tfrac{\partial}{\partial x}(y^{4/3}-x^2) - \tfrac{\partial}{\partial y}(x^{2/3}+y^2)\right] dA \\ &= -\int_0^2\int_{y^2}^4 (-2x-2y)\,dx\,dy = \int_0^2\int_{y^2}^4 (2x+2y)\,dx\,dy \\ &= \int_0^2 \left[x^2+2xy\right]_{x=y^2}^{x=4} dy = \int_0^2 (16+8y-y^4-2y^3)\,dy \\ &= \left[16y+4y^2-\tfrac{y^5}{5}-\tfrac{y^4}{2}\right]_0^2 = 16(2)+4(2^2)-\tfrac{2^5}{5}-\tfrac{2^4}{2}-0 = \tfrac{168}{5}\end{aligned}$$

15. $\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4 - 2x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy \\
&= -\iint_D \left[\tfrac{\partial}{\partial x}\,(xy + x\cos x) - \tfrac{\partial}{\partial y}\,(y\cos x - xy\sin x)\right] dA \\
&= -\iint_D (y - x\sin x + \cos x - \cos x + x\sin x)\,dA = -\int_0^2 \int_0^{4-2x} y\,dy\,dx \\
&= -\int_0^2 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx = -\int_0^2 \tfrac{1}{2}(4-2x)^2\,dx = -\int_0^2 (8 - 8x + 2x^2)\,dx \\
&= -\left[8x - 4x^2 + \tfrac{2}{3}x^3\right]_0^2 = -\left(16 - 16 + \tfrac{16}{3} - 0\right) = -\tfrac{16}{3}
\end{aligned}$$

16. $\mathbf{F}(x,y) = \left\langle e^{-x} + y^2, e^{-y} + x^2\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid -\pi/2 \le x \le \pi/2, 0 \le y \le \cos x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}\left(e^{-x} + y^2\right)dx + \left(e^{-y} + x^2\right)dy = -\iint_D \left[\tfrac{\partial}{\partial x}\left(e^{-y} + x^2\right) - \tfrac{\partial}{\partial y}\left(e^{-x} + y^2\right)\right] dA \\
&= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x - 2y)\,dy\,dx = -\int_{-\pi/2}^{\pi/2}\left[2xy - y^2\right]_{y=0}^{y=\cos x} dx \\
&= -\int_{-\pi/2}^{\pi/2}(2x\cos x - \cos^2 x)\,dx = -\int_{-\pi/2}^{\pi/2}\left[2x\cos x - \tfrac{1}{2}(1 + \cos 2x)\right] dx \\
&= -\left[2x\sin x + 2\cos x - \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2} \qquad \text{[integrate by parts in the first term]} \\
&= -\left(\pi - \tfrac{1}{4}\pi - \pi - \tfrac{1}{4}\pi\right) = \tfrac{1}{2}\pi
\end{aligned}$$

17. $\mathbf{F}(x,y) = \langle y - \cos y, x\sin y\rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3,-4)$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(y - \cos y)\,dx + (x\sin y)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}\,(x\sin y) - \tfrac{\partial}{\partial y}\,(y - \cos y)\right] dA \\
&= -\iint_D (\sin y - 1 - \sin y)\,dA = \iint_D dA = \text{area of } D = \pi(2)^2 = 4\pi
\end{aligned}$$

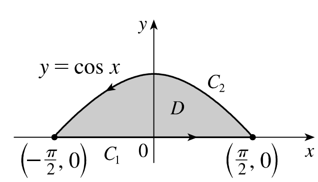
18. $\mathbf{F}(x,y) = \left\langle \sqrt{x^2+1}, \tan^{-1} x\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$. C is oriented positively, so

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \sqrt{x^2+1}\,dx + \tan^{-1} x\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(\tan^{-1} x\right) - \frac{\partial}{\partial y}\left(\sqrt{x^2+1}\right)\right] dA \\
&= \int_0^1 \int_x^1 \left(\frac{1}{1+x^2} - 0\right) dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\,y\,\right]_{y=x}^{y=1} dx = \int_0^1 \frac{1}{1+x^2}\,(1-x)\,dx \\
&= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = \left[\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

19. Here $C = C_1 + C_2$ where

C_1 can be parametrized as $x = t,\ \ y = 0,\ \ -\pi/2 \le t \le \pi/2$, and

C_2 is given by $x = -t,\ \ y = \cos t,\ \ -\pi/2 \le t \le \pi/2$.



[continued]

Then the line integral is

$$\oint_{C_1+C_2} x^3y^4\,dx + x^5y^4\,dy = \int_{-\pi/2}^{\pi/2}(0+0)\,dt + \int_{-\pi/2}^{\pi/2}[(-t)^3(\cos t)^4(-1) + (-t)^5(\cos t)^4(-\sin t)]\,dt$$
$$= 0 + \int_{-\pi/2}^{\pi/2}(t^3\cos^4 t + t^5\cos^4 t\sin t)\,dt = \tfrac{1}{15}\pi^4 - \tfrac{4144}{1125}\pi^2 + \tfrac{7{,}578{,}368}{253{,}125} \approx 0.0779$$

according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(5x^4y^4 - 4x^3y^3)\,dy\,dx = \tfrac{1}{15}\pi^4 - \tfrac{4144}{1125}\pi^2 + \tfrac{7{,}578{,}368}{253{,}125} \approx 0.0779,$$

verifying Green's Theorem in this case.

20. We can parametrize C as $x = \cos\theta,\ \ y = 2\sin\theta,\ \ 0 \le \theta \le 2\pi$. Then the line integral is

$$\oint_C P\,dx + Q\,dy = \int_0^{2\pi}\left[2\cos\theta - (\cos\theta)^3(2\sin\theta)^5\right](-\sin\theta)\,d\theta + \int_0^{2\pi}(\cos\theta)^3(2\sin\theta)^8\cdot 2\cos\theta\,d\theta$$
$$= \int_0^{2\pi}(-2\cos\theta\sin\theta + 32\cos^3\theta\sin^6\theta + 512\cos^4\theta\sin^8\theta)\,d\theta = 7\pi,$$

according to a CAS. The double integral is $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}(3x^2y^8 + 5x^3y^4)\,dy\,dx = 7\pi.$

21. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dA$ where C is the path described in the question and D is the triangle bounded by C. So

$$W = \int_0^1\int_0^{1-x}(y^2 - x)\,dy\,dx = \int_0^1\left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x}dx = \int_0^1\left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right)dx$$
$$= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}$$

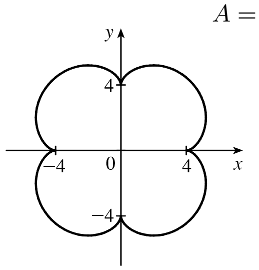
22. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \sin x\,dx + \left(\sin y + xy^2 + \tfrac{1}{3}x^3\right)dy = \iint_D (y^2 + x^2 - 0)\,dA$, where D is the region (a quarter-disk) bounded by C. Converting to polar coordinates, we have

$W = \int_0^{\pi/2}\int_0^5 r^2\cdot r\,dr\,d\theta = \left[\,\theta\,\right]_0^{\pi/2}\left[\tfrac{1}{4}r^4\right]_0^5 = \tfrac{1}{2}\pi\left(\tfrac{625}{4}\right) = \tfrac{625}{8}\pi.$

23. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt)$$
$$= \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$$

24.



$$A = \oint_C x\,dy = \int_0^{2\pi}(5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$
$$= \int_0^{2\pi}(25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$
$$= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi}$$

[Use Formula 80 in the Table of Integrals]

$$= 30\pi$$

25. (a) Using Equation 16.2.8, we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\begin{aligned}\int_C x\,dy - y\,dx &= \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt \\ &= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt \\ &= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1\end{aligned}$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\begin{aligned}\text{area of polygon} = A(D) &= \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx \\ &= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)\end{aligned}$$

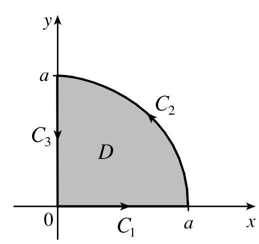
To evaluate these integrals we use the formula from (a) to get

$$A(D) = \tfrac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$$

(c) $A = \frac{1}{2}[(0 \cdot 1 - 2 \cdot 0) + (2 \cdot 3 - 1 \cdot 1) + (1 \cdot 2 - 0 \cdot 3) + (0 \cdot 1 - (-1) \cdot 2) + (-1 \cdot 0 - 0 \cdot 1)]$
$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

26. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

27. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2 dx$.

Here $C = C_1 + C_2 + C_3$ where

C_1: $x = t,\ \ y = 0,\ \ 0 \le t \le a$;

C_2: $x = a\cos t,\ \ y = a\sin t,\ \ 0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0$, $y = a - t$, $0 \le t \le a$. Then

$$\begin{aligned}\oint_C x^2\,dy &= \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = \int_0^a 0\,dt + \int_0^{\pi/2}(a\cos t)^2(a\cos t)\,dt + \int_0^a 0\,dt \\ &= \int_0^{\pi/2} a^3\cos^3 t\,dt = a^3\int_0^{\pi/2}(1 - \sin^2 t)\cos t\,dt = a^3\left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3\end{aligned}$$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy = \dfrac{4a}{3\pi}$.

$$\begin{aligned}\oint_C y^2 dx &= \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = \int_0^a 0\,dt + \int_0^{\pi/2}(a\sin t)^2(-a\sin t)\,dt + \int_0^a 0\,dt \\ &= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt = -a^3\int_0^{\pi/2}(1 - \cos^2 t)\sin t\,dt = -a^3\left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,\end{aligned}$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2 dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x}, \overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

28. Here $A = \frac{1}{2}ab$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le a$;

C_2: $x = a$, $y = y$, $0 \le y \le b$; and C_3: $x = x$, $y = \frac{b}{a}x$, $x = a$ to $x = 0$. Then

$$\oint_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_0^b a^2\,dy + \int_a^0 (x^2)\left(\tfrac{b}{a}\,dx\right)$$
$$= a^2b + \tfrac{b}{a}\left[\tfrac{1}{3}x^3\right]_a^0 = a^2b - \tfrac{1}{3}a^2b = \tfrac{2}{3}a^2b.$$

Similarly, $\oint_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + 0 + \int_a^0 \left(\frac{b}{a}x\right)^2\,dx = \frac{b^2}{a^2} \cdot \frac{1}{3}x^3\Big]_a^0 = -\frac{1}{3}ab^2$. Thus

$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{ab} \cdot \frac{2}{3}a^2b = \frac{2}{3}a$ and $\overline{y} = -\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{ab}\left(-\frac{1}{3}ab^2\right) = \frac{1}{3}b$, so $(\overline{x}, \overline{y}) = \left(\frac{2}{3}a, \frac{1}{3}b\right)$.

29. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D (-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and

$\frac{1}{3}\rho\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D (3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

30. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho\oint_C x^3\,dy = \tfrac{1}{3}\rho\int_0^{2\pi}(a\cos t)^3(a\cos t\,dt) = \tfrac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)\,dt$$
$$= \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left[\tfrac{1}{2}(1+\cos 2t)\right]^2 dt = \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left(\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right) dt$$
$$= \tfrac{1}{3}a^4\rho\left[\tfrac{3}{8}t + \tfrac{1}{4}\sin 2t + \tfrac{1}{32}\sin 4t\right]_0^{2\pi} = \tfrac{1}{3}a^4\rho \cdot \tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

31. As in Example 5, let C' be a counterclockwise-oriented circle with center the origin and radius a, where a is chosen to be small enough so that C' lies inside C, and D the region bounded by C and C'. Here

$P = \dfrac{2xy}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial P}{\partial y} = \dfrac{2x(x^2+y^2)^2 - 2xy \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4} = \dfrac{2x^3 - 6xy^2}{(x^2+y^2)^3}$ and

$Q = \dfrac{y^2 - x^2}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial Q}{\partial x} = \dfrac{-2x(x^2+y^2)^2 - (y^2-x^2) \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4} = \dfrac{2x^3 - 6xy^2}{(x^2+y^2)^3}$. Thus, as in the example,

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\,dA = 0$$

and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$. We parametrize C' as $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. Then

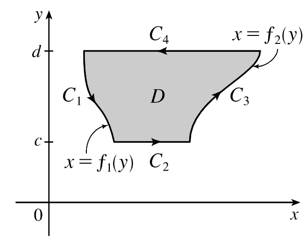
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{2\,(a\cos t)\,(a\sin t)\,\mathbf{i} + \left(a^2\sin^2 t - a^2\cos^2 t\right)\mathbf{j}}{\left(a^2\cos^2 t + a^2\sin^2 t\right)^2} \cdot \Big(-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}\Big)\,dt$$
$$= \frac{1}{a}\int_0^{2\pi}\left(-\cos t\sin^2 t - \cos^3 t\right) dt = \frac{1}{a}\int_0^{2\pi}\left(-\cos t\sin^2 t - \cos t\left(1 - \sin^2 t\right)\right) dt$$
$$= -\frac{1}{a}\int_0^{2\pi}\cos t\,dt = -\frac{1}{a}\sin t\Big]_0^{2\pi} = 0$$

32. P and Q have continuous partial derivatives on $\mathbb{R}^2$, so by Green's Theorem we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left[\frac{\partial}{\partial x}(3x - y^2) - \frac{\partial}{\partial y}(x^2 + y)\right] dA$$
$$= \iint_D (3-1)\,dA = 2\iint_D dA = 2 \cdot A(D) = 2 \cdot 6 = 12$$

33. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 16.3.41(a), $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D 0\,dA = 0$.

34. We express D as a type II region: $D = \{(x,y) \mid f_1(y) \le x \le f_2(y),\, c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\displaystyle\oint_C Q\,dy = \oint_{C_1+C_2+C_3+C_4} Q\,dy$.



Then $\int_{C_1} Q\,dy = \int_d^c Q(f_1(y),y)\,dy$, $\int_{C_2} Q\,dy = \int_{C_4} Q\,dy = 0$, and $\int_{C_3} Q\,dy = \int_c^d Q(f_2(y),y)\,dy$. Hence

$\oint_C Q\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy = \iint_D (\partial Q/\partial x)\,dA$.

35. Using the first part of Equation 5, we have that $\iint_R dx\,dy = A(R) = \int_{\partial R} x\,dy$. But $x = g(u,v)$, and $dy = \dfrac{\partial h}{\partial u}\,du + \dfrac{\partial h}{\partial v}\,dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) = \int_{\partial S} g(u,v)\,\frac{\partial h}{\partial u}\,du + g(u,v)\,\frac{\partial h}{\partial v}\,dv$$

$$= \pm\iint_S \left[\tfrac{\partial}{\partial u}\left(g(u,v)\,\tfrac{\partial h}{\partial v}\right) - \tfrac{\partial}{\partial v}\left(g(u,v)\,\tfrac{\partial h}{\partial u}\right)\right] dA \qquad \text{[using Green's Theorem in the } uv\text{-plane]}$$

$$= \pm\iint_S \left(\tfrac{\partial g}{\partial u}\tfrac{\partial h}{\partial v} + g(u,v)\,\tfrac{\partial^2 h}{\partial u\,\partial v} - \tfrac{\partial g}{\partial v}\tfrac{\partial h}{\partial u} - g(u,v)\,\tfrac{\partial^2 h}{\partial v\,\partial u}\right) dA \qquad \text{[using the Chain Rule]}$$

$$= \pm\iint_S \left(\tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial v} - \tfrac{\partial x}{\partial v}\tfrac{\partial y}{\partial u}\right) dA \quad \text{[by the equality of mixed partials]} \quad = \pm\iint_S \tfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial(x,y)}{\partial(u,v)}$.

Therefore, $A(R) = \displaystyle\iint_R dx\,dy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

16.5 Curl and Divergence

1. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(x^2y^2z) - \frac{\partial}{\partial z}(x^2yz^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right]\mathbf{k}$$

$$= (2x^2yz - 2x^2yz)\,\mathbf{i} - (2xy^2z - 2xy^2z)\,\mathbf{j} + (2xyz^2 - 2xyz^2)\,\mathbf{k} = \mathbf{0}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\,(xy^2z^2) + \dfrac{\partial}{\partial y}\,(x^2yz^2) + \dfrac{\partial}{\partial z}\,(x^2y^2z) = y^2z^2 + x^2z^2 + x^2y^2$

2. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(y^4z^3) - \frac{\partial}{\partial z}(x^3yz^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(y^4z^3) - \frac{\partial}{\partial z}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(0)\right]\mathbf{k}$$

$$= (4y^3z^3 - 2x^3yz)\,\mathbf{i} - (0-0)\,\mathbf{j} + (3x^2yz^2 - 0)\,\mathbf{k}$$

$$= (4y^3z^3 - 2x^3yz)\,\mathbf{i} + 3x^2yz^2\,\mathbf{k}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\,(0) + \dfrac{\partial}{\partial y}\,(x^3yz^2) + \dfrac{\partial}{\partial z}\,(y^4z^3) = 0 + x^3z^2 + 3y^4z^2 = x^3z^2 + 3y^4z^2$

3. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xye^z & 0 & yze^x \end{vmatrix} = (ze^x - 0)\,\mathbf{i} - (yze^x - xye^z)\,\mathbf{j} + (0 - xe^z)\,\mathbf{k}$

$$= ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\,(xye^z) + \dfrac{\partial}{\partial y}\,(0) + \dfrac{\partial}{\partial z}\,(yze^x) = ye^z + 0 + ye^x = y(e^z + e^x)$

4. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin yz & \sin zx & \sin xy \end{vmatrix}$

$$= (x\cos xy - x\cos zx)\,\mathbf{i} - (y\cos xy - y\cos yz)\,\mathbf{j} + (z\cos zx - z\cos yz)\,\mathbf{k}$$

$$= x(\cos xy - \cos zx)\,\mathbf{i} + y(\cos yz - \cos xy)\,\mathbf{j} + z(\cos zx - \cos yz)\,\mathbf{k}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\,(\sin yz) + \dfrac{\partial}{\partial y}\,(\sin zx) + \dfrac{\partial}{\partial z}\,(\sin xy) = 0 + 0 + 0 = 0$

5. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{\sqrt{x}}{1+z} & \dfrac{\sqrt{y}}{1+x} & \dfrac{\sqrt{z}}{1+y} \end{vmatrix}$

$$= \left[\sqrt{z}\,(-1)(1+y)^{-2} - 0\right]\mathbf{i} - \left[0 - \sqrt{x}(-1)(1+z)^{-2}\right]\mathbf{j} + \left[\sqrt{y}\,(-1)(1+x)^{-2} - 0\right]\mathbf{k}$$

$$= -\frac{\sqrt{z}}{(1+y)^2}\,\mathbf{i} - \frac{\sqrt{x}}{(1+z)^2}\,\mathbf{j} - \frac{\sqrt{y}}{(1+x)^2}\,\mathbf{k}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{\sqrt{x}}{1+z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\sqrt{y}}{1+x}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\sqrt{z}}{1+y}\right)$

$$= \frac{1}{2\sqrt{x}\,(1+z)} + \frac{1}{2\sqrt{y}\,(1+x)} + \frac{1}{2\sqrt{z}\,(1+y)}$$

6. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln(2y+3z) & \ln(x+3z) & \ln(x+2y) \end{vmatrix}$

$$= \left(\frac{2}{x+2y} - \frac{3}{x+3z}\right)\mathbf{i} - \left(\frac{1}{x+2y} - \frac{3}{2y+3z}\right)\mathbf{j} + \left(\frac{1}{x+3z} - \frac{2}{2y+3z}\right)\mathbf{k}$$

$$= \left(\frac{2}{x+2y} - \frac{3}{x+3z}\right)\mathbf{i} + \left(\frac{3}{2y+3z} - \frac{1}{x+2y}\right)\mathbf{j} + \left(\frac{1}{x+3z} - \frac{2}{2y+3z}\right)\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\big[\ln(2y+3z)\big] + \dfrac{\partial}{\partial y}\big[\ln(x+3z)\big] + \dfrac{\partial}{\partial z}\big[\ln(x+2y)\big] = 0+0+0 = 0$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} = (0 - e^y \cos z)\,\mathbf{i} - (e^z \cos x - 0)\,\mathbf{j} + (0 - e^x \cos y)\,\mathbf{k}$

$$= \langle -e^y \cos z, -e^z \cos x, -e^x \cos y \rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(e^x \sin y) + \dfrac{\partial}{\partial y}(e^y \sin z) + \dfrac{\partial}{\partial z}(e^z \sin x) = e^x \sin y + e^y \sin z + e^z \sin x$

8. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \arctan(xy) & \arctan(yz) & \arctan(zx) \end{vmatrix}$

$$= \left(0 - \frac{y}{1+(yz)^2}\right)\mathbf{i} - \left(\frac{z}{1+(zx)^2} - 0\right)\mathbf{j} + \left(0 - \frac{x}{1+(xy)^2}\right)\mathbf{k}$$

$$= \left\langle -\frac{y}{1+y^2z^2}, -\frac{z}{1+x^2z^2}, -\frac{x}{1+x^2y^2} \right\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\big[\arctan(xy)\big] + \dfrac{\partial}{\partial y}\big[\arctan(yz)\big] + \dfrac{\partial}{\partial z}\big[\arctan(zx)\big] = \dfrac{y}{1+x^2y^2} + \dfrac{z}{1+y^2z^2} + \dfrac{x}{1+x^2z^2}$

9. (a) $\operatorname{div}\mathbf{F}$ is negative because the vectors that start near P are shorter than those that end near P. Intuitively, if $\mathbf{F}$ represents a velocity field of fluid flow, then the net flow at P is inward.

(b) $\operatorname{curl}\mathbf{F}$ is zero because we can see that if $\mathbf{F}$ represents a velocity field of fluid flow, then a paddle wheel placed at P moves with the fluid, but does not rotate.

10. (a) $\operatorname{div}\mathbf{F}$ is positive because the vectors that start near P are longer than those that end near P. Intuitively, if $\mathbf{F}$ represents a velocity field of fluid flow, then the net flow at P is outward.

(b) $\operatorname{curl}\mathbf{F}$ is zero because we can see that if $\mathbf{F}$ represents a velocity field of fluid flow, then a paddle wheel placed at P moves with the fluid, but does not rotate.

11. (a) $\operatorname{div}\mathbf{F}$ is zero because the vectors that start near P are the same length as those that end near P. Intuitively, if $\mathbf{F}$ represents a velocity field of fluid flow, then the net flow at P is zero.

(b) $\operatorname{curl}\mathbf{F} \neq 0$ because we can see that if $\mathbf{F}$ represents a velocity field of fluid flow, then a paddle wheel placed at P would rotate clockwise about its axis, and hence the curl vector there points in the direction of $-\mathbf{k}$.

12. (a) $\operatorname{div}\mathbf{F}$ is zero because the vectors that start near P are the same length as those that end near P. Intuitively, if $\mathbf{F}$ represents a velocity field of fluid flow, then the net flow at P is zero.

(b) $\operatorname{curl}\mathbf{F} \neq 0$ because we can see that if $\mathbf{F}$ represents a velocity field of fluid flow, then a paddle wheel placed at P would rotate counterclockwise about its axis, and hence the curl vector there points in the direction of $\mathbf{k}$.

13. (a) We need to verify $\operatorname{curl}(\nabla f) = \mathbf{0}$ for $f(x, y, z) = \sin xyz$. First, $\nabla f = yz\cos xyz\,\mathbf{i} + xz\cos xyz\,\mathbf{j} + xy\cos xyz\,\mathbf{k}$. Then

$$\begin{aligned}\operatorname{curl}(\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\cos xyz & xz\cos xyz & xy\cos xyz \end{vmatrix} \\ &= \left[-x^2yz\sin xyz + x\cos xyz - (-x^2yz\sin xyz + x\cos xyz)\right]\mathbf{i} \\ &\quad - \left[-xy^2z\sin xyz + y\cos xyz - (-xy^2z\sin xyz + y\cos xyz)\right]\mathbf{j} \\ &\quad + \left[-xyz^2\sin xyz + z\cos xyz - (-xyz^2\sin xyz + z\cos xyz)\right]\mathbf{k} = \mathbf{0}\end{aligned}$$

(b) We need to verify that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$ for $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^3\,\mathbf{j} + y^2\,\mathbf{k}$. First,

$$\begin{aligned}\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & x^2yz^3 & y^2 \end{vmatrix} \\ &= (2y - 3x^2yz^2)\,\mathbf{i} - (0 - 2xyz)\,\mathbf{j} + (2xyz^3 - xz^2)\,\mathbf{k} \\ &= (2y - 3x^2yz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (2xyz^3 - xz^2)\,\mathbf{k}\end{aligned}$$

Then

$$\begin{aligned}\operatorname{div}\operatorname{curl}\mathbf{F} = \nabla\cdot(\nabla\times\mathbf{F}) &= \tfrac{\partial}{\partial x}(2y - 3x^2yz^2) + \tfrac{\partial}{\partial y}(2xyz) + \tfrac{\partial}{\partial z}(2xyz^3 - xz^2) \\ &= -6xyz^2 + 2xz + 6xyz^2 - 2xz = 0\end{aligned}$$

14. (a) $\operatorname{curl} f = \nabla\times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div}\mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad}\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

(i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$ is a vector field.

(j) $\operatorname{div}(\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(k) $(\operatorname{grad} f)\times(\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

15. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^2 & 3x^2y^2z^2 & 2x^2y^3z \end{vmatrix}$

$$= (6x^2y^2z - 6x^2y^2z)\,\mathbf{i} - (4xy^3z - 4xy^3z)\,\mathbf{j} + (6xy^2z^2 - 6xy^2z^2)\,\mathbf{k} = \mathbf{0}$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives, so, by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$. Now $f_x(x,y,z) = 2xy^3z^2$ implies that $f(x,y,z) = x^2y^3z^2 + g(y,z)$ and then $f_y(x,y,z) = 3x^2y^2z^2 + g_y(y,z)$. But $f_y(x,y,z) = 3x^2y^2z^2$, so $g(y,z) = h(z)$ and $f(x,y,z) = x^2y^3z^2 + h(z)$. Thus, $f_z(x,y,z) = 2x^2y^3z + h'(z)$, but $f_z(x,y,z) = 2x^2y^3z$, so $h(z) = K$, a constant. Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = x^2y^3z^2 + K$.

16. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz+y & xy-x \end{vmatrix} = (x-x)\,\mathbf{i} - (y-1-y)\,\mathbf{j} + (z-z)\,\mathbf{k} = \mathbf{j} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

17. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln y & (x/y)+\ln z & y/z \end{vmatrix} = (1/z - 1/z)\,\mathbf{i} - (0-0)\,\mathbf{j} + (1/y - 1/y)\,\mathbf{k} = \mathbf{0}$

and the partial derivatives of the component functions are defined and continuous on the open set $\{(x,y,z) \mid y,z > 0\}$, so, by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$. Now $f_x(x,y,z) = \ln y$ implies that $f(x,y,z) = x\ln y + g(y,z)$ and then $f_y(x,y,z) = (x/y) + g_y(y,z)$. But $f_y(x,y,z) = (x/y) + \ln z$, so $g(y,z) = y\ln z + h(z)$ and $f(x,y,z) = x\ln y + y\ln z + h(z)$. Thus, $f_z(x,y,z) = (y/z) + h'(z)$, but $f_z(x,y,z) = y/z$, so $h(z) = K$, a constant. Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = x\ln y + y\ln z + K$.

18. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\sin(xy) & xz\sin(xy) & -\cos(xy) \end{vmatrix}$

$$\begin{aligned} &= [x\sin(xy) - x\sin(xy)]\,\mathbf{i} - [y\sin(xy) - y\sin(xy)]\,\mathbf{j} \\ &\quad + [z\sin(xy) + xyz\cos(xy) - z\sin(xy) - xyz\cos(xy)]\,\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives, so, by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$. Now $f_x(x,y,z) = yz\sin(xy)$ implies that $f(x,y,z) = -z\cos(xy) + g(y,z)$ and then $f_y(x,y,z) = xz\sin(xy) + g_y(y,z)$. But $f_y(x,y,z) = xz\sin(xy)$, so $g(y,z) = h(z)$ and $f(x,y,z) = -z\cos(xy) + h(z)$. Thus, $f_z(x,y,z) = -\cos(xy) + h'(z)$, but $f_z(x,y,z) = -\cos(xy)$, so $h(z) = K$, a constant. Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = -z\cos(xy) + K$.

19. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz^2e^{xz} & ze^{xz} & xyze^{xz} \end{vmatrix}$

$$= (xze^{xz} - e^{xz} - xze^{xz})\,\mathbf{i} - (yze^{xz} + xyz^2e^{xz} - 2yze^{xz} - xyz^2e^{xz})\,\mathbf{j} + (z^2e^{xz} - z^2e^{xz})\,\mathbf{k}$$

$$= -e^{xz}\,\mathbf{i} + yze^{xz}\,\mathbf{j} \neq \mathbf{0},$$

so $\mathbf{F}$ is not conservative.

20. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z\cos x & e^y\cos z & e^z\sin x - e^y\sin z \end{vmatrix}$

$$= [-e^y\sin z - (-e^y\sin z)]\,\mathbf{i} - (e^z\cos x - e^z\cos x)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives, so, by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$. Now $f_x(x,y,z) = e^z\cos x$ implies that $f(x,y,z) = e^z\sin x + g(y,z)$ and then $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = e^y\cos z$, so $g(y,z) = e^y\cos z + h(z)$ and $f(x,y,z) = e^z\sin x + e^y\cos z + h(z)$. Thus, $f_z(x,y,z) = e^z\sin x - e^y\sin z + h'(z)$, but $f_z(x,y,z) = e^z\sin x - e^y\sin z$, so $h(z) = K$, a constant. Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = e^z\sin x + e^y\cos z + K$.

21. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \dfrac{\partial}{\partial x}(x\sin y) + \dfrac{\partial}{\partial y}(\cos y) + \dfrac{\partial}{\partial z}(z - xy) = \sin y - \sin y + 1 \neq 0$, which contradicts Theorem 11.

22. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 \neq 0$ which contradicts Theorem 11.

23. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$. Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.

24. $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(f(y,z)) + \dfrac{\partial}{\partial y}(g(x,z)) + \dfrac{\partial}{\partial z}(h(x,y)) = 0$ so $\mathbf{F}$ is incompressible.

For Exercises 25–31, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

25. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\langle P_1 + P_2, Q_1 + Q_2, R_1 + R_2\rangle = \dfrac{\partial(P_1+P_2)}{\partial x} + \dfrac{\partial(Q_1+Q_2)}{\partial y} + \dfrac{\partial(R_1+R_2)}{\partial z}$

$$= \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_1}{\partial z} + \frac{\partial R_2}{\partial z} = \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)$$

$$= \operatorname{div}\langle P_1, Q_1, R_1\rangle + \operatorname{div}\langle P_2, Q_2, R_2\rangle = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$$

26. $$\operatorname{curl}\mathbf{F}+\operatorname{curl}\mathbf{G}=\left[\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\mathbf{k}\right]$$
$$+\left[\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\mathbf{k}\right]$$
$$=\left[\frac{\partial(R_1+R_2)}{\partial y}-\frac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i}+\left[\frac{\partial(P_1+P_2)}{\partial z}-\frac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j}$$
$$+\left[\frac{\partial(Q_1+Q_2)}{\partial x}-\frac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k}=\operatorname{curl}(\mathbf{F}+\mathbf{G})$$

27. $$\operatorname{div}(f\mathbf{F})=\operatorname{div}(f\langle P_1,Q_1,R_1\rangle)=\operatorname{div}\langle fP_1,fQ_1,fR_1\rangle=\frac{\partial(fP_1)}{\partial x}+\frac{\partial(fQ_1)}{\partial y}+\frac{\partial(fR_1)}{\partial z}$$
$$=\left(f\,\frac{\partial P_1}{\partial x}+P_1\,\frac{\partial f}{\partial x}\right)+\left(f\,\frac{\partial Q_1}{\partial y}+Q_1\,\frac{\partial f}{\partial y}\right)+\left(f\,\frac{\partial R_1}{\partial z}+R_1\,\frac{\partial f}{\partial z}\right)$$
$$=f\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+\langle P_1,Q_1,R_1\rangle\cdot\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle=f\operatorname{div}\mathbf{F}+\mathbf{F}\cdot\nabla f$$

28. $$\operatorname{curl}(f\mathbf{F})=\left[\frac{\partial(fR_1)}{\partial y}-\frac{\partial(fQ_1)}{\partial z}\right]\mathbf{i}+\left[\frac{\partial(fP_1)}{\partial z}-\frac{\partial(fR_1)}{\partial x}\right]\mathbf{j}+\left[\frac{\partial(fQ_1)}{\partial x}-\frac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$$
$$=\left[f\,\frac{\partial R_1}{\partial y}+R_1\,\frac{\partial f}{\partial y}-f\,\frac{\partial Q_1}{\partial z}-Q_1\,\frac{\partial f}{\partial z}\right]\mathbf{i}+\left[f\,\frac{\partial P_1}{\partial z}+P_1\,\frac{\partial f}{\partial z}-f\,\frac{\partial R_1}{\partial x}-R_1\,\frac{\partial f}{\partial x}\right]\mathbf{j}$$
$$+\left[f\,\frac{\partial Q_1}{\partial x}+Q_1\,\frac{\partial f}{\partial x}-f\,\frac{\partial P_1}{\partial y}-P_1\,\frac{\partial f}{\partial y}\right]\mathbf{k}$$
$$=f\left[\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right]\mathbf{i}+f\left[\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right]\mathbf{j}+f\left[\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right]\mathbf{k}$$
$$+\left[R_1\frac{\partial f}{\partial y}-Q_1\frac{\partial f}{\partial z}\right]\mathbf{i}+\left[P_1\frac{\partial f}{\partial z}-R_1\frac{\partial f}{\partial x}\right]\mathbf{j}+\left[Q_1\frac{\partial f}{\partial x}-P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$
$$=f\operatorname{curl}\mathbf{F}+(\nabla f)\times\mathbf{F}$$

29. $$\operatorname{div}(\mathbf{F}\times\mathbf{G})=\nabla\cdot(\mathbf{F}\times\mathbf{G})=\begin{vmatrix}\partial/\partial x & \partial/\partial y & \partial/\partial z\\ P_1 & Q_1 & R_1\\ P_2 & Q_2 & R_2\end{vmatrix}=\frac{\partial}{\partial x}\begin{vmatrix}Q_1 & R_1\\ Q_2 & R_2\end{vmatrix}-\frac{\partial}{\partial y}\begin{vmatrix}P_1 & R_1\\ P_2 & R_2\end{vmatrix}+\frac{\partial}{\partial z}\begin{vmatrix}P_1 & Q_1\\ P_2 & Q_2\end{vmatrix}$$
$$=\left[Q_1\,\frac{\partial R_2}{\partial x}+R_2\,\frac{\partial Q_1}{\partial x}-Q_2\,\frac{\partial R_1}{\partial x}-R_1\,\frac{\partial Q_2}{\partial x}\right]-\left[P_1\,\frac{\partial R_2}{\partial y}+R_2\,\frac{\partial P_1}{\partial y}-P_2\,\frac{\partial R_1}{\partial y}-R_1\,\frac{\partial P_2}{\partial y}\right]$$
$$+\left[P_1\,\frac{\partial Q_2}{\partial z}+Q_2\,\frac{\partial P_1}{\partial z}-P_2\,\frac{\partial Q_1}{\partial z}-Q_1\,\frac{\partial P_2}{\partial z}\right]$$
$$=\left[P_2\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)+Q_2\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)+R_2\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\right]$$
$$-\left[P_1\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)+Q_1\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)+R_1\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\right]$$
$$=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

30. $\operatorname{div}(\nabla f\times\nabla g)=\nabla g\cdot\operatorname{curl}(\nabla f)-\nabla f\cdot\operatorname{curl}(\nabla g)$ [by Exercise 29] $=0$ [by Theorem 3]

31. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla\times(\nabla\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$

$$= \left(\frac{\partial^2 Q_1}{\partial y\partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z\partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x\partial y}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x\partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y\partial z}\right)\mathbf{k}$$

Now let's consider $\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$ and compare with the above. (Note that $\nabla^2\mathbf{F}$ is defined in the discussion following Example 16.5.5.)

$$\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x\partial y} + \frac{\partial^2 R_1}{\partial x\partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z\partial x} + \frac{\partial^2 Q_1}{\partial z\partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$= \left(\frac{\partial^2 Q_1}{\partial x\partial y} + \frac{\partial^2 R_1}{\partial x\partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\partial x} + \frac{\partial^2 R_1}{\partial y\partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial z\partial x} + \frac{\partial^2 Q_1}{\partial z\partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

32. (a) $\nabla\cdot\mathbf{r} = \left(\dfrac{\partial}{\partial x}\,\mathbf{i} + \dfrac{\partial}{\partial y}\,\mathbf{j} + \dfrac{\partial}{\partial z}\,\mathbf{k}\right)\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 1 + 1 + 1 = 3$

(b) $\nabla\cdot(r\mathbf{r}) = \nabla\cdot\sqrt{x^2+y^2+z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$$= \left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$+ \left(\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\,(4x^2 + 4y^2 + 4z^2) = 4\sqrt{x^2+y^2+z^2} = 4r$$

Another method:

By Exercise 27, $\nabla\cdot(r\mathbf{r}) = \operatorname{div}(r\mathbf{r}) = r\operatorname{div}\mathbf{r} + \mathbf{r}\cdot\nabla r = 3r + \mathbf{r}\cdot\dfrac{\mathbf{r}}{r}$ [see Exercise 33(a) below] $= 4r$.

(c) $\nabla^2 r^3 = \nabla^2 \left(x^2+y^2+z^2\right)^{3/2}$

$$= \tfrac{\partial}{\partial x}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right] + \tfrac{\partial}{\partial y}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right] + \tfrac{\partial}{\partial z}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]$$

$$= 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x) + (x^2+y^2+z^2)^{1/2}\right]$$

$$+ 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y) + (x^2+y^2+z^2)^{1/2}\right]$$

$$+ 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z) + (x^2+y^2+z^2)^{1/2}\right]$$

$$= 3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2) = 12(x^2+y^2+z^2)^{1/2} = 12r$$

Another method: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2} = 3x\sqrt{x^2+y^2+z^2} \;\Rightarrow\; \nabla r^3 = 3r(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}) = 3r\,\mathbf{r}$,
so $\nabla^2 r^3 = \nabla\cdot\nabla r^3 = \nabla\cdot(3r\,\mathbf{r}) = 3(4r) = 12r$ by part (b).

33. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{r}$

(b) $\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left[\dfrac{\partial}{\partial y}(z) - \dfrac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\dfrac{\partial}{\partial z}(x) - \dfrac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\dfrac{\partial}{\partial x}(y) - \dfrac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$

(c) $\nabla\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla \ln r = \nabla\ln(x^2+y^2+z^2)^{1/2} = \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

34. $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} \;\Rightarrow\; r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2)-px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2-px^2}{r^{p+2}}$. Similarly,

$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-pz^2}{r^{p+2}}$. Thus

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{r^2-px^2}{r^{p+2}} + \frac{r^2-py^2}{r^{p+2}} + \frac{r^2-pz^2}{r^{p+2}} = \frac{3r^2-px^2-py^2-pz^2}{r^{p+2}}$$

$$= \frac{3r^2-p(x^2+y^2+z^2)}{r^{p+2}} = \frac{3r^2-pr^2}{r^{p+2}} = \frac{3-p}{r^p}$$

Consequently, if $p=3$ we have $\operatorname{div}\mathbf{F} = 0$.

35. By (13), $\oint_C f(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 27. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

36. By Exercise 35, $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$ and
$\iint_D g\nabla^2 f\,dA = \oint_C g(\nabla f)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$. Hence
$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dA = \oint_C [f(\nabla g)\cdot\mathbf{n} - g(\nabla f)\cdot\mathbf{n}]\,ds + \iint_D (\nabla f\cdot\nabla g - \nabla g\cdot\nabla f)\,dA = \oint_C [f\nabla g - g\nabla f]\cdot\mathbf{n}\,ds$.

37. Let $f(x,y) = 1$. Then $\nabla f = \mathbf{0}$ and Green's first identity (see Exercise 35) says
$\iint_D \nabla^2 g\,dA = \oint_C (\nabla g)\cdot\,\mathbf{n}\,ds - \iint_D \mathbf{0}\cdot\nabla g\,dA \quad\Rightarrow\quad \iint_D \nabla^2 g\,dA = \oint_C \nabla g\cdot\,\mathbf{n}\,ds$. But g is harmonic on D, so
$\nabla^2 g = 0 \quad\Rightarrow\quad \oint_C \nabla g\cdot\,\mathbf{n}\,ds = 0$ and $\oint_C D_{\mathbf{n}} g\,ds = \oint_C (\nabla g\cdot\,\mathbf{n})\,ds = 0$.

38. Let $g = f$. Then Green's first identity (see Exercise 35) says $\iint_D f\,\nabla^2 f\,dA = \oint_C\,(f)(\nabla f)\cdot\,\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla f\,dA$.
But f is harmonic, so $\nabla^2 f = 0$, and $\nabla f\cdot\nabla f = |\nabla f|^2$, so we have $0 = \oint_C\,(f)\,(\nabla f)\cdot\,\mathbf{n}\,ds - \iint_D |\nabla f|^2\,dA \quad\Rightarrow$
$\iint_D |\nabla f|^2\,dA = \oint_C\,(f)\,(\nabla f)\cdot\,\mathbf{n}\,ds = 0$ since $f\,(x,y) = 0$ on C.

39. (a) We know that $\omega = v/d$ (where v is the tangential speed), and from the diagram $\sin\theta = d/r$ (where $r = |\mathbf{r}|$) $\Rightarrow$
$v = d\omega = (\sin\theta)r\omega = |\mathbf{w}\times\mathbf{r}|$ (by 12.4.9). But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From part (a), $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} - (0\cdot z - \omega x)\,\mathbf{j} + (0\cdot y - 0\cdot x)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.

(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k}$$
$$= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}$$

40. Let $\mathbf{H} = \langle h_1, h_2, h_3\rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3\rangle$.

(a) $\nabla\times(\nabla\times\mathbf{E}) = \nabla\times(\operatorname{curl}\mathbf{E}) = \nabla\times\left(-\dfrac{1}{c}\dfrac{\partial\mathbf{H}}{\partial t}\right) = -\dfrac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\,\partial t} - \frac{\partial^2 h_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z\,\partial t} - \frac{\partial^2 h_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x\,\partial t} - \frac{\partial^2 h_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

(b) $\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (\operatorname{curl} \mathbf{H}) = \nabla \times \left(\dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}\right) = \dfrac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$

$$= \frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\,\partial t} - \frac{\partial^2 E_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z\,\partial t} - \frac{\partial^2 E_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x\,\partial t} - \frac{\partial^2 E_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(c) Using Exercise 31, we have that $\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \quad \Rightarrow$

$$\nabla^2\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}0 + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{[from part (a)]} \quad = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2\mathbf{H} = \operatorname{grad}\operatorname{div}\mathbf{H} - \operatorname{curl}\operatorname{curl}\mathbf{H} = \operatorname{grad}0 + \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$ [using part (b)] $= \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$.

41. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x,y,z) = \langle g(x,y,z), 0, 0\rangle$ where $g(x,y,z) = \int_0^x f(t,y,z)\,dt$. Then $\operatorname{div}\mathbf{G} = \dfrac{\partial}{\partial x}(g(x,y,z)) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(0) = \dfrac{\partial}{\partial x}\int_0^x f(t,y,z)\,dt = f(x,y,z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

16.6 Parametric Surfaces and Their Areas

1. $P(4,-5,1)$ lies on the parametric surface $\mathbf{r}(u,v) = \langle u+v, u-2v, 3+u-v\rangle$ if and only if there are values for u and v where $u+v=4$, $u-2v=-5$, and $3+u-v=1$. From the first equation we have $u=4-v$ and substituting into the second equation gives $4-v-2v=-5 \quad \Leftrightarrow \quad v=3$. Then $u=1$, and these values satisfy the third equation, so P does lie on the surface.

$Q(0,4,6)$ lies on $\mathbf{r}(u,v)$ if and only if $u+v=0$, $u-2v=4$, and $3+u-v=6$, but solving the first two equations simultaneoulsy gives $u=\frac{4}{3}$, $v=-\frac{4}{3}$ and these values do not satisfy the third equation, so Q does not lie on the surface.

2. $P(1,2,1)$ lies on the parametric surface $\mathbf{r}(u,v) = \left\langle 1+u-v, u+v^2, u^2-v^2\right\rangle$ if and only if there are values for u and v where $1+u-v=1$, $u+v^2=2$, and $u^2-v^2=1$. From the first equation we have $u=v$ and substituting into the third equation gives $0=1$, an impossibility, so P does not lie on the surface.

$Q(2,3,3)$ lies on $\mathbf{r}(u,v)$ if and only if $1+u-v=2$, $u+v^2=3$, and $u^2-v^2=3$. From the first equation we have $u=v+1$ and substituting into the second equation gives $v+1+v^2=3 \quad \Leftrightarrow \quad v^2+v-2=0 \quad \Leftrightarrow \quad (v+2)(v-1)=0$, so $v=-2 \quad \Rightarrow \quad u=-1$ or $v=1 \quad \Rightarrow \quad u=2$. The third equation is satisfied by $u=2$, $v=1$ so Q does lie on the surface.

3. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (3-v)\,\mathbf{j} + (1+4u+5v)\,\mathbf{k} = \langle 0,3,1\rangle + u\,\langle 1,0,4\rangle + v\,\langle 1,-1,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0,3,1)$ and containing vectors $\mathbf{a} = \langle 1,0,4\rangle$ and $\mathbf{b} = \langle 1,-1,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x-0)-(y-3)-(z-1)=0$ or $4x-y-z=-4$.

4. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = u^2,\ y = u\cos v,$ $z = u\sin v$. For any point (x,y,z) on the surface, we have $y^2+z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = x$. Since no restrictions are placed on the parameters, the surface is $x = y^2+z^2$, which we recognize as a circular paraboloid whose axis is the x-axis.

5. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s\rangle$, so the corresponding parametric equations for the surface are $x = s\cos t,\ \ y = s\sin t,\ \ z = s$. For any point (x,y,z) on the surface, we have $x^2+y^2 = s^2\cos^2 t + s^2\sin^2 t = s^2 = z^2$. Since no restrictions are placed on the parameters, the surface is $z^2 = x^2+y^2$, which we recognize as a circular cone with axis the z-axis.

6. $\mathbf{r}(s,t) = \langle 3\cos t, s, \sin t\rangle$, so the corresponding parametric equations for the surface are $x = 3\cos t,\ \ y = s,\ \ z = \sin t$. For any point (x,y,z) on the surface, we have $(x/3)^2 + z^2 = \cos^2 t + \sin^2 t = 1$, so vertical cross-sections parallel to the xz-plane are all identical ellipses. Since $y = s$ and $-1 \le s \le 1$, the surface is the portion of the elliptic cylinder $\frac{1}{9}x^2 + z^2 = 1$ corresponding to $-1 \le y \le 1$.

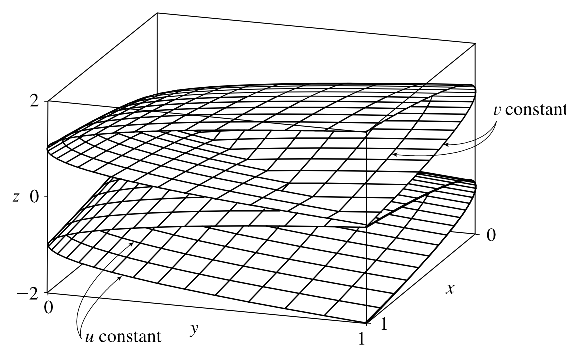
7. $\mathbf{r}(u,v) = \left\langle u^2, v^2, u+v\right\rangle,\ \ -1\le u\le 1,\ \ -1\le v\le 1$.

The surface has parametric equations $x = u^2$, $y = v^2$, $z = u+v$, $-1\le u\le 1$, $-1\le v\le 1$.

In Maple, the surface can be graphed by entering

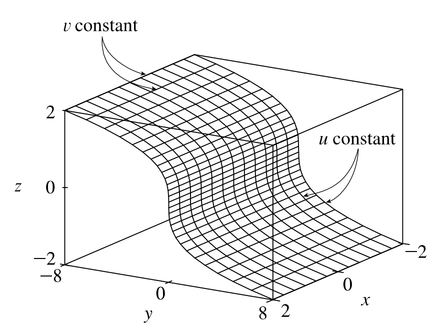
```
plot3d([u^2,v^2,u+v],u=-1..1,v=-1..1);
```

In Mathematica we use the `ParametricPlot3D` command. If we keep u constant at u_0, $x = u_0^2$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^2$, a constant, so these grid curves are the curves parallel to the xz-plane.

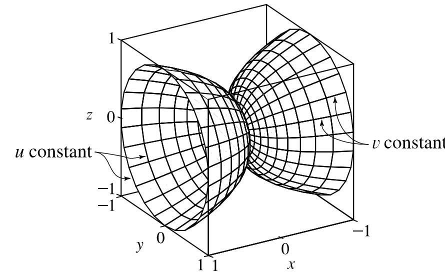


8. $\mathbf{r}(u,v) = \left\langle u, v^3, -v\right\rangle,\ \ -2\le u\le 2,\ \ -2\le v\le 2$.

The surface has parametric equations $x = u$, $y = v^3$, $z = -v$, $-2\le u\le 2$, $-2\le v\le 2$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $y = v_0^3 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane.

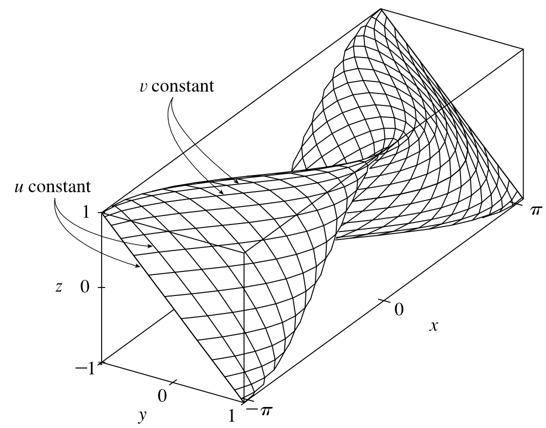


9. $\mathbf{r}(u,v) = \langle u^3, u\sin v, u\cos v\rangle,\ -1 \le u \le 1,\ 0 \le v \le 2\pi$



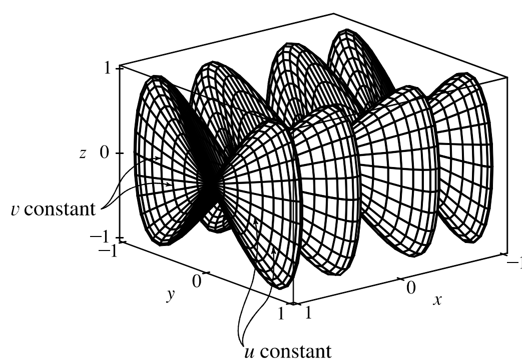
The surface has parametric equations $x = u^3,\ y = u\sin v,$ $z = u\cos v,\ -1 \le u \le 1,\ 0 \le v \le 2\pi$. Note that if $u = u_0$ is constant then $x = u_0^3$ is constant and $y = u_0\sin v$, $z = u_0\cos v$ describe a circle in y, z of radius $|u_0|$, so the corresponding grid curves are circles parallel to the yz-plane. If $v = v_0$, a constant, the parametric equations become $x = u^3$, $y = u\sin v_0$, $z = u\cos v_0$. Then $y = (\tan v_0)z$, so these are the grid curves we see that lie in planes $y = kz$ that pass through the x-axis.

10. $\mathbf{r}(u,v) = \langle u, \sin(u+v), \sin v\rangle,\ -\pi \le u \le \pi,\ -\pi \le v \le \pi.$



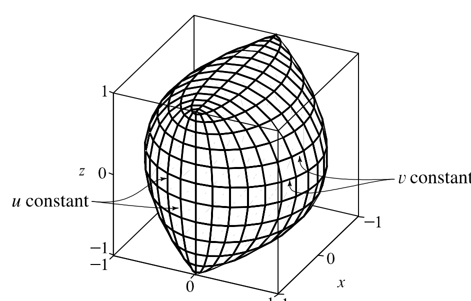
The surface has parametric equations $x = u$, $y = \sin(u+v)$, $z = \sin v$, $-\pi \le u \le \pi$, $-\pi \le v \le \pi$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $z = \sin v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.

11. $x = \sin v,\quad y = \cos u \sin 4v,\quad z = \sin 2u\ \sin 4v,\quad 0 \le u \le 2\pi,\quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}.$



Note that if $v = v_0$ is constant, then $x = \sin v_0$ is constant, so the corresponding grid curves must be parallel to the yz-plane. These are the vertically oriented grid curves we see, each shaped like a “figure-eight.” When $u = u_0$ is held constant, the parametric equations become $x = \sin v$, $y = \cos u_0 \sin 4v$, $z = \sin 2u_0 \sin 4v$. Since z is a constant multiple of y, the corresponding grid curves are the curves contained in planes $z = ky$ that pass through the x-axis.

12. $x = \cos u,\ y = \sin u\ \sin v,\ z = \cos v,\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi.$



If $u = u_0$ is constant, then $x = \cos u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, then $z = \cos v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.

13. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph IV.

14. $\mathbf{r}(u,v) = uv^2\,\mathbf{i} + u^2v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$. The parametric equations for the surface are $x = uv^2,\ y = u^2v,\ z = u^2 - v^2$. If $u = u_0$ is held constant, then $x = u_0v^2$, $y = u_0^2v$ so $x = u_0(y/u_0^2)^2 = (1/u_0^3)y^2$, and $z = u_0^2 - v^2 = u_0^2 - (1/u_0)x$. Thus each grid curve corresponding to $u = u_0$ lies in the plane $z = u_0^2 - (1/u_0)x$ and its projection onto the xy-plane is a parabola $x = ky^2$ with axis the x-axis. Similarly, if $v = v_0$ is held constant, then $x = uv_0^2$, $y = u^2v_0 \quad\Rightarrow$ $y = (x/v_0^2)^2v_0 = (1/v_0^3)x^2$, and $z = u^2 - v_0^2 = (1/v_0)y - v_0^2$. Each grid curve lies in the plane $z = (1/v_0)y - v_0^2$ and its projection onto the xy-plane is a parabola $y = kx^2$ with axis the y-axis. The surface is graph VI.

15. $\mathbf{r}(u,v) = (u^3 - u)\,\mathbf{i} + v^2\,\mathbf{j} + u^2\,\mathbf{k}$. The parametric equations for the surface are $x = u^3 - u,\ y = v^2,\ z = u^2$. If we fix u then x and z are constant so each corresponding grid curve is contained in a line parallel to the y-axis. (Since $y = v^2 \geq 0$, the grid curves are half-lines.) If v is held constant, then $y = v^2 =$ constant, so each grid curve is contained in a plane parallel to the xz-plane. Since x and z are functions of u only, the grid curves all have the same shape. The surface is the cylinder shown in graph I.

16. $x = (1-u)(3+\cos v)\cos 4\pi u$, $y = (1-u)(3+\cos v)\sin 4\pi u$, $z = 3u + (1-u)\sin v$. These equations correspond to graph V: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3,0,0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2}\cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. $x = \cos^3 u\,\cos^3 v$, $y = \sin^3 u\,\cos^3 v$, $z = \sin^3 v$. If $v = v_0$ is held constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curve lies in a horizontal plane. Several of the graphs exhibit horizontal grid curves, but the curves for this surface are neither ellipses nor straight lines, so graph III is the only possibility. (In fact, the horizontal grid curves here are members of the family $x = a\cos^3 u$, $y = a\sin^3 u$ and are called astroids.) The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, as then we have $x = \cos^3 u_0\,\cos^3 v$, $y = \sin^3 u_0\,\cos^3 v$ so the corresponding grid curve lies in the vertical plane $y = (\tan^3 u_0)x$ through the z-axis.

18. $x = \sin u,\ y = \cos u\,\sin v,\ z = \sin v$. If $v = v_0$ is fixed, then $z = \sin v_0$ is constant, and $x = \sin u$, $y = (\sin v_0)\cos u$ describe an ellipse that is contained in the horizontal plane $z = \sin v_0$. If $u = u_0$ is fixed, then $x = \sin u_0$ is constant, and $y = (\cos u_0)\sin v$, $z = \sin v \quad\Rightarrow\quad y = (\cos u_0)z$, so the grid curves are portions of lines through the x-axis contained in the plane $x = \sin u_0$ (parallel to the yz-plane). The surface is graph II.

19. From Example 3, parametric equations for the plane through the point $(0,0,0)$ that contains the vectors $\mathbf{a} = \langle 1,-1,0 \rangle$ and $\mathbf{b} = \langle 0,1,-1 \rangle$ are $x = 0 + u(1) + v(0) = u,\ y = 0 + u(-1) + v(1) = v - u,\ z = 0 + u(0) + v(-1) = -v$.

20. From Example 3, parametric equations for the plane through the point $(0,-1,5)$ that contains the vectors $\mathbf{a} = \langle 2,1,4 \rangle$ and $\mathbf{b} = \langle -3,2,5 \rangle$ are $x = 0 + u(2) + v(-3) = 2u - 3v,\ y = -1 + u(1) + v(2) = -1 + u + 2v$, $z = 5 + u(4) + v(5) = 5 + 4u + 5v$.

21. Solving the equation $4x^2 - 4y^2 - z^2 = 4$ for x gives $x^2 = 1 + y^2 + \frac{1}{4}z^2 \quad\Rightarrow\quad x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $x \geq 0$.) If we let y and z be the parameters, parametric equations are $y = y,\ z = z,\ x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$.

22. Solving the equation $x^2 + 2y^2 + 3z^2 = 1$ for y gives $y^2 = \frac{1}{2}(1 - x^2 - 3z^2) \quad\Rightarrow\quad y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$ (since we want the part of the ellipsoid that corresponds to $y \leq 0$). If we let x and z be the parameters, parametric equations are $x = x,\ z = z,\ y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$.

Alternate solution: The equation can be rewritten as $x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} + \dfrac{z^2}{\left(1/\sqrt{3}\right)^2} = 1$, and if we let $x = u\cos v$ and $z = \dfrac{1}{\sqrt{3}}u\sin v$, then $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)} = -\sqrt{\frac{1}{2}(1 - u^2\cos^2 v - u^2\sin^2 v)} = -\sqrt{\frac{1}{2}(1 - u^2)}$, where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

Second alternate solution: We can adapt the formulas for converting from spherical to rectangular coordinates as follows. We let $x = \sin\phi\cos\theta,\ y = \dfrac{1}{\sqrt{2}}\sin\phi\sin\theta,\ z = \dfrac{1}{\sqrt{3}}\cos\phi$; the surface is generated for $0 \leq \phi \leq \pi,\ \pi \leq \theta \leq 2\pi$.

23. Since the cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 4$ in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$.
Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

24. We can parametrize the cylinder as $x = 3\cos\theta,\ y = y,\ z = 3\sin\theta$. To restrict the surface to that portion above the xy-plane and between the planes $y = -4$ and $y = 4$ we require $0 \leq \theta \leq \pi$, $-4 \leq y \leq 4$.

25. In spherical coordinates, parametric equations are $x = 6\sin\phi\cos\theta$, $y = 6\sin\phi\sin\theta$, $z = 6\cos\phi$. The intersection of the sphere with the plane $z = 3\sqrt{3}$ corresponds to $z = 6\cos\phi = 3\sqrt{3} \quad\Rightarrow\quad \cos\phi = \frac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \frac{\pi}{6}$, and the plane $z = 0$ (the xy-plane) corresponds to $\phi = \frac{\pi}{2}$. Thus the surface is described by $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq 2\pi$.

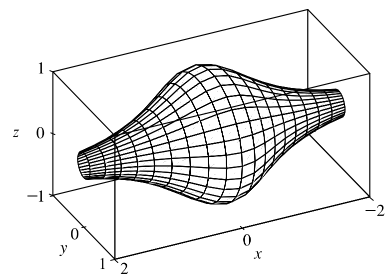
26. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \le x^2 + y^2 \le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \le s \le 1$ and $0 \le \theta \le 2\pi$.

27. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \le x \le 5$, $y \le 0$ to obtain the portion shown. To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter domain $0 \le u \le 5$, $\frac{\pi}{2} \le v \le \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \le x \le 5$ and $-3 \le z \le 3$.

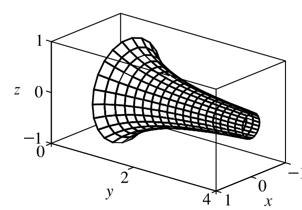
28. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$ will give the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$.

29. Using Equations 3, we have the parametrization $x = x$,

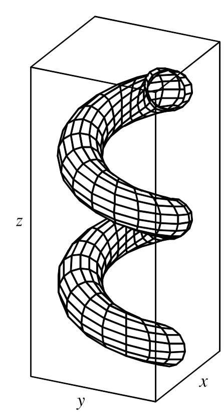
$$y = \frac{1}{1+x^2}\cos\theta, \quad z = \frac{1}{1+x^2}\sin\theta, \quad -2 \le x \le 2, \quad 0 \le \theta \le 2\pi.$$



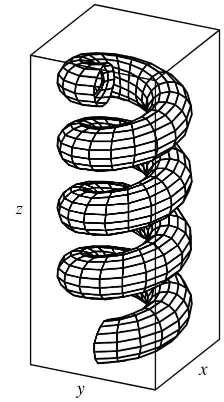
30. Letting θ be the angle of rotation about the y-axis (adapting Equations 3), we have the parametrization $x = (1/y)\cos\theta$, $y = y$, $z = (1/y)\sin\theta$, $y \ge 1$, $0 \le \theta \le 2\pi$.



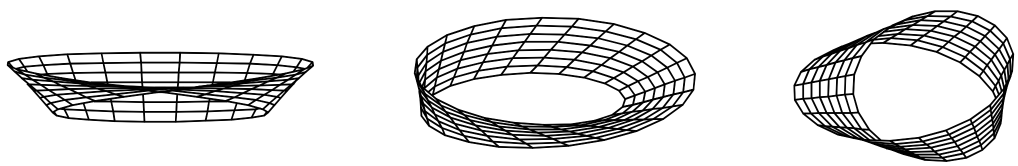
31. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v) \cos 2u$, $y = (2 + \sin v) \sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v) \cos 2u$, $y = (2 + \sin v) \sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \leq u \leq \pi$ while the original surface requires $0 \leq u \leq 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.



32. First we graph the surface as viewed from the front, then from two additional viewpoints.



The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 16.7.)

33. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.

34. $\mathbf{r}(u, v) = (u^2 + 1)\,\mathbf{i} + (v^3 + 1)\,\mathbf{j} + (u + v)\,\mathbf{k}$.

$\mathbf{r}_u = 2u\,\mathbf{i} + \mathbf{k}$ and $\mathbf{r}_v = 3v^2\,\mathbf{j} + \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -3v^2\,\mathbf{i} - 2u\,\mathbf{j} + 6uv^2\,\mathbf{k}$. Since the point $(5, 2, 3)$ corresponds to $u = 2$, $v = 1$, a normal vector to the surface at $(5, 2, 3)$ is $-3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$, and an equation of the tangent plane is $-3(x - 5) - 4(y - 2) + 12(z - 3) = 0$ or $3x + 4y - 12z = -13$.

35. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}\left(1, \frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

$\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is

$\mathbf{r}_u\left(1, \frac{\pi}{3}\right) \times \mathbf{r}_v\left(1, \frac{\pi}{3}\right) = \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) \times \left(-\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \mathbf{k}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j} + \mathbf{k}$. Thus an equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is $\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + 1\left(z - \frac{\pi}{3}\right) = 0$ or $\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3}$.

36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u\,\sin v\,\mathbf{j} + \sin v\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$.

$\mathbf{r}_u = \cos u\,\mathbf{i} - \sin u\sin v\,\mathbf{j}$ and $\mathbf{r}_v = \cos u\cos v\,\mathbf{j} + \cos v\,\mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is

$\mathbf{r}_u\left(\frac{\pi}{6},\frac{\pi}{6}\right)\times\mathbf{r}_v\left(\frac{\pi}{6},\frac{\pi}{6}\right)=\left(\frac{\sqrt{3}}{2}\,\mathbf{i}-\frac{1}{4}\,\mathbf{j}\right)\times\left(\frac{3}{4}\,\mathbf{j}+\frac{\sqrt{3}}{2}\,\mathbf{k}\right)=-\frac{\sqrt{3}}{8}\,\mathbf{i}-\frac{3}{4}\,\mathbf{j}+\frac{3\sqrt{3}}{8}\,\mathbf{k}.$

Thus an equation of the tangent plane at $\left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$ is $-\frac{\sqrt{3}}{8}\left(x-\frac{1}{2}\right)-\frac{3}{4}\left(y-\frac{\sqrt{3}}{4}\right)+\frac{3\sqrt{3}}{8}\left(z-\frac{1}{2}\right)=0$ or

$\sqrt{3}\,x+6y-3\sqrt{3}\,z=\frac{\sqrt{3}}{2}$ or $2x+4\sqrt{3}\,y-6z=1$.

37. $\mathbf{r}(u,v)=u^2\,\mathbf{i}+2u\sin v\,\mathbf{j}+u\cos v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(1,0)=(1,0,1)$.

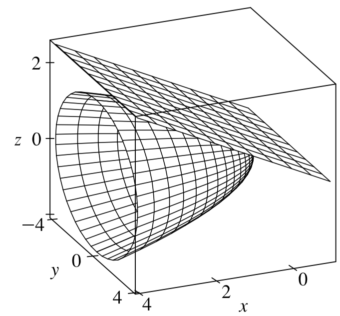
$\mathbf{r}_u=2u\,\mathbf{i}+2\sin v\,\mathbf{j}+\cos v\,\mathbf{k}$ and $\mathbf{r}_v=2u\cos v\,\mathbf{j}-u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1,0,1)$ is

$\mathbf{r}_u(1,0)\times\mathbf{r}_v(1,0)=(2\,\mathbf{i}+\mathbf{k})\times(2\,\mathbf{j})=-2\,\mathbf{i}+4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1,0,1)$ is

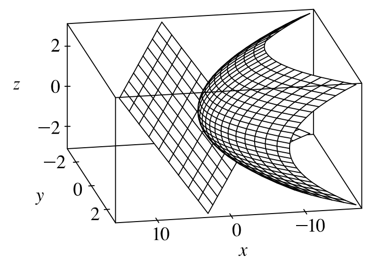
$-2(x-1)+0(y-0)+4(z-1)=0$ or $-x+2z=1$.



38. $\mathbf{r}(u,v)=(1-u^2-v^2)\,\mathbf{i}-v\,\mathbf{j}-u\,\mathbf{k}$.

$\mathbf{r}_u=-2u\,\mathbf{i}-\mathbf{k}$ and $\mathbf{r}_v=-2v\,\mathbf{i}-\mathbf{j}$. Since the point $(-1,-1,-1)$ corresponds to $u=1$, $v=1$, a normal vector to the surface at $(-1,-1,-1)$ is

$\mathbf{r}_u(1,1)\times\mathbf{r}_v(1,1)=(-2\,\mathbf{i}-\mathbf{k})\times(-2\,\mathbf{i}-\mathbf{j})=-\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k}$.



Thus an equation of the tangent plane is $-1(x+1)+2(y+1)+2(z+1)=0$ or $-x+2y+2z=-3$.

39. The surface S is given by $z=f(x,y)=6-3x-2y$ which intersects the xy-plane in the line $3x+2y=6$, so D is the triangular region given by $\left\{(x,y)\;\middle|\;0\le x\le 2,\ 0\le y\le 3-\frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$\begin{aligned}A(S)&=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\&=\iint_D\sqrt{1+(-3)^2+(-2)^2}\,dA=\sqrt{14}\iint_D dA=\sqrt{14}\,A(D)=\sqrt{14}\left(\tfrac{1}{2}\cdot 2\cdot 3\right)=3\sqrt{14}.\end{aligned}$$

40. $\mathbf{r}(u,v)=\langle u+v,2-3u,1+u-v\rangle \quad\Rightarrow\quad \mathbf{r}_u=\langle 1,-3,1\rangle$, $\mathbf{r}_v=\langle 1,0,-1\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v=\langle 3,2,3\rangle$. Then by Definition 6,

$$A(S)=\iint_D|\,\mathbf{r}_u\times\mathbf{r}_v\,|\,dA=\int_0^2\int_{-1}^1|\,\langle 3,2,3\rangle\,|\,dv\,du=\sqrt{22}\int_0^2 du\int_{-1}^1 dv=\sqrt{22}\,(2)(2)=4\sqrt{22}$$

41. Here we can write $z=f(x,y)=\frac{1}{3}-\frac{1}{3}x-\frac{2}{3}y$ and D is the disk $x^2+y^2\le 3$, so by Formula 9 the area of the surface is

$$\begin{aligned}A(S)&=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\iint_D\sqrt{1+\left(-\tfrac{1}{3}\right)^2+\left(-\tfrac{2}{3}\right)^2}\,dA=\tfrac{\sqrt{14}}{3}\iint_D dA\\&=\tfrac{\sqrt{14}}{3}\,A(D)=\tfrac{\sqrt{14}}{3}\cdot\pi\left(\sqrt{3}\right)^2=\sqrt{14}\,\pi\end{aligned}$$

42. $z=f(x,y)=\sqrt{x^2+y^2} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x}=\dfrac{1}{2}\left(x^2+y^2\right)^{-1/2}\cdot 2x=\dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial z}{\partial y}=\dfrac{y}{\sqrt{x^2+y^2}}$, and

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}=\sqrt{2}$$

[continued]

Here D is given by $\{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\}$, so by Formula 9, the surface area of S is

$$A(S) = \iint_D \sqrt{2}\,dA = \int_0^1 \int_{x^2}^{x} \sqrt{2}\,dy\,dx = \sqrt{2}\int_0^1 \left(x - x^2\right) dx = \sqrt{2}\,\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \sqrt{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{6}$$

43. $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (\sqrt{x}\,)^2 + \left(\sqrt{y}\,\right)^2}\,dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\,dy\,dx \\ &= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx \\ &= \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

44. $z = f(x,y) = 4 - 2x^2 + y$ and $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (-4x)^2 + (1)^2}\,dA = \int_0^1 \int_0^x \sqrt{16x^2 + 2}\,dy\,dx = \int_0^1 x\,\sqrt{16x^2 + 2}\,dx \\ &= \tfrac{1}{32} \cdot \tfrac{2}{3}(16x^2 + 2)^{3/2}\Big]_0^1 = \tfrac{1}{48}(18^{3/2} - 2^{3/2}) = \tfrac{1}{48}(54\sqrt{2} - 2\sqrt{2}\,) = \tfrac{13}{12}\sqrt{2} \end{aligned}$$

45. $z = f(x,y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2}\,dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}\left(r^2 + 1\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right) \end{aligned}$$

46. A parametric representation of the surface is $x = z^2 + y$, $y = y$, $z = z$ with $0 \le y \le 2$, $0 \le z \le 2$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - \mathbf{j} - 2z\,\mathbf{k}$. Then

$$\begin{aligned} A(S) &= \iint_D |\,\mathbf{r}_y \times \mathbf{r}_z\,|\,dA = \int_0^2 \int_0^2 \sqrt{1 + 1 + 4z^2}\,dy\,dz = \int_0^2 2\sqrt{2 + 4z^2}\,dz \\ &= \left[2 \cdot \tfrac{1}{2}\left(z\sqrt{2 + 4z^2} + \ln\left(2z + \sqrt{2 + 4z^2}\,\right)\right)\right]_0^2 \quad \left[\begin{array}{c}\text{Use trigonometric substitution} \\ \text{or Formula 21 in the Table of Integrals}\end{array}\right] \\ &= 6\sqrt{2} + \ln\left(4 + 3\sqrt{2}\,\right) - \ln\sqrt{2} \ \text{ or } \ 6\sqrt{2} + \ln\frac{4 + 3\sqrt{2}}{\sqrt{2}} = 6\sqrt{2} + \ln\left(2\sqrt{2} + 3\right) \end{aligned}$$

Note: In general, if $x = f(y,z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\,\mathbf{j} - \dfrac{\partial f}{\partial z}\,\mathbf{k}$ and $A\,(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA.$

47. A parametric representation of the surface is $x = x$, $y = x^2 + z^2$, $z = z$ with $0 \le x^2 + z^2 \le 16$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x,z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA.$

Then

$$\begin{aligned} A(S) &= \iint\limits_{0 \le x^2 + z^2 \le 16} \sqrt{1 + 4x^2 + 4z^2}\,dA = \int_0^{2\pi} \int_0^4 \sqrt{1 + 4r^2}\ r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta\,\int_0^4 r\,\sqrt{1 + 4r^2}\,dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^4 = \tfrac{\pi}{6}\left(65^{3/2} - 1\right) \end{aligned}$$

48. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi \quad \Rightarrow$

$\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$\begin{aligned} A(S) &= \int_0^\pi \int_0^1 \sqrt{1+u^2}\,du\,dv = \int_0^\pi dv \int_0^1 \sqrt{1+u^2}\,du \\ &= \pi\left[\tfrac{u}{2}\sqrt{u^2+1} + \tfrac{1}{2}\ln\left|u+\sqrt{u^2+1}\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] \end{aligned}$$

49. $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2 \quad \Rightarrow \quad \mathbf{r}_u = \langle 2u, v, 0\rangle$, $\mathbf{r}_v = \langle 0, u, v\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \left\langle v^2, -2uv, 2u^2\right\rangle$.

Then

$$\begin{aligned} A(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^1\int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dv\,du = \int_0^1\int_0^2 \sqrt{(v^2+2u^2)^2}\,dv\,du \\ &= \int_0^1\int_0^2 (v^2+2u^2)\,dv\,du = \int_0^1 \left[\tfrac{1}{3}v^3 + 2u^2v\right]_{v=0}^{v=2} du = \int_0^1 \left(\tfrac{8}{3} + 4u^2\right) du = \left[\tfrac{8}{3}u + \tfrac{4}{3}u^3\right]_0^1 = 4 \end{aligned}$$

50. The cylinder encloses separate portions of the sphere in the upper and lower halves. The top half of the sphere is $z = f(x,y) = \sqrt{b^2 - x^2 - y^2}$ and D is given by $\left\{(x,y) \mid x^2 + y^2 \le a^2\right\}$. By Formula 9, the surface area of the upper enclosed portion is

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{-x}{\sqrt{b^2-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{b^2-x^2-y^2}}\right)^2}\,dA = \iint_D \sqrt{1 + \frac{x^2+y^2}{b^2-x^2-y^2}}\,dA \\ &= \iint_D \sqrt{\frac{b^2}{b^2-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^a \frac{b}{\sqrt{b^2-r^2}}\,r\,dr\,d\theta = b\int_0^{2\pi} d\theta \int_0^a \frac{r}{\sqrt{b^2-r^2}}\,dr \\ &= b\left[\theta\right]_0^{2\pi}\left[-\sqrt{b^2-r^2}\,\right]_0^a = 2\pi b\left(-\sqrt{b^2-a^2} + \sqrt{b^2-0}\right) = 2\pi b\left(b - \sqrt{b^2-a^2}\,\right) \end{aligned}$$

The lower portion of the sphere enclosed by the cylinder has identical shape, so the total area is $2A = 4\pi b\left(b - \sqrt{b^2-a^2}\,\right)$.

51. From Formula 9 with $z = f(x,y)$, we have $A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$. Since $|f_x| \le 1$ and $|f_y| \le 1$, we know $0 \le (f_x)^2 \le 1$ and $0 \le (f_y)^2 \le 1$, so $1 \le 1 + (f_x)^2 + (f_y)^2 \le 3 \quad \Rightarrow \quad 1 \le \sqrt{1 + (f_x)^2 + (f_y)^2} \le \sqrt{3}$. By Property 15.2.10, $\iint_D 1\,dA \le \iint_D \sqrt{1+(f_x)^2+(f_y)^2}\,dA \le \iint_D \sqrt{3}\,dA \quad \Rightarrow \quad A(D) \le A(S) \le \sqrt{3}\,A(D) \quad \Rightarrow$ $\pi R^2 \le A(S) \le \sqrt{3}\pi R^2$.

52. $z = f(x,y) = \cos(x^2+y^2)$ with $x^2 + y^2 \le 1$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (-2x\sin(x^2+y^2))^2 + (-2y\sin(x^2+y^2))^2}\,dA \\ &= \iint_D \sqrt{1 + 4x^2\sin^2(x^2+y^2) + 4y^2\sin^2(x^2+y^2)}\,dA = \iint_D \sqrt{1 + 4(x^2+y^2)\sin^2(x^2+y^2)}\,dA \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2\sin^2(r^2)}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr \\ &= 2\pi\int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr \approx 4.1073 \end{aligned}$$

53. $z = f(x,y) = \ln(x^2+y^2+2)$ with $x^2 + y^2 \le 1$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{2x}{x^2+y^2+2}\right)^2 + \left(\frac{2y}{x^2+y^2+2}\right)^2}\,dA = \iint_D \sqrt{1 + \frac{4x^2+4y^2}{(x^2+y^2+2)^2}}\,dA \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1 + \frac{4r^2}{(r^2+2)^2}}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{\frac{(r^2+2)^2+4r^2}{(r^2+2)^2}}\,dr = 2\pi\int_0^1 \frac{r\sqrt{r^4+8r^2+4}}{r^2+2}\,dr \\ &\approx 3.5618 \end{aligned}$$

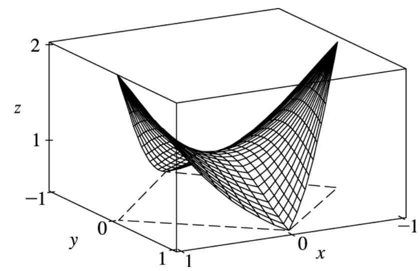
54. Let $f(x,y)=\dfrac{1+x^2}{1+y^2}$. Then $f_x=\dfrac{2x}{1+y^2}$,

$f_y=(1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right]=-\dfrac{2y(1+x^2)}{(1+y^2)^2}$.

We use a CAS to estimate

$\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{1+f_x^2+f_y^2}\,dy\,dx\approx 2.6959$.

In order to graph only the part of the surface above the square, we use $-(1-|x|)\le y\le 1-|x|$ as the y-range in our plot command.



55. (a) $z=\dfrac{1}{1+x^2+y^2}\quad\Rightarrow\quad\dfrac{\partial z}{\partial x}=\dfrac{-2x}{(1+x^2+y^2)^2}$ and $\dfrac{\partial z}{\partial y}=\dfrac{-2y}{(1+x^2+y^2)^2}$.

$$A(S)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\int_0^6\int_0^4\sqrt{1+\frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\,dy\,dx.$$

Using the Midpoint Rule with $f(x,y)=\sqrt{1+\dfrac{4x^2+4y^2}{(1+x^2+y^2)^4}}$, $m=3$, $n=2$ we have

$$A(S)\approx\sum_{i=1}^{3}\sum_{j=1}^{2}f(\overline{x}_i,\overline{y}_j)\,\Delta A=4\,[f(1,1)+f(1,3)+f(3,1)+f(3,3)+f(5,1)+f(5,3)]\approx 24.2055$$

(b) Using a CAS, we have $A(S)=\displaystyle\int_0^6\int_0^4\sqrt{1+\frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\,dy\,dx\approx 24.2476$. This agrees with the estimate in part (a) to the first decimal place.

56. $\mathbf{r}(u,v)=\langle\cos^3 u\,\cos^3 v,\sin^3 u\,\cos^3 v,\sin^3 v\rangle$, so $\mathbf{r}_u=\langle -3\cos^2 u\,\sin u\cos^3 v,3\sin^2 u\,\cos u\cos^3 v,0\rangle$,

$\mathbf{r}_v=\langle -3\cos^3 u\,\cos^2 v\,\sin v,-3\sin^3 u\,\cos^2 v\,\sin v,3\sin^2 v\,\cos v\rangle$, and

$\mathbf{r}_u\times\mathbf{r}_v=\langle 9\cos u\,\sin^2 u\,\cos^4 v\,\sin^2 v,9\cos^2 u\,\sin u\,\cos^4 v\,\sin^2 v,9\cos^2 u\,\sin^2 u\,\cos^5 v\,\sin v\rangle$. Then

$$\begin{aligned}|\mathbf{r}_u\times\mathbf{r}_v|&=9\sqrt{\cos^2 u\,\sin^4 u\,\cos^8 v\,\sin^4 v+\cos^4 u\,\sin^2 u\,\cos^8 v\,\sin^4 v+\cos^4 u\,\sin^4 u\,\cos^{10} v\,\sin^2 v}\\&=9\sqrt{\cos^2 u\,\sin^2 u\,\cos^8 v\,\sin^2 v\,(\sin^2 v+\cos^2 u\,\sin^2 u\,\cos^2 v)}\\&=9\cos^4 v\,|\cos u\,\sin u\,\sin v|\sqrt{\sin^2 v+\cos^2 u\,\sin^2 u\,\cos^2 v}\end{aligned}$$

Using a CAS, we have $A(S)=\int_0^{\pi}\int_0^{2\pi}9\cos^4 v\,|\cos u\,\sin u\,\sin v|\sqrt{\sin^2 v+\cos^2 u\,\sin^2 u\,\cos^2 v}\,dv\,du\approx 4.4506$.

57. $z=1+2x+3y+4y^2$, so

$$A(S)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\int_1^4\int_0^1\sqrt{1+4+(3+8y)^2}\,dy\,dx=\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx.$$

Using a CAS, we have

$\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx=\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right)-\frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)$

or $\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$.

58. (a) $x = au\cos v$, $y = bu\sin v$, $z = u^2$, $0 \le u \le 2$, $0 \le v \le 2\pi \quad \Rightarrow \quad \mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$,

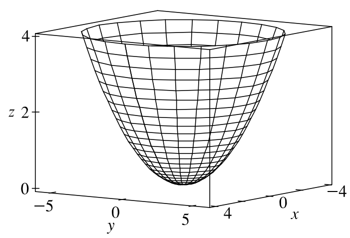
$\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and $\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$A(S) = \int_0^{2\pi}\int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \quad \Rightarrow \quad x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D, notice that $0 \le u \le 2 \quad \Rightarrow \quad 0 \le z \le 4 \quad \Rightarrow \quad 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have

$$A(S) = \int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx.$$

(c)



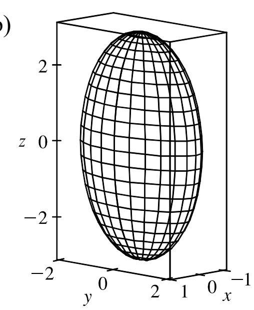
(d) We substitute $a = 2$, $b = 3$ in the integral in part (a) to get $A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set `Digits:=7;` (in Maple) or use the approximation command `N` (in Mathematica). We find that $A(S) \approx 115.6596$.

59. (a) $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u \quad \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2$$
$$= \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

(b)



(c) From the parametric equations (with $a = 1$, $b = 2$, and $c = 3$), we calculate $\mathbf{r}_u = \cos u\cos v\,\mathbf{i} + 2\cos u\sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and

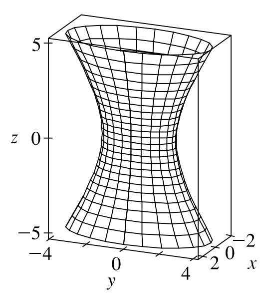
$\mathbf{r}_v = -\sin u\sin v\,\mathbf{i} + 2\sin u\cos v\,\mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6\sin^2 u\cos v\,\mathbf{i} + 3\sin^2 u\sin v\,\mathbf{j} + 2\sin u\cos u\,\mathbf{k}$, and the surface area is given by $A(S) = \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$

60. (a) $x = a\cosh u\,\cos v$, $\quad y = b\cosh u\,\sin v$, $\quad z = c\sinh u \quad \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u\,\cos^2 v + \cosh^2 u\,\sin^2 v - \sinh^2 u$$
$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b)



(c) $\mathbf{r}_u = \sinh u\,\cos v\,\mathbf{i} + 2\sinh u\,\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and

$\mathbf{r}_v = -\cosh u\,\sin v\,\mathbf{i} + 2\cosh u\,\cos v\,\mathbf{j}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6\cosh^2 u\,\cos v\,\mathbf{i} - 3\cosh^2 u\,\sin v\,\mathbf{j} + 2\cosh u\,\sinh u\,\mathbf{k}$.

We integrate between $u = \sinh^{-1}(-1) = -\ln\left(1 + \sqrt{2}\right)$ and $u = \sinh^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$, since then z varies between

-3 and 3, as desired. So the surface area is

$$A(S) = \int_0^{2\pi} \int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

$$= \int_0^{2\pi} \int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \sqrt{36 \cosh^4 u \, \cos^2 v + 9 \cosh^4 u \, \sin^2 v + 4 \cosh^2 u \, \sinh^2 u} \, du \, dv$$

61. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \{(x, y) \mid x^2 + y^2 \leq 3\}$.

$$A(S) = \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2} \, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}} \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r \, dr}{\sqrt{4 - r^2}} \, d\theta = \int_0^{2\pi} \Big[-2(4 - r^2)^{1/2}\Big]_{r=0}^{r=\sqrt{3}} \, d\theta$$

$$= \int_0^{2\pi} (-2 + 4) \, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

62. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x$, $y = \sqrt{1 - z^2}$, $z = z$ with $x^2 + z^2 \leq 1$. Then $\mathbf{r}(x, z) = \langle x, \sqrt{1 - z^2}, z \rangle \Rightarrow \mathbf{r}_x = \langle 1, 0, 0 \rangle$, $\mathbf{r}_z = \langle 0, -z/\sqrt{1 - z^2}, 1 \rangle$ and $\mathbf{r}_x \times \mathbf{r}_z = \langle 0, -1, -z/\sqrt{1 - z^2} \rangle \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{1 + \dfrac{z^2}{1 - z^2}} = \dfrac{1}{\sqrt{1 - z^2}}$.

$$A(S) = \iint\limits_{x^2 + z^2 \leq 1} |\mathbf{r}_x \times \mathbf{r}_z| \, dA = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}} \, dx \, dz = 4 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}} \, dx \, dz \quad \begin{bmatrix}\text{by the symmetry} \\ \text{of the surface}\end{bmatrix}$$

This integral is improper [when $z = 1$], so

$$A(S) = \lim_{t \to 1^-} 4 \int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}} \, dx \, dz = \lim_{t \to 1^-} 4 \int_0^t \frac{\sqrt{1 - z^2}}{\sqrt{1 - z^2}} \, dz = \lim_{t \to 1^-} 4 \int_0^t dz = \lim_{t \to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

Alternate solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x, \theta) = \langle x, \cos\theta, \sin\theta \rangle$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $x^2 + z^2 \leq 1 \Leftrightarrow x^2 + \sin^2\theta \leq 1 \Leftrightarrow$ $-\sqrt{1 - \sin^2\theta} \leq x \leq \sqrt{1 - \sin^2\theta} \Leftrightarrow -\cos\theta \leq x \leq \cos\theta$. Then $\mathbf{r}_x = \langle 1, 0, 0 \rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta \rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta \rangle \Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = 1$, so

$A(S) = \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta} 1 \, dx \, d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta \, d\theta = 2\sin\theta\Big]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.

63. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$.

Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$,

$z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or

$\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or

$\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\!\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$.

Hence $D = \left\{(\phi, \theta) \mid 0 \le \phi \le \frac{\pi}{2},\ \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi} a^2\sin\phi\,d\theta\,d\phi = a^2\int_0^{\pi/2}(\pi - 2\phi)\sin\phi\,d\phi \\ &= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$.

Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$\begin{aligned} A(S_1) &= \iint\limits_{0 \le (x-(a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} -a(a^2 - r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} a^2\left[1 - (1 - \cos^2\theta)^{1/2}\right]d\theta \\ &= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\,d\theta = 2a^2\int_0^{\pi/2}(1 - \sin\theta)\,d\theta = 2a^2\left(\tfrac{\pi}{2} - 1\right) \end{aligned}$$

Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

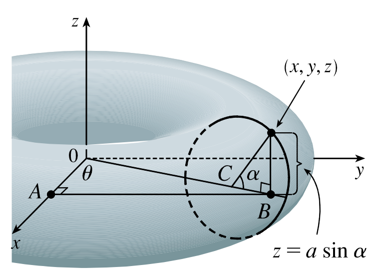
(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

64. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But

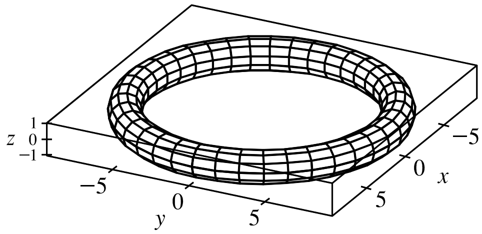
$|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that

$y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so

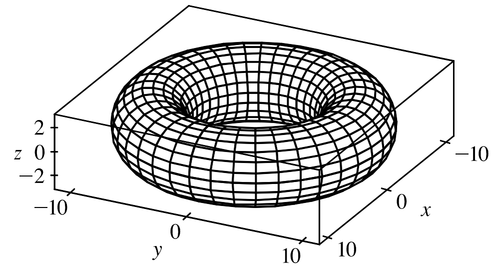
$x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



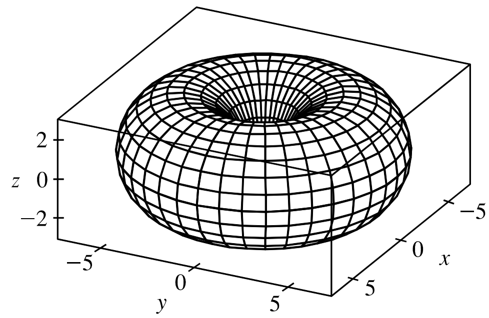
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$,

$\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k} \\ &= -a\,(b + a\cos\alpha)\left[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A\,(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab.$

16.7 Surface Integrals

1. The box is a cube where each face has surface area 4. The centers of the faces are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. For each face we take the point P_{ij}^* to be the center of the face and $f(x, y, z) = \cos(x + 2y + 3z)$, so by Definition 1,

$$\begin{aligned}\iint_S f(x, y, z)\,dS &\approx [f(1, 0, 0)](4) + [f(-1, 0, 0)](4) + [f(0, 1, 0)](4) \\ &\quad + [f(0, -1, 0)](4) + [f(0, 0, 1)](4) + [f(0, 0, -1)](4) \\ &= 4\,[\cos 1 + \cos(-1) + \cos 2 + \cos(-2) + \cos 3 + \cos(-3)] \approx -6.93\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0, 0, 1)$ as a sample point in the top disk, $(0, 0, -1)$ in the bottom disk, and $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ in the four

quarter-cylinders. Then $\iint_S f(x,y,z)\,dS$ can be approximated by the Riemann sum

$$f(1,0,0)(\pi)+f(-1,0,0)(\pi)+f(0,1,0)\,(\pi)+f(0,-1,0)(\pi)+f(0,0,1)(\pi)+f(0,0,-1)(\pi)$$
$$=(2+2+3+3+4+4)\pi=18\pi\approx 56.5.$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S=\frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2=25\pi$. Then $(\pm 3,\pm 4,5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\iint_H f(x,y,z)\,dS &\approx f(3,4,5)\,\Delta S+f(3,-4,5)\,\Delta S+f(-3,4,5)\,\Delta S+f(-3,-4,5)\,\Delta S\\ &=(7+8+9+12)(25\pi)=900\pi\approx 2827\end{aligned}$$

4. On the surface, $f(x,y,z)=g\left(\sqrt{x^2+y^2+z^2}\right)=g(2)=-5$. So since the area of a sphere is $4\pi r^2$,

$\iint_S f(x,y,z)\,dS=\iint_S g(2)\,dS=-5\iint_S dS=-5[4\pi(2)^2]=-80\pi$.

5. $\mathbf{r}(u,v)=(u+v)\,\mathbf{i}+(u-v)\,\mathbf{j}+(1+2u+v)\,\mathbf{k}$, $0\le u\le 2$, $0\le v\le 1$ and

$\mathbf{r}_u\times\mathbf{r}_v=(\mathbf{i}+\mathbf{j}+2\,\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})=3\,\mathbf{i}+\mathbf{j}-2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{3^2+1^2+(-2)^2}=\sqrt{14}$. Then by Formula 2,

$$\begin{aligned}\iint_S(x+y+z)\,dS&=\iint_D(u+v+u-v+1+2u+v)\ |\mathbf{r}_u\times\mathbf{r}_v|\ dA=\int_0^1\int_0^2(4u+v+1)\cdot\sqrt{14}\,du\,dv\\ &=\sqrt{14}\int_0^1\left[2u^2+uv+u\right]_{u=0}^{u=2}dv=\sqrt{14}\int_0^1(2v+10)\,dv=\sqrt{14}\left[v^2+10v\right]_0^1=11\sqrt{14}\end{aligned}$$

6. $\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+u\,\mathbf{k}$, $0\le u\le 1$, $0\le v\le \pi/2$ and

$\mathbf{r}_u\times\mathbf{r}_v=(\cos v\,\mathbf{i}+\sin v\,\mathbf{j}+\mathbf{k})\times(-u\sin v\,\mathbf{i}+u\cos v\,\mathbf{j})=-u\cos v\,\mathbf{i}-u\sin v\,\mathbf{j}+u\,\mathbf{k}\quad\Rightarrow$

$|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{u^2\cos^2 v+u^2\sin^2 v+u^2}=\sqrt{2u^2}=\sqrt{2}\,u$ [since $u\ge 0$]. Then by Formula 2,

$$\begin{aligned}\iint_S xyz\,dS&=\iint_D(u\cos v)(u\sin v)(u)\ |\mathbf{r}_u\times\mathbf{r}_v|\ dA=\int_0^1\int_0^{\pi/2}(u^3\sin v\cos v)\cdot\sqrt{2}\,u\,dv\,du\\ &=\sqrt{2}\int_0^1u^4\,du\ \int_0^{\pi/2}\sin v\cos v\,dv=\sqrt{2}\left[\tfrac15u^5\right]_0^1\ \left[\tfrac12\sin^2 v\right]_0^{\pi/2}=\sqrt{2}\cdot\tfrac15\cdot\tfrac12=\tfrac{1}{10}\sqrt{2}\end{aligned}$$

7. $\mathbf{r}(u,v)=\langle u\cos v,u\sin v,v\rangle,\ 0\le u\le 1,\ 0\le v\le\pi$ and

$\mathbf{r}_u\times\mathbf{r}_v=\langle\cos v,\sin v,0\rangle\times\langle -u\sin v,u\cos v,1\rangle=\langle\sin v,-\cos v,u\rangle\quad\Rightarrow$

$|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{\sin^2 v+\cos^2 v+u^2}=\sqrt{u^2+1}$. Then

$$\begin{aligned}\iint_S y\,dS&=\iint_D(u\sin v)\ |\mathbf{r}_u\times\mathbf{r}_v|\ dA=\int_0^1\int_0^\pi(u\sin v)\cdot\sqrt{u^2+1}\,dv\,du=\int_0^1u\sqrt{u^2+1}\,du\ \int_0^\pi\sin v\,dv\\ &=\left[\tfrac13(u^2+1)^{3/2}\right]_0^1\ \left[-\cos v\right]_0^\pi=\tfrac13(2^{3/2}-1)\cdot 2=\tfrac23(2\sqrt{2}-1)\end{aligned}$$

8. $\mathbf{r}(u,v)=\left\langle 2uv,u^2-v^2,u^2+v^2\right\rangle$, $u^2+v^2\le 1$ and

$\mathbf{r}_u\times\mathbf{r}_v=\langle 2v,2u,2u\rangle\times\langle 2u,-2v,2v\rangle=\left\langle 8uv,4u^2-4v^2,-4u^2-4v^2\right\rangle$, so

$$\begin{aligned}|\mathbf{r}_u\times\mathbf{r}_v|&=\sqrt{(8uv)^2+(4u^2-4v^2)^2+(-4u^2-4v^2)^2}=\sqrt{64u^2v^2+32u^4+32v^4}\\ &=\sqrt{32(u^2+v^2)^2}=4\sqrt{2}\,(u^2+v^2)\end{aligned}$$

[continued]

Then

$$\iint_S (x^2+y^2)\,dS = \iint_D \left[(2uv)^2+(u^2-v^2)^2\right]\,|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D (4u^2v^2+u^4-2u^2v^2+v^4)\cdot 4\sqrt{2}\,(u^2+v^2)\,dA$$
$$= 4\sqrt{2}\iint_D (u^4+2u^2v^2+v^4)\,(u^2+v^2)\,dA = 4\sqrt{2}\iint_D (u^2+v^2)^3\,dA = 4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3\,r\,dr\,d\theta$$
$$= 4\sqrt{2}\int_0^{2\pi} d\theta\int_0^1 r^7\,dr = 4\sqrt{2}\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{8}r^8\big]_0^1 = 4\sqrt{2}\cdot 2\pi\cdot\tfrac{1}{8} = \sqrt{2}\,\pi$$

9. $z=1+2x+3y$, so $\dfrac{\partial z}{\partial x}=2$ and $\dfrac{\partial z}{\partial y}=3$. The surface is the graph of a function, so by Formula 4,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\,\sqrt{4+9+1}\,dy\,dx$$
$$= \sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3\left[\tfrac{1}{2}x^2y^2+x^3y^2+x^2y^3\right]_{y=0}^{y=2}dx$$
$$= \sqrt{14}\int_0^3 (10x^2+4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3+x^4\right]_0^3 = 171\sqrt{14}$$

10. The surface S is given by $z=4-2x-2y$, which intersects the xy-plane in the line $2x+2y=4$, so $D=\{(x,y)\mid 0\le x\le 2,\ 0\le y\le 2-x\}$. The surface is the graph of a function, so by Formula 4,

$$\iint_S xz\,dS = \iint_D x(4-2x-2y)\,\sqrt{(-2)^2+(-2)^2+1}\,dA = 3\int_0^2\int_0^{2-x}\big(4x-2x^2-2xy\big)\,dy\,dx$$
$$= 3\int_0^2\left[4xy-2x^2y-xy^2\right]_{y=0}^{y=2-x}dx = 3\int_0^2\left[4x(2-x)-2x^2(2-x)-x(2-x)^2\right]dx$$
$$= 3\int_0^2\big(x^3-4x^2+4x\big)\,dx = 3\left[\tfrac{1}{4}x^4-\tfrac{4}{3}x^3+2x^2\right]_0^2 = 3\left(4-\tfrac{32}{3}+8\right) = 4$$

11. An equation of the plane through the points $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$ is $4x-2y+z=4$ (see Example 12.5.5), so S is the region in the plane $z=4-4x+2y$ over $D=\{(x,y)\mid 0\le x\le 1,\ 2x-2\le y\le 0\}$. Thus, by Formula 4,

$$\iint_S x\,dS = \iint_D x\,\sqrt{(-4)^2+(2)^2+1}\,dA = \sqrt{21}\int_0^1\int_{2x-2}^0 x\,dy\,dx = \sqrt{21}\int_0^1\,[xy]_{y=2x-2}^{y=0}\,dx$$
$$= \sqrt{21}\int_0^1(-2x^2+2x)\,dx = \sqrt{21}\left[-\tfrac{2}{3}x^3+x^2\right]_0^1 = \sqrt{21}\left(-\tfrac{2}{3}+1\right) = \tfrac{\sqrt{21}}{3}$$

12. $z=\frac{2}{3}(x^{3/2}+y^{3/2})$ and by Formula 4,

$$\iint_S y\,dS = \iint_D y\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA = \int_0^1\int_0^1 y\,\sqrt{x+y+1}\,dx\,dy$$
$$= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1}dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2}-(y+1)^{3/2}\right]dy$$

Substituting $u=y+2$ in the first term and $t=y+1$ in the second, we have

$$\iint_S y\,dS = \tfrac{2}{3}\int_2^3(u-2)u^{3/2}\,du-\tfrac{2}{3}\int_1^2(t-1)t^{3/2}\,dt = \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2}-\tfrac{4}{5}u^{5/2}\right]_2^3-\tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2}-\tfrac{2}{5}t^{5/2}\right]_1^2$$
$$= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2}-2^{7/2})-\tfrac{4}{5}(3^{5/2}-2^{5/2})-\tfrac{2}{7}(2^{7/2}-1)+\tfrac{2}{5}(2^{5/2}-1)\right]$$
$$= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3}+\tfrac{8}{35}\sqrt{2}-\tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3}+4\sqrt{2}-2\right)$$

13. Using y and z as parameters, we have $\mathbf{r}(y,z)=(y^2+z^2)\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, $y^2+z^2\le 1$. Then $\mathbf{r}_y\times\mathbf{r}_z=(2y\,\mathbf{i}+\mathbf{j})\times(2z\,\mathbf{i}+\mathbf{k})=\mathbf{i}-2y\,\mathbf{j}-2z\,\mathbf{k}$ and $|\mathbf{r}_y\times\mathbf{r}_z|=\sqrt{1+4y^2+4z^2}=\sqrt{1+4(y^2+z^2)}$. Thus, by

Formula 2,

$$\iint_S z^2\,dS = \iint\limits_{y^2+z^2\le 1} z^2\sqrt{1+4(y^2+z^2)}\,dA = \int_0^{2\pi}\int_0^1 (r\sin\theta)^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1 r^3\sqrt{1+4r^2}\,dr \qquad \left[\text{let } u=1+4r^2 \;\Rightarrow\; r^2=\tfrac14(u-1) \text{ and } r\,dr=\tfrac18 du\right]$$
$$= \left[\tfrac12\theta-\tfrac14\sin 2\theta\right]_0^{2\pi}\int_1^5 \tfrac14(u-1)\sqrt{u}\cdot\tfrac18 du = \pi\cdot\tfrac{1}{32}\int_1^5 (u^{3/2}-u^{1/2})\,du = \tfrac{1}{32}\pi\left[\tfrac25 u^{5/2}-\tfrac23 u^{3/2}\right]_1^5$$
$$= \tfrac{1}{32}\pi\left[\tfrac25(5)^{5/2}-\tfrac23(5)^{3/2}-\tfrac25+\tfrac23\right] = \tfrac{1}{32}\pi\left(\tfrac{20}{3}\sqrt5+\tfrac{4}{15}\right) = \tfrac{1}{120}\pi\left(25\sqrt5+1\right)$$

14. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i}+\sqrt{x^2+z^2}\,\mathbf{j}+z\,\mathbf{k}$, $x^2+z^2\le 25$. Then

$\mathbf{r}_x\times\mathbf{r}_z = \left(\mathbf{i}+\dfrac{x}{\sqrt{x^2+z^2}}\,\mathbf{j}\right)\times\left(\dfrac{z}{\sqrt{x^2+z^2}}\,\mathbf{j}+\mathbf{k}\right) = \dfrac{x}{\sqrt{x^2+z^2}}\,\mathbf{i}-\mathbf{j}+\dfrac{z}{\sqrt{x^2+z^2}}\,\mathbf{k}$ and

$|\mathbf{r}_x\times\mathbf{r}_z| = \sqrt{\dfrac{x^2}{x^2+z^2}+1+\dfrac{z^2}{x^2+z^2}} = \sqrt{\dfrac{x^2+z^2}{x^2+z^2}+1} = \sqrt2$. Thus, by Formula 2,

$$\iint_S y^2z^2\,dS = \iint\limits_{x^2+z^2\le 25}(x^2+z^2)z^2\sqrt2\,dA = \sqrt2\int_0^{2\pi}\int_0^5 r^2(r\sin\theta)^2\,r\,dr\,d\theta$$
$$= \sqrt2\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^5 r^5\,dr = \sqrt2\left[\tfrac12\theta-\tfrac14\sin 2\theta\right]_0^{2\pi}\left[\tfrac16 r^6\right]_0^5$$
$$= \sqrt2\,(\pi)\cdot\tfrac16(15{,}625-0) = \frac{15{,}625\sqrt2}{6}\pi$$

15. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i}+(x^2+4z)\,\mathbf{j}+z\,\mathbf{k}$, $0\le x\le 1, 0\le z\le 1$. Then

$\mathbf{r}_x\times\mathbf{r}_z = (\mathbf{i}+2x\,\mathbf{j})\times(4\,\mathbf{j}+\mathbf{k}) = 2x\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ and $|\mathbf{r}_x\times\mathbf{r}_z| = \sqrt{4x^2+1+16} = \sqrt{4x^2+17}$. Thus, by Formula 2,

$$\iint_S x\,dS = \int_0^1\int_0^1 x\sqrt{4x^2+17}\,dz\,dx = \int_0^1 x\sqrt{4x^2+17}\,dx = \left[\tfrac18\cdot\tfrac23(4x^2+17)^{3/2}\right]_0^1$$
$$= \tfrac{1}{12}(21^{3/2}-17^{3/2}) = \tfrac{1}{12}\left(21\sqrt{21}-17\sqrt{17}\right) = \tfrac74\sqrt{21}-\tfrac{17}{12}\sqrt{17}$$

16. The sphere intersects the cone in the circle $x^2+y^2=\frac12$, $z=\frac{1}{\sqrt2}$, so S is the portion of the sphere where $z\ge\frac{1}{\sqrt2}$.

Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = \sin\phi\,\cos\theta\,\mathbf{i}+\sin\phi\,\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k}$, and

$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = \sin\phi$ (as in Example 1). The portion where $z\ge\frac{1}{\sqrt2}$ corresponds to $0\le\phi\le\frac{\pi}{4}$, $0\le\theta\le 2\pi$, so by Formula 2,

$$\iint_S y^2\,dS = \int_0^{2\pi}\int_0^{\pi/4}(\sin\phi\sin\theta)^2(\sin\phi)\,d\phi\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/4}\sin^3\phi\,d\phi = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\,d\phi$$
$$= \left[\tfrac12\theta-\tfrac14\sin 2\theta\right]_0^{2\pi}\left[\tfrac13\cos^3\phi-\cos\phi\right]_0^{\pi/4} = \pi\left(\tfrac{\sqrt2}{12}-\tfrac{\sqrt2}{2}-\tfrac13+1\right) = \left(\tfrac23-\tfrac{5\sqrt2}{12}\right)\pi$$

17. Using spherical coordinates to parametrize the sphere (see Example 16.6.4), we have

$\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i}+2\sin\phi\sin\theta\,\mathbf{j}+2\cos\phi\,\mathbf{k}$ and $|\mathbf{r}_\phi\times\mathbf{r}_\theta| = 4\sin\phi$ (see Example 16.6.10). Here S is the portion of the sphere corresponding to $0\le\phi\le\pi/2$, so by Formula 2,

$$\iint_S(x^2z+y^2z)\,dS = \iint_S(x^2+y^2)z\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta$$
$$= 32\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi = 32\,(2\pi)\left[\tfrac14\sin^4\phi\right]_0^{\pi/2} = 16\pi(1-0) = 16\pi$$

18. S is given by $\mathbf{r}(u,v) = \cos v\,\mathbf{i} + u\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le \pi$ (see Example 16.6.5). Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} \times (-\sin v\,\mathbf{i} + \cos v\,\mathbf{k}) = \cos v\,\mathbf{i} + \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\begin{aligned}\iint_S (x+y+z)\,dS &= \int_0^\pi \int_0^2 (\cos v + u + \sin v)(1)\,du\,dv = \int_0^\pi \left[u(\cos v + \sin v) + \tfrac{1}{2}u^2\right]_{u=0}^{u=2} dv \\ &= \int_0^\pi (2\cos v + 2\sin v + 2)\,dv = [2\sin v - 2\cos v + 2v]_0^\pi = 2 + 2\pi + 2 = 4 + 2\pi\end{aligned}$$

19. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 5$; and the back, S_3, in the plane $x = 0$.

On S_1: the surface is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\cos v\,\mathbf{j} + 3\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$ (see Example 16.6.5), and $0 \le x \le 5 - y \;\Rightarrow\; 0 \le u \le 5 - 3\cos v$. Then $\mathbf{r}_u \times \mathbf{r}_v = -3\cos v\,\mathbf{j} - 3\sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{9\cos^2 v + 9\sin^2 v} = 3$, so

$$\begin{aligned}\iint_{S_1} xz\,dS &= \int_0^{2\pi}\int_0^{5-3\cos v} u(3\sin v)(3)\,du\,dv = 9\int_0^{2\pi} \left[\tfrac{1}{2}u^2\right]_{u=0}^{u=5-3\cos v} \sin v\,dv \\ &= \tfrac{9}{2}\int_0^{2\pi} (5 - 3\cos v)^2 \sin v\,dv = \tfrac{9}{2}\left[\tfrac{1}{9}(5 - 3\cos v)^3\right]_0^{2\pi} = 0.\end{aligned}$$

On S_2: $\mathbf{r}(y,z) = (5-y)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = |\mathbf{i} + \mathbf{j}| = \sqrt{2}$, where $y^2 + z^2 \le 9$ and

$$\begin{aligned}\iint_{S_2} xz\,dS &= \iint_{y^2+z^2\le 9} (5-y)z\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^3 (5 - r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_0^3 (5r^2 - r^3\cos\theta)(\sin\theta)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\tfrac{5}{3}r^3 - \tfrac{1}{4}r^4\cos\theta\right]_{r=0}^{r=3} \sin\theta\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left(45 - \tfrac{81}{4}\cos\theta\right)\sin\theta\,d\theta = \sqrt{2}\left(\tfrac{4}{81}\right)\cdot\tfrac{1}{2}\left(45 - \tfrac{81}{4}\cos\theta\right)^2\Big]_0^{2\pi} = 0\end{aligned}$$

On S_3: $x = 0$ so $\iint_{S_3} xz\,dS = 0$. Hence $\iint_S xz\,dS = 0 + 0 + 0 = 0$.

20. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,

$\iint_{S_1} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^2 (9 + z^2)\,3\,dz\,d\theta = 2\pi(54 + 8) = 124\pi$.

On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_2} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi$.

On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_3} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi$.

Hence $\iint_S \left(x^2 + y^2 + z^2\right)dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

21. From Exercise 5, $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1 + 2u + v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$, and $\mathbf{r}_u \times \mathbf{r}_v = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$. Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(u,v)) &= (1 + 2u + v)e^{(u+v)(u-v)}\,\mathbf{i} - 3(1 + 2u + v)e^{(u+v)(u-v)}\,\mathbf{j} + (u+v)(u-v)\,\mathbf{k} \\ &= (1 + 2u + v)e^{u^2 - v^2}\,\mathbf{i} - 3(1 + 2u + v)e^{u^2 - v^2}\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}\end{aligned}$$

[continued]

Because the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative we use $-(\mathbf{r}_u \times \mathbf{r}_v)$ in Formula 9 for the upward orientation:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_u \times \mathbf{r}_v))\,dA = \int_0^1\int_0^2 \left[-3(1+2u+v)e^{u^2-v^2} + 3(1+2u+v)e^{u^2-v^2} + 2(u^2-v^2)\right] du\,dv \\ &= \int_0^1\int_0^2 2(u^2-v^2)\,du\,dv = 2\int_0^1 \left[\tfrac{1}{3}u^3 - uv^2\right]_{u=0}^{u=2} dv = 2\int_0^1 \left(\tfrac{8}{3} - 2v^2\right) dv \\ &= 2\left[\tfrac{8}{3}v - \tfrac{2}{3}v^3\right]_0^1 = 2\left(\tfrac{8}{3} - \tfrac{2}{3}\right) = 4\end{aligned}$$

22. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle$. Since $\mathbf{F}(x,y,z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$,

$\mathbf{F}(\mathbf{r}(u,v)) = v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$, and by Formula 9,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \int_0^1\int_0^\pi (v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du \\ &= \int_0^1 \left[\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi} du = \int_0^1 \pi\,du = \pi u\big]_0^1 = \pi\end{aligned}$$

23. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1 [2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx \\ &= \int_0^1 \left[x^2y^2 + \tfrac{8}{3}y^3 - \tfrac{2}{3}x^2y^3 - \tfrac{2}{5}y^5 + 4xy - x^3y - \tfrac{1}{3}xy^3\right]_{y=0}^{y=1} dx \\ &= \int_0^1 \left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right) dx = \left[\tfrac{1}{9}x^3 + \tfrac{11}{6}x^2 - \tfrac{1}{4}x^4 + \tfrac{34}{15}x\right]_0^1 = \tfrac{713}{180}\end{aligned}$$

24. $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the annular region $\{(x,y) \mid 1 \le x^2+y^2 \le 9\}$. Since S has downward orientation, we have, by Equation 10,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-(-x)\left(\frac{x}{\sqrt{x^2+y^2}}\right) - (-y)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^3\right] dA \\ &= -\iint_D \left[\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA = -\int_0^{2\pi}\int_1^3 \left(\frac{r^2}{r} + r^3\right) r\,dr\,d\theta \\ &= -\int_0^{2\pi} d\theta \int_1^3 (r^2 + r^4)\,dr = -\big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{3}r^3 + \tfrac{1}{5}r^5\right]_1^3 \\ &= -2\pi\left(9 + \tfrac{243}{5} - \tfrac{1}{3} - \tfrac{1}{5}\right) = -\tfrac{1712}{15}\pi\end{aligned}$$

25. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, and using spherical coordinates, S is given by $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (\sin\phi\cos\theta)\,\mathbf{i} + (\sin\phi\sin\theta)\,\mathbf{j} + (\cos^2\phi)\,\mathbf{k}$ and, from Example 4,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$. Thus,

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^3\phi = \sin^3\phi + \sin\phi\cos^3\phi$$

and

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)]\,dA = \int_0^{2\pi}\int_0^\pi (\sin^3\phi + \sin\phi\cos^3\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^\pi (1 - \cos^2\phi + \cos^3\phi)\sin\phi\,d\phi = (2\pi)\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi - \tfrac{1}{4}\cos^4\phi\right]_0^\pi \\ &= 2\pi\left(1 - \tfrac{1}{3} - \tfrac{1}{4} + 1 - \tfrac{1}{3} + \tfrac{1}{4}\right) = \tfrac{8}{3}\pi\end{aligned}$$

26. $\mathbf{F}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, $z = g(x,y) = \sqrt{4 - x^2 - y^2}$ and D is the disk $\{(x,y) \mid x^2 + y^2 \le 4\}$. S has downward orientation, so by Equation 10,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= -\iint_D [-y \cdot \tfrac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2x) - (-x) \cdot \tfrac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2y) + 2z]\,dA \\ &= -\iint_D \left(\frac{xy}{\sqrt{4 - x^2 - y^2}} - \frac{xy}{\sqrt{4 - x^2 - y^2}} + 2\sqrt{4 - x^2 - y^2} \right) dA \\ &= -\iint_D 2\sqrt{4 - x^2 - y^2}\,dA = -2\int_0^{2\pi} \int_0^2 \sqrt{4 - r^2}\, r\,dr\,d\theta = -2\int_0^{2\pi} d\theta \int_0^2 r\sqrt{4 - r^2}\,dr \\ &= -2(2\pi) \left[-\tfrac{1}{2} \cdot \tfrac{2}{3}(4 - r^2)^{3/2} \right]_0^2 = -4\pi \left[0 + \tfrac{1}{3}(4)^{3/2} \right] = -4\pi \cdot \tfrac{8}{3} = -\tfrac{32}{3}\pi\end{aligned}$$

27. $\mathbf{F}(x,y,z) = y\,\mathbf{j} - z\,\mathbf{k}$. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. On S_1 we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + z\,\mathbf{k}$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2 + z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then by Formula 9,

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2 + z^2 \le 1} [0 - (x^2 + z^2) - 2z^2]\,dA = -\int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \sin^2\theta)\, r\,dr\,d\theta \\ &= -\int_0^{2\pi} \int_0^1 r^3(1 + 2\sin^2\theta)\,dr\,d\theta = -\int_0^{2\pi} (1 + 1 - \cos 2\theta)\,d\theta \int_0^1 r^3\,dr \\ &= -\left[2\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^1 = -4\pi \cdot \tfrac{1}{4} = -\pi\end{aligned}$$

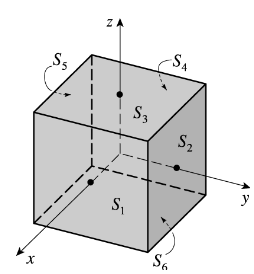
On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + z^2 \le 1} (1)\,dA = \pi$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = -\pi + \pi = 0$.

28. $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$, $z = g(x,y) = x\sin y$, and D is the rectangle $[0,2] \times [0,\pi]$, so by Equation 10

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-yz(\sin y) - zx(x\cos y) + xy]\,dA = \int_0^\pi \int_0^2 (-xy\sin^2 y - x^3 \sin y\cos y + xy)\,dx\,dy \\ &= \int_0^\pi \left[-\tfrac{1}{2}x^2 y\sin^2 y - \tfrac{1}{4}x^4 \sin y\cos y + \tfrac{1}{2}x^2 y\right]_{x=0}^{x=2} dy \\ &= \int_0^\pi \left(-2y\sin^2 y - 4\sin y\cos y + 2y\right) dy \qquad \text{[integrate by parts in the first term]} \\ &= \left[\left(-\tfrac{1}{2}y^2 + \tfrac{1}{2}y\sin 2y + \tfrac{1}{4}\cos 2y\right) - 2\sin^2 y + y^2\right]_0^\pi = -\tfrac{1}{2}\pi^2 + \tfrac{1}{4} + \pi^2 - \tfrac{1}{4} = \tfrac{1}{2}\pi^2\end{aligned}$$

29. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 dy\,dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i} + 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i} - 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 3\,dx\,dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.

30. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$. Here S consists of three surfaces: S_1, the lateral surface of the cylinder $x^2+z^2=1$; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$.

On S_1: $\mathbf{r}(\theta,y) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + \cos\,\theta\,\mathbf{k}$. $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \;\;\Rightarrow$

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2\,-\,\sin\theta} (\sin^2\theta + 5\cos\theta)\,dy\,d\theta \\ &= \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\,\cos\theta)\,d\theta = 2\pi\end{aligned}$$

On S_2: $\mathbf{r}(x,z) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + z\,\mathbf{k}$. $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{x^2+z^2\le 1} [x + (2-x)]\,dA = 2\pi$$

On S_3: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + 5\,\mathbf{k}$. The surface is oriented in the negative y-direction so that $\mathbf{n} = -\mathbf{j}$ and by (8), $\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = 0$. Hence, $\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi$.

31. $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$. Here S consists of four surfaces: S_1, the top surface (a portion of the circular cylinder $y^2+z^2=1$); S_2, the bottom surface (a portion of the xy-plane); S_3, the front half-disk in the plane $x=2$, and S_4, the back half-disk in the plane $x=0$.

On S_1: The surface is $z=\sqrt{1-y^2}$ for $0\le x\le 2$, $-1\le y\le 1$ with upward orientation, so by Equation 10,

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \int_0^2\int_{-1}^1 \left[-x^2\,(0) - y^2\left(-\frac{y}{\sqrt{1-y^2}}\right) + z^2\right] dy\,dx = \int_0^2\int_{-1}^1 \left(\frac{y^3}{\sqrt{1-y^2}} + 1 - y^2\right) dy\,dx \\ &= \int_0^2 \left[-\sqrt{1-y^2} + \tfrac{1}{3}(1-y^2)^{3/2} + y - \tfrac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_0^2 \tfrac{4}{3}\,dx = \tfrac{8}{3}\end{aligned}$$

On S_2: The surface is $z=0$ for $0\le x\le 2$, $-1\le y\le 1$ with downward orientation, so that $\mathbf{n}=-\mathbf{k}$ and by (8),

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^2\int_{-1}^1 \left(-z^2\right) dy\,dx = \int_0^2\int_{-1}^1 (0)\,dy\,dx = 0$$

On S_3: The surface is $x=2$ for $-1\le y\le 1$, $0\le z\le\sqrt{1-y^2}$, oriented in the positive x-direction, so that $\mathbf{n}=\mathbf{i}$ and by (8),

$$\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{-1}^1\int_0^{\sqrt{1-y^2}} x^2\,dz\,dy = \int_{-1}^1\int_0^{\sqrt{1-y^2}} 4\,dz\,dy = 4A\,(S_3) = 2\pi$$

On S_4: The surface is $x=0$ for $-1\le y\le 1$, $0\le z\le\sqrt{1-y^2}$, oriented in the negative x-direction, so that $\mathbf{n}=-\mathbf{i}$ and by (8),

$$\iint_{S_4} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{-1}^1\int_0^{\sqrt{1-y^2}} (-x^2)\,dz\,dy = \int_{-1}^1\int_0^{\sqrt{1-y^2}} (0)\;dz\,dy = 0$$

Thus, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{8}{3} + 0 + 2\pi + 0 = 2\pi + \frac{8}{3}$.

32. $\mathbf{F}(x,y,z) = y\,\mathbf{i} + (z-y)\,\mathbf{j} + x\,\mathbf{k}$. Here S consists of four surfaces: S_1, the triangular face with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$; S_2, the face of the tetrahedron in the xy-plane; S_3, the face in the xz-plane; and S_4, the face in the yz-plane.

On S_1: The face is the portion of the plane $z=1-x-y$ for $0\le x\le 1$, $0\le y\le 1-x$ with upward orientation,

so by Equation 10,

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{1-x}\left[-y\,(-1)-(z-y)\,(-1)+x\right]dy\,dx = \int_0^1\int_0^{1-x}(z+x)\,dy\,dx = \int_0^1\int_0^{1-x}(1-y)\,dy\,dx$$

$$= \int_0^1\left[y-\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x}dx = \tfrac{1}{2}\int_0^1\left(1-x^2\right)dx = \tfrac{1}{2}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

On S_2: The surface is $z=0$ for $0\le x\le 1$, $0\le y\le 1-x$ with downward orientation, so that that $\mathbf{n}=-\mathbf{k}$ and by (8),

$$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-x}(-x)\,dy\,dx = -\int_0^1 x\,(1-x)\,dx = -\left[\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right]_0^1 = -\tfrac{1}{6}$$

On S_3: The surface is $y=0$ for $0\le x\le 1$, $0\le z\le 1-x$, oriented in the negative y-direction, so that $\mathbf{n}=-\mathbf{j}$ and by (8),

$$\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-x}-(z-y)\,dz\,dx = -\int_0^1\int_0^{1-x}z\,dz\,dx = -\int_0^1\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x}dx$$

$$= -\tfrac{1}{2}\int_0^1(1-x)^2\,dx = \tfrac{1}{6}\left[(1-x)^3\right]_0^1 = -\tfrac{1}{6}$$

On S_4: The surface is $x=0$ for $0\le y\le 1$, $0\le z\le 1-y$, oriented in the negative x-direction, so that $\mathbf{n}=-\mathbf{i}$ and by (8),

$$\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_4}\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-y}(-y)\,dz\,dy = -\int_0^1 y\,(1-y)\,dy = -\left[\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\right]_0^1 = -\tfrac{1}{6}$$

Thus, $\iint_S\mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}-\frac{1}{6}-\frac{1}{6}-\frac{1}{6} = -\frac{1}{6}$.

33. $z=xe^y \quad\Rightarrow\quad \partial z/\partial x = e^y$, $\partial z/\partial y = xe^y$, so by Formula 4, a CAS gives

$\iint_S(x^2+y^2+z^2)\,dS = \int_0^1\int_0^1(x^2+y^2+x^2e^{2y})\sqrt{e^{2y}+x^2e^{2y}+1}\,dx\,dy \approx 4.5822.$

34. $z=x^2y^2 \quad\Rightarrow\quad \partial z/\partial x = 2xy^2,\ \partial z/\partial y = 2x^2y$, so by Formula 4, a CAS gives

$$\iint_S xyz\,dS = \int_0^2\int_0^1 xy(x^2y^2)\sqrt{(2xy^2)^2+(2x^2y)^2+1}\,dx\,dy$$

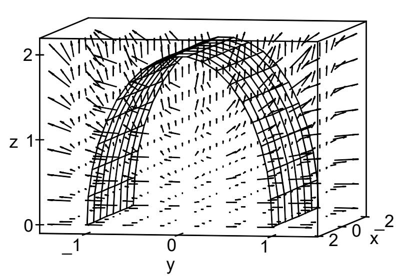
$$= \int_0^2\int_0^1 x^3y^3\sqrt{4x^2y^4+4x^4y^2+1}\,dx\,dy = -\tfrac{151}{33}-\tfrac{1}{220}\sqrt{3}\,\pi+\tfrac{1977}{176}\ln 7-\tfrac{9891}{880}\ln 3+\tfrac{3}{440}\sqrt{3}\tan^{-1}\tfrac{5}{\sqrt{3}}$$

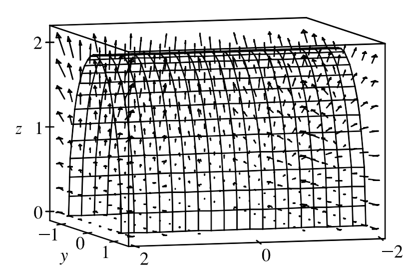
35. We use Formula 4 with $z=3-2x^2-y^2 \quad\Rightarrow\quad \partial z/\partial x=-4x$, $\partial z/\partial y=-2y$. The boundaries of the region $3-2x^2-y^2\ge 0$ are $-\sqrt{\frac{3}{2}}\le x\le\sqrt{\frac{3}{2}}$ and $-\sqrt{3-2x^2}\le y\le\sqrt{3-2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation may take a long time) to calculate

$$\iint_S x^2y^2z^2\,dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}}\int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3-2x^2-y^2)^2\sqrt{16x^2+4y^2+1}\,dy\,dx \approx 3.4895$$

36. The flux of $\mathbf{F}$ across S is given by $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_S\mathbf{F}\cdot\mathbf{n}\,dS$. Now on S, $z=g(x,y)=2\sqrt{1-y^2}$, so $\partial g/\partial x=0$ and $\partial g/\partial y=-2y(1-y^2)^{-1/2}$. Therefore, by Equation 10,

$$\iint_S\mathbf{F}\cdot d\mathbf{S} = \int_{-2}^2\int_{-1}^1\left(-x^2y\left[-2y(1-y^2)^{-1/2}\right]+\left[2\sqrt{1-y^2}\right]^2e^{x/5}\right)dy\,dx = \tfrac{1}{3}(16\pi+80e^{2/5}-80e^{-2/5})$$





37. If S is given by $y = h(x,z)$, then S is also the level surface $f(x,y,z) = y - h(x,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (10), using Formula 4 to evaluate

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot(-\mathbf{n})\,dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\frac{\dfrac{\partial h}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\,\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of S onto the xz-plane. Therefore $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P\,\frac{\partial h}{\partial x} - Q + R\,\frac{\partial h}{\partial z}\right)dA.$

38. If S is given by $x = k(y,z)$, then S is also the level surface $f(x,y,z) = x - k(y,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{\mathbf{i} - k_y\,\mathbf{j} - k_z\,\mathbf{k}}{\sqrt{1 + k_y^2 + k_z^2}}$, and since the x-component is positive this is the unit normal that points forward.

Now we proceed as in the derivation of (10), using Formula 4 for

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\frac{\mathbf{i} - \dfrac{\partial k}{\partial y}\,\mathbf{j} - \dfrac{\partial k}{\partial z}\,\mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial k}{\partial y}\right)^2 + \left(\dfrac{\partial k}{\partial z}\right)^2}}\sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}\,dA$$

where D is the projection of S onto the yz-plane. Therefore $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P - Q\,\frac{\partial k}{\partial y} - R\,\frac{\partial k}{\partial z}\right)dA.$

39. $m = \iint_S K\,dS = K\cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zK\,dS = K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)\,d\phi\,d\theta = 2\pi K a^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi K a^3.$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{2}a\right)$.

40. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2+y^2}{x^2+y^2}} = \sqrt{2}$ so

$$m = \iint_S (10 - z)\,dS = \iint_S \left(10 - \sqrt{x^2+y^2}\right)dS = \iint_{1\le x^2+y^2\le 16}\left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$
$$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10 - r)\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\,\pi$$

41. (a) $I_z = \iint_S (x^2+y^2)\rho(x,y,z)\,dS$ (see 15.6.16).

(b) $I_z = \iint_S (x^2+y^2)\left(10 - \sqrt{x^2+y^2}\right)dS = \displaystyle\iint_{1\le x^2+y^2\le 16}(x^2+y^2)\left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA$

$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10r^3 - r^4)\,dr\,d\theta = 2\sqrt{2}\,\pi\left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\,\pi$

42. Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 25\sin\phi$ (see Example 16.6.10). S is the portion of the sphere where $z \ge 4$, so $0 \le \phi \le \tan^{-1}(3/4)$ and $0 \le \theta \le 2\pi$.

[continued]

(a) $m = \iint_S \rho(x,y,z)\,dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin\phi)\,d\phi\,d\theta = 25k\int_0^{2\pi} d\theta \int_0^{\tan^{-1}(3/4)} \sin\phi\,d\phi$

$= 25k(2\pi)\left[-\cos\left(\tan^{-1}\frac{3}{4}\right)+1\right] = 50\pi k\left(-\frac{4}{5}+1\right) = 10\pi k.$

Because S has constant density, $\overline{x} = \overline{y} = 0$ by symmetry, and

$\overline{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS = \frac{1}{10\pi k}\int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(5\cos\phi)(25\sin\phi)\,d\phi\,d\theta$

$= \frac{1}{10\pi k}(125k)\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)}\sin\phi\cos\phi\,d\phi = \frac{1}{10\pi k}(125k)(2\pi)\left[\frac{1}{2}\sin^2\phi\right]_0^{\tan^{-1}(3/4)} = 25\cdot\frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{9}{2},$

so the center of mass is $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{9}{2}\right)$.

(b) $I_z = \iint_S (x^2+y^2)\rho(x,y,z)\,dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin^2\phi)(25\sin\phi)\,d\phi\,d\theta$

$= 625k\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)}\sin^3\phi\,d\phi = 625k(2\pi)\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\tan^{-1}(3/4)}$

$= 1250\pi k\left[\frac{1}{3}\left(\frac{4}{5}\right)^3 - \frac{4}{5} - \frac{1}{3} + 1\right] = 1250\pi k\left(\frac{14}{375}\right) = \frac{140}{3}\pi k$

43. The rate of flow through the cylinder is the flux $\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS = \iint_S \rho\mathbf{v}\cdot d\mathbf{S}$ (see Formula 7). We use the parametric representation of the cylinder $\mathbf{r}(u,v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\,\mathbf{k}$ for S, where $0 \le u \le 2\pi$, $0 \le v \le 1$, so $\mathbf{r}_u = -2\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j}$, $\mathbf{r}_v = \mathbf{k}$, and the outward orientation is given by $\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}$. Then

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \rho\int_0^{2\pi}\int_0^1 \left(v\,\mathbf{i} + 4\sin^2 u\,\mathbf{j} + 4\cos^2 u\,\mathbf{k}\right)\cdot(2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j})\,dv\,du \\ &= \rho\int_0^{2\pi}\int_0^1 \left(2v\cos u + 8\sin^3 u\right)dv\,du = \rho\int_0^{2\pi}\left(\cos u + 8\sin^3 u\right)du \\ &= \rho\left[\sin u + 8\left(-\tfrac{1}{3}\right)(2+\sin^2 u)\cos u\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

44. A parametric representation for the hemisphere S is $\mathbf{r}(\phi,\theta) = 3\sin\phi\,\cos\theta\,\mathbf{i} + 3\sin\phi\,\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\,\cos\theta\,\mathbf{i} + 3\cos\phi\,\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\,\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}$. The rate of flow through S is [by (7)]

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\,\sin\theta\,\mathbf{i} + 3\sin\phi\,\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi \\ &= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta \\ &= 54\rho\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

45. S consists of the hemisphere S_1 given by $z = \sqrt{a^2-x^2-y^2}$ and the disk S_2 given by $0 \le x^2+y^2 \le a^2$, $z = 0$.

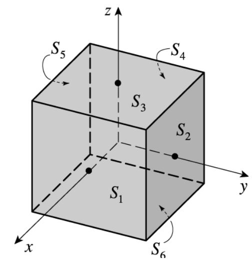
On S_1: As in Example 4, we use a parametric representation to get $\mathbf{E} = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$, $\mathbf{T}_\phi \times \mathbf{T}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. Thus

$$\begin{aligned}\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi + 2a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi + a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta = (2\pi)a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3\end{aligned}$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = 0$.

Hence, by (11), the total charge is $q = \varepsilon_0\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\varepsilon_0$.

46. Referring to the figure, on



S_1: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dx\,dz = 4$;

S_3: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dx\,dy = 4$;

S_4: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{E} \cdot d\mathbf{S} = 4$.

Similarly, $\iint_{S_5} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_6} \mathbf{E} \cdot d\mathbf{S} = 4$. Hence, by (11), $q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \varepsilon_0 \sum\limits_{i=1}^{6} \iint_{S_i} \mathbf{E} \cdot d\mathbf{S} = 24\varepsilon_0$.

47. The heat flow for $K = 6.5$ and $u(x,y,z) = 2y^2 + 2z^2$ is $-K\,\nabla u = -6.5(4y\,\mathbf{j} + 4z\,\mathbf{k})$. As in Example 16.6.5, we can parametrize the cylindrical surface S: $y^2 + z^2 = 6$, $0 \le x \le 4$ as $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\,\cos\theta\,\mathbf{j} + \sqrt{6}\,\sin\theta\,\mathbf{k}$, $0 \le x \le 4$, $0 \le \theta \le 2\pi$. Since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6}\,\cos\theta\,\mathbf{j} - \sqrt{6}\,\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by

$$\begin{aligned}
\iint_S (-K\,\nabla u) \cdot d\mathbf{S} &= -K \iint \nabla u \cdot (\mathbf{r}_x \times \mathbf{r}_\theta)\,dA \\
&= -6.5 \int_0^{2\pi} \int_0^4 \left(4\sqrt{6}\cos\theta\,\mathbf{j} + 4\sqrt{6}\sin\theta\,\mathbf{k}\right) \cdot \left(-\sqrt{6}\cos\theta\,\mathbf{j} - \sqrt{6}\sin\theta\,\mathbf{k}\right) dx\,d\theta \\
&= -6.5 \int_0^{2\pi} \int_0^4 (-24\cos^2\theta - 24\sin^2\theta)\,dx\,d\theta = -6.5(-24)(4)(2\pi) = 1248\pi
\end{aligned}$$

48. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\begin{aligned}
\mathbf{F} = -K\,\nabla u &= -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right] \\
&= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})
\end{aligned}$$

and the outward unit normal is $\mathbf{n} = \dfrac{1}{a}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

Thus $\mathbf{F} \cdot \mathbf{n} = \dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2 = a^2$ so $\mathbf{F} \cdot \mathbf{n} = \dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{cK}{a^2} \iint_S dS = \frac{cK}{a^2}\,(4\pi a^2) = 4\pi Kc$.

49. Let S be a sphere of radius a centered at the origin. Then $|\mathbf{r}| = a$ and $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = \left(c/a^3\right)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$. A parametric representation for S is $\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. The flux of $\mathbf{F}$ across S is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot\, d\mathbf{S} &= \int_0^{\pi} \int_0^{2\pi} \frac{c}{a^3}\,(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}) \\
&\qquad \cdot \left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right) d\theta\,d\phi \\
&= \frac{c}{a^3} \int_0^{\pi} \int_0^{2\pi} a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right) d\theta\,d\phi = c\int_0^{\pi} \int_0^{2\pi} \sin\phi\,d\theta\,d\phi = 4\pi c
\end{aligned}$$

Thus the flux does not depend on the radius a.

16.8 Stokes' Theorem

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ (where C is the boundary curve).

2. $\mathbf{F}(x, y, z) = x^2 \sin z\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = 1$, $z = 0$. This boundary curve C should be oriented in the counterclockwise direction when viewed from above, so a vector equation of C is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$,

$$\mathbf{F}(\mathbf{r}(t)) = (\cos t)^2 (\sin 0)\,\mathbf{i} + (\sin t)^2\,\mathbf{j} + (\cos t)(\sin t)\,\mathbf{k} = \sin^2 t\,\mathbf{j} + \sin t\ \cos t\,\mathbf{k},$$

and by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (\sin^2 t\,\mathbf{j} + \sin t\ \cos t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt \\ &= \int_0^{2\pi} (0 + \sin^2 t \cos t + 0)\,dt = \left[\tfrac{1}{3} \sin^3 t\right]_0^{2\pi} = 0\end{aligned}$$

3. $\mathbf{F}(x, y, z) = ze^y\,\mathbf{i} + x \cos y\,\mathbf{j} + xz \sin y\,\mathbf{k}$. The boundary curve C is the circle $x^2 + z^2 = 16$, $y = 0$ where the hemisphere intersects the xz-plane. The curve should be oriented in the counterclockwise direction when viewed from the right (from the positive y-axis), so a vector equation of C is $\mathbf{r}(t) = 4\cos(-t)\,\mathbf{i} + 4\sin(-t)\,\mathbf{k} = 4\cos t\,\mathbf{i} - 4\sin t\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{k}$ and

$$\mathbf{F}(\mathbf{r}(t)) = (-4\sin t)e^0\,\mathbf{i} + (4\cos t)(\cos 0)\,\mathbf{j} + (4\cos t)(-4\sin t)(\sin 0)\,\mathbf{k} = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j},$$

and by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j}) \cdot (-4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{k})\,dt \\ &= \int_0^{2\pi} (16\sin^2 t + 0 + 0)\,dt = 16\left[\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 16\pi\end{aligned}$$

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\,\mathbf{i} + x^2y\,\mathbf{j} + x^2z^2\,\mathbf{k}$. The boundary curve C is the circle $y^2 + z^2 = 4$, $x = 2$ which should be oriented in the counterclockwise direction when viewed from the front, so a vector equation of C is $\mathbf{r}(t) = 2\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \tan^{-1}(32\cos t \sin^2 t)\,\mathbf{i} + 8\cos t\,\mathbf{j} + 16\sin^2 t\,\mathbf{k}$, $\mathbf{r}'(t) = -2\sin t\,\mathbf{j} + 2\cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\sin t\cos t + 32\sin^2 t\cos t$. Thus,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-16\sin t\cos t + 32\sin^2 t\cos t)\,dt \\ &= \left[-8\sin^2 t + \tfrac{32}{3}\sin^3 t\right]_0^{2\pi} = 0\end{aligned}$$

5. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2yz\,\mathbf{k}$. C is the square in the plane $z = -1$. Rather than evaluating a line integral around C we can use Equation 3: $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl} \mathbf{F} = x^2z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus, by (16.7.8), $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} (x + y)\,dx\,dy = 0$ so $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

6. $\mathbf{F}(x,y,z) = e^{xy}\,\mathbf{i} + e^{xz}\,\mathbf{j} + x^2 z\,\mathbf{k}$. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ which should be oriented in the counterclockwise direction when viewed from the right, so a vector equation of C is $\mathbf{r}(t) = \cos(-t)\,\mathbf{i} + \sin(-t)\,\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \mathbf{i} + e^{-\cos t \sin t}\,\mathbf{j} - \cos^2 t\,\sin t\,\mathbf{k}$, $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\sin t + \cos^3 t\,\sin t$. Thus,

$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-\sin t + \cos^3 t\,\sin t)\,dt = \left[\cos t - \frac{1}{4}\cos^4 t\right]_0^{2\pi} = 0.$

7. $\mathbf{F}(x,y,z) = (x+y^2)\,\mathbf{i} + (y+z^2)\,\mathbf{j} + (z+x^2)\,\mathbf{k}$ and $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$. We take the surface to be the planar region enclosed by C and D to be the projection of S onto the xy-plane, so S is the portion of the plane $x + y + z = 1$ over $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 16.7.10, we have $z = g(x,y) = 1 - x - y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-(-2z)(-1) - (-2x)(-1) + (-2y)\right] dA \\
&= \int_0^1 \int_0^{1-x} (-2)\,dy\,dx = -2\int_0^1 (1-x)\,dx = -1
\end{aligned}$$

8. $\mathbf{F}(x,y,z) = \mathbf{i} + (x+yz)\,\mathbf{j} + \left(xy - \sqrt{z}\right)\mathbf{k}$ and $\operatorname{curl}\mathbf{F} = (x-y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$. We take the surface S to be the planar region enclosed by C and D to be the projection of S onto the xy-plane, so S is the portion of the plane $3x + 2y + z = 1$ over $D = \left\{(x,y) \mid 0 \le x \le \frac{1}{3}, 0 \le y \le \frac{1}{2}(1-3x)\right\}$. We orient S upward and use Equation 16.7.10 with $z = g(x,y) = 1 - 3x - 2y$:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-(x-y)(-3) - (-y)(-2) + 1\right] dA = \int_0^{1/3} \int_0^{(1-3x)/2} (1 + 3x - 5y)\,dy\,dx \\
&= \int_0^{1/3} \left[(1+3x)y - \tfrac{5}{2}y^2\right]_{y=0}^{y=(1-3x)/2} dx = \int_0^{1/3} \left[\tfrac{1}{2}(1+3x)(1-3x) - \tfrac{5}{2} \cdot \tfrac{1}{4}(1-3x)^2\right] dx \\
&= \int_0^{1/3} \left(-\tfrac{81}{8}x^2 + \tfrac{15}{4}x - \tfrac{1}{8}\right) dx = \left[-\tfrac{27}{8}x^3 + \tfrac{15}{8}x^2 - \tfrac{1}{8}x\right]_0^{1/3} = -\tfrac{1}{8} + \tfrac{5}{24} - \tfrac{1}{24} = \tfrac{1}{24}
\end{aligned}$$

9. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$. $\operatorname{curl}\mathbf{F} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$ and we take S to be the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant. Since C is oriented counterclockwise (from above), we orient S upward. Then using Equation 16.7.10 with $z = g(x,y) = 1 - x^2 - y^2$ we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-(-y)(-2x) - (-z)(-2y) + (-x)\right] dA \\
&= \iint_D \left[-2xy - 2y(1 - x^2 - y^2) - x\right] dA \\
&= \int_0^{\pi/2} \int_0^1 \left[-2(r\cos\theta)(r\sin\theta) - 2(r\sin\theta)(1 - r^2) - r\cos\theta\right] r\,dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^1 \left[-2r^3 \sin\theta\cos\theta - 2(r^2 - r^4)\sin\theta - r^2\cos\theta\right] dr\,d\theta \\
&= \int_0^{\pi/2} \left[-\tfrac{1}{2}r^4 \sin\theta\cos\theta - 2\left(\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right)\sin\theta - \tfrac{1}{3}r^3\cos\theta\right]_{r=0}^{r=1} d\theta \\
&= \int_0^{\pi/2} \left(-\tfrac{1}{2}\sin\theta\cos\theta - \tfrac{4}{15}\sin\theta - \tfrac{1}{3}\cos\theta\right) d\theta = \left[-\tfrac{1}{4}\sin^2\theta + \tfrac{4}{15}\cos\theta - \tfrac{1}{3}\sin\theta\right]_0^{\pi/2} \\
&= -\tfrac{1}{4} - \tfrac{4}{15} - \tfrac{1}{3} = -\tfrac{17}{20}
\end{aligned}$$

10. $\mathbf{F}(x,y,z) = 2y\,\mathbf{i} + xz\,\mathbf{j} + (x+y)\,\mathbf{k}$. The curve of intersection is an ellipse in the plane $z = y + 2$.

$\operatorname{curl}\mathbf{F} = (1-x)\,\mathbf{i} - \mathbf{j} + (z-2)\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C with upward orientation. From Equation 16.7.10 with $z = g(x,y) = y+2$ we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint\limits_{x^2+y^2\le 1} [-(1-x)(0) - (-1)(1) + (y+2-2)]\,dA \\ &= \iint\limits_{x^2+y^2\le 1} (y+1)\,dA = \int_0^{2\pi}\int_0^1 (r\sin\theta + 1)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{3}r^3\sin\theta + \tfrac{1}{2}r^2\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi}\left(\tfrac{1}{3}\sin\theta + \tfrac{1}{2}\right) d\theta = \left[-\tfrac{1}{3}\cos\theta + \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi\end{aligned}$$

11. $\mathbf{F}(x,y,z) = \langle -yx^2, xy^2, e^{xy}\rangle$ and $\operatorname{curl}\mathbf{F} = xe^{xy}\,\mathbf{i} - ye^{xy}\,\mathbf{j} + (x^2+y^2)\,\mathbf{k}$. C is the circle in the xy-plane centered at the origin with radius 2. Choose S to be the portion of the xy-plane enclosed by C. So $S = D = \{(x,y) \mid x^2+y^2 \le 4\}$. C is oriented counterclockwise, so we orient S upward and the normal vector to S is $\mathbf{n} = \mathbf{k}$. By Stokes' Theorem (See Solution 2 of Example 2), we get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\mathbf{S} = \iint_S [xe^{xy}\,\mathbf{i} - ye^{xy}\,\mathbf{j} + (x^2+y^2)\,\mathbf{k}]\cdot\mathbf{k}\,dS \\ &= \iint_D (x^2+y^2)\,dA = \int_0^2\int_0^{2\pi} r^2\,r\,d\theta\,dr = 2\pi\int_0^2 r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_0^2 = 8\pi\end{aligned}$$

12. $\mathbf{F}(x,y,z) = ze^x\,\mathbf{i} + (z-y^3)\,\mathbf{j} + (x-z^3)\,\mathbf{k}$ and $\operatorname{curl}\mathbf{F} = -\,\mathbf{i} - (1-e^x)\,\mathbf{j}$. C is the circle $y^2+z^2 = 4$, $x = 3$, and we choose the surface S to be the portion of the plane $x = 3$ enclosed by C. The projection of S onto the yz-plane is the disk $D = \{(y,z) \mid y^2+z^2 \le 4\}$. C is oriented clockwise, so we orient S to have normal vector $\mathbf{n} = \mathbf{i}$. By Stokes' Theorem (see Solution 2 of Example 2), we get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\mathbf{S} = \iint_S [-\mathbf{i} - (1-e^x)\,\mathbf{j}]\cdot\mathbf{i}\,dS \\ &= -\iint_D dA = -A(D) = -\pi(2^2) = -4\pi\end{aligned}$$

13. $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + x^3\,\mathbf{j} + e^z\tan^{-1}z\,\mathbf{k}$ and $\operatorname{curl}\mathbf{F} = 2x^2\,\mathbf{k}$. Note that the curve $C = \langle\cos t, \sin t, \sin t\rangle$ is contained in the plane $y = z$ because the **j** and **k** components of the curve are equal. We choose the surface S to be the portion of the plane $y = z$ enclosed by C. The projection of C onto the xy-plane is the circle $\langle\cos t, \sin t, 0\rangle$, and D is the disk in the xy-plane enclosed by C. We orient S upward and use Equation 16.7.10 with $z = g(x,y) = y$:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D [-(0)(0) - (0)(1) + 2x^2]\,dA = \int_0^{2\pi}\int_0^1 2(r\cos\theta)^2\,r\,dr\,d\theta \\ &= \int_0^{2\pi}(1+\cos 2\theta)\,d\theta\int_0^1 r^3\,dr = \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 = (2\pi + 0 - 0)\frac{1}{4} = \frac{\pi}{2}\end{aligned}$$

14. $\mathbf{F}(x,y,z) = \langle x^3 - z, xy, y + z^2\rangle$ and $\operatorname{curl}\mathbf{F} = \mathbf{i} - \mathbf{j} + y\,\mathbf{k}$. C is the curve of intersection of the paraboloid $z = x^2+y^2$ and the plane $z = x$. Let S be the portion of the plane $z = x$ enclosed by C. To find the projection of S onto the xy-plane, note that $x = x^2+y^2$. Converting to polar coordinates, we get $r\cos\theta = r^2 \quad\Rightarrow\quad r = \cos\theta$. So D is the region in the

xy-plane enclosed by the circle $r = \cos\theta$; that is, $D = \{(r,\theta) \mid 0 \le r \le \cos\theta, 0 \le \theta \le \pi\}$. We orient S upward and use Equation 16.7.10 with $z = g(x,y) = x$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D [-(1)(1) - (-1)(0) + y] \, dA = \iint_D (-1 + y) \, dA$$

$$= \int_0^\pi \int_0^{\cos\theta} (-1 + r\sin\theta) \, r \, dr \, d\theta = \int_0^\pi \left[-\frac{r^2}{2} + \frac{r^3}{3} \sin\theta \right]_0^{\cos\theta} d\theta$$

$$= \int_0^\pi \left(-\frac{\cos^2\theta}{2} + \frac{\cos^3\theta}{3} \sin\theta \right) d\theta = \int_0^\pi \left(-\frac{1 + \cos 2\theta}{4} + \frac{\cos^3\theta}{3} \sin\theta \right) d\theta$$

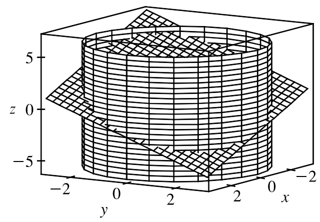
$$= \left[-\frac{1}{4}\theta - \frac{\sin 2\theta}{8} - \frac{\cos^4\theta}{12} \right]_0^\pi = \left(-\frac{\pi}{4} - 0 - \frac{1}{12} \right) - \left(0 - 0 - \frac{1}{12} \right) = -\frac{\pi}{4}$$

15. (a) $\mathbf{F}(x,y,z) = x^2 z \, \mathbf{i} + xy^2 \, \mathbf{j} + z^2 \, \mathbf{k}$. The curve of intersection is an ellipse in the plane $x + y + z = 1$. The unit normal is $\mathbf{n} = \frac{1}{\sqrt{3}} (\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\operatorname{curl} \mathbf{F} = x^2 \, \mathbf{j} + y^2 \, \mathbf{k}$, and $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then, by Stokes' Theorem,

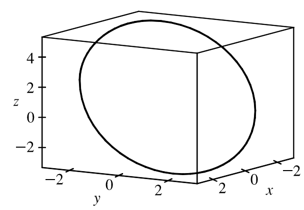
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \tfrac{1}{\sqrt{3}} (x^2 + y^2) \, dS$$

$$= \iint_{x^2 + y^2 \le 9} (x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^3 r^3 \, dr \, d\theta = 2\pi \left(\tfrac{81}{4} \right) = \tfrac{81\pi}{2}$$

(b)



(c) One possible parametrization is $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3\cos t - 3\sin t$, $0 \le t \le 2\pi$.

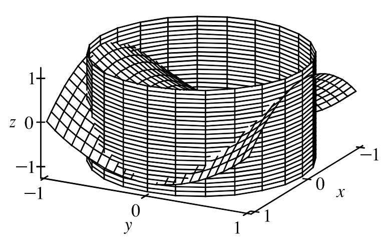


16. (a) $\mathbf{F}(x,y,z) = x^2 y \, \mathbf{i} + \frac{1}{3} x^3 \, \mathbf{j} + xy \, \mathbf{k}$. S is the part of the surface $z = y^2 - x^2$ that lies above the unit disk D. $\operatorname{curl} \mathbf{F} = x \, \mathbf{i} - y \, \mathbf{j} + (x^2 - x^2) \, \mathbf{k} = x \, \mathbf{i} - y \, \mathbf{j}$. Using Equation 16.7.10 with $z = g(x,y) = y^2 - x^2$, by Stokes' Theorem we have

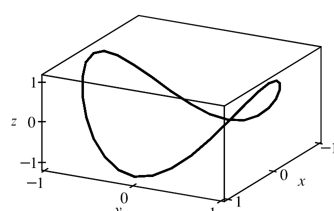
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D [-x(-2x) - (-y)(2y)] \, dA = 2 \iint_D (x^2 + y^2) \, dA$$

$$= 2 \int_0^{2\pi} \int_0^1 r^2 r \, dr \, d\theta = 2(2\pi) \left[\tfrac{1}{4} r^4 \right]_0^1 = \pi$$

(b)



(c) One possible set of parametric equations is $x = \cos t$, $y = \sin t$, $z = \sin^2 t - \cos^2 t$, $0 \le t \le 2\pi$.



17. $\mathbf{F}(x,y,z) = -y\,\mathbf{i} + x\,\mathbf{j} - 2\,\mathbf{k}$. The boundary curve C is the circle $x^2 + y^2 = 16$, $z = 4$ oriented in the clockwise direction as viewed from above (since S is oriented downward). We can parametrize C by $\mathbf{r}(t) = 4\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = 4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} - 2\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\sin^2 t - 16\cos^2 t = -16$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-16)\,dt = -16\,(2\pi) = -32\pi$$

Now curl $\mathbf{F} = 2\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 16$, so by Equation 16.7.10 with $z = g(x,y) = \sqrt{x^2 + y^2}$ [and multiplying by -1 for the downward orientation] we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -\iint_D (-0 - 0 + 2)\,dA = -2 \cdot A(D) = -2 \cdot \pi(4^2) = -32\pi$$

18. $\mathbf{F}(x,y,z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k}$. The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \;\Leftrightarrow\; x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -4\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 6\cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 8\sin^2 t + 4\sin t\cos t$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t\cos t)\,dt = 8\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 2\sin^2 t\Big]_0^{2\pi} = 8\pi$$

Now curl $\mathbf{F} = (-3 - 2y)\,\mathbf{j} + 2z\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 4$, so by Equation 16.7.10 with $z = g(x,y) = 5 - x^2 - y^2$ we have

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z]\,dA = \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2)]\,dA \\ &= \int_0^{2\pi} \int_0^2 \left[-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)\right] r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \left(-16\sin\theta - 16\sin^2\theta + 20 - 8\right) d\theta \\ &= 16\cos\theta - 16\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + 12\theta\Big]_0^{2\pi} = 8\pi \end{aligned}$$

19. $\mathbf{F}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y-axis. Then C can be described by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\cos^2 t$, and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t)\,dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t\Big]_0^{2\pi} = -\pi$. Now curl $\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 16.6.10) by $\mathbf{r}(\phi,\theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$ and

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1} \operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^\pi \int_0^\pi (-\sin^2\phi\,\cos\theta - \sin^2\phi\,\sin\theta - \sin\phi\,\cos\phi)\,d\theta\,d\phi \\ &= \int_0^\pi (-2\sin^2\phi - \pi\sin\phi\,\cos\phi)\,d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi \end{aligned}$$

20. Let S be the surface in the plane $x + y + z = 1$ with upward orientation enclosed by C. Then an upward unit normal vector for S is $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. Orient C in the counterclockwise direction, as viewed from above. $\int_C z\,dx - 2x\,dy + 3y\,dz$ is equivalent to $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x,y,z) = z\,\mathbf{i} - 2x\,\mathbf{j} + 3y\,\mathbf{k}$, and the components of $\mathbf{F}$ are polynomials, which have continuous

partial derivatives throughout $\mathbb{R}^3$. We have curl $\mathbf{F} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so by Stokes' Theorem,

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) \cdot \tfrac{1}{\sqrt{3}}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})\ dS$$
$$= \tfrac{2}{\sqrt{3}} \iint_S dS = \tfrac{2}{\sqrt{3}}\,(\text{surface area of } S)$$

Thus the value of $\int_C z\,dx - 2x\,dy + 3y\,dz$ is always $\frac{2}{\sqrt{3}}$ times the area of the region enclosed by C, regardless of its shape or location. [Notice that because $\mathbf{n}$ is normal to a plane, it is constant. But curl $\mathbf{F}$ is also constant, so the dot product curl $\mathbf{F} \cdot \mathbf{n}$ is constant and we could have simply argued that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$ is a constant multple of $\iint_S dS$, the surface area of S.]

21. $\mathbf{F}(x,y,z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$. It is easier to use Stokes' Theorem than to compute the work directly. Let S be the planar region enclosed by the path of the particle, so S is the portion of the plane $z = \frac{1}{2}y$ for $0 \le x \le 1$, $0 \le y \le 2$, with upward orientation. curl $\mathbf{F} = 8y\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ and by Stokes' Theorem and Equation 16.7.10, the work done is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-8y\,(0) - 2z\left(\tfrac{1}{2}\right) + 2y\right] dA$$
$$= \int_0^1 \int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\,dx = \int_0^1 \int_0^2 \tfrac{3}{2}y\,dy\,dx = \int_0^1 \left[\tfrac{3}{4}y^2\right]_{y=0}^{y=2} dx = \int_0^1 3\,dx = 3$$

22. $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz = \int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y + \sin x)\,\mathbf{i} + (z^2 + \cos y)\,\mathbf{j} + x^3\,\mathbf{k}$ $\Rightarrow$ curl $\mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2\sin t\,\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D $[x^2 + y^2 \le 1]$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 16.7.10 with $g(x,y) = 2xy$, $P = -2z = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$ and multiplying by -1 for the downward orientation, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA$$
$$= -\iint_D (8xy^2 + 6x^3 - 1)\,dA = -\int_0^{2\pi} \int_0^1 (8r^3 \cos\theta \sin^2\theta + 6r^3 \cos^3\theta - 1)\,r\,dr\,d\theta$$
$$= -\int_0^{2\pi} \left(\tfrac{8}{5}\cos\theta \sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right) d\theta = -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi$$

23. Assume S is a sphere centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

24. (a) By Exercise 16.5.28, $\operatorname{curl}(f\nabla g) = f\operatorname{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\operatorname{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$.

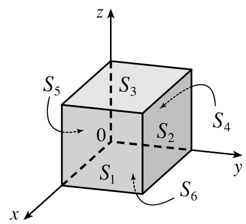
(b) As in (a), $\operatorname{curl}(f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f) \cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f\nabla f)] \cdot d\mathbf{S} = 0$.

(c) As in part (a),

$$\begin{aligned} \operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) && [\text{by Exercise 16.5.26}] \\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} && [\text{since } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})] \end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f) \cdot d\mathbf{S} = 0$.

16.9 The Divergence Theorem



1. $\mathbf{F}(x,y,z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k} \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, on

S_1: $x = 1$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, $\mathbf{n} = \mathbf{i}$, and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{i}\,d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

S_2: $y = 1$, $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$, and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot\mathbf{j}\,d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

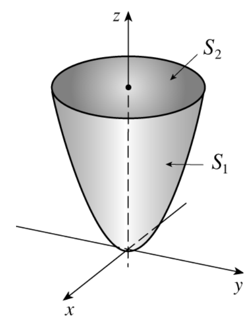
S_3: $z = 1$, $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$, and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3}\mathbf{F}\cdot\mathbf{k}\,d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $x = 0$, $\mathbf{F} = \mathbf{0}$, $\mathbf{n} = -\mathbf{i}$, $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 0$;

S_5: $y = 0$, $\mathbf{F} = 3x\,\mathbf{i} + 2xz\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$, and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_5}\mathbf{F}\cdot(-\mathbf{j})\,d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $z = 0$, $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$, and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_6}\mathbf{F}\cdot(-\mathbf{k})\,d\mathbf{S} = \iint_{S_6} 0\,dS = 0$.

Thus, $\iint_S \mathbf{F}\cdot d\mathbf{S} = 3 + \frac{1}{2} + 1 + 0 + 0 + 0 = \frac{9}{2}$.



2. $\mathbf{F}(x,y,z) = y^2z^3\,\mathbf{i} + 2yz\,\mathbf{j} + 4z^2\,\mathbf{k} \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = 0 + 2z + 8z = 10z$,

so, using cylindrical coordinates,

$$\begin{aligned}
\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E 10z\,dV = \int_0^{2\pi}\int_0^3\int_{r^2}^9 (10z)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^3 \left[5rz^2\right]_{z=r^2}^{z=9}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (405r - 5r^5)\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^3 (405r - 5r^5)\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{405}{2}r^2 - \tfrac{5}{6}r^6\right]_0^3 \\
&= 2\pi\left(\tfrac{3645}{2} - \tfrac{1215}{2}\right) = 2430\pi
\end{aligned}$$

On S_1: The surface is $z = x^2 + y^2$, $x^2 + y^2 \le 9$, with downward orientation. Then, by Equation 16.7.10,

$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= -\iint_D [-(y^2z^3)(2x) - (2yz)(2y) + (4z^2)]\,dA \\
&= \iint_D \left[2xy^2(x^2+y^2)^3 + 4y^2(x^2+y^2) - 4(x^2+y^2)^2\right] dA \\
&= \int_0^{2\pi}\int_0^3 (2r^3\cos\theta\sin^2\theta\cdot r^6 + 4r^2\sin^2\theta\cdot r^2 - 4r^4)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^3 (2r^{10}\sin^2\theta\cos\theta + 4r^5\sin^2\theta - 4r^5)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\tfrac{2}{11}r^{11}\sin^2\theta\cos\theta + \tfrac{2}{3}r^6\sin^2\theta - \tfrac{2}{3}r^6\right]_{r=0}^{r=3} d\theta \\
&= \int_0^{2\pi}\left(\tfrac{354{,}294}{11}\sin^2\theta\cos\theta + 486\sin^2\theta - 486\right) d\theta \\
&= \left[\tfrac{354{,}294}{11}\cdot\tfrac{1}{3}\sin^3\theta + 486\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) - 486\theta\right]_0^{2\pi} \\
&= 0 + 486(\pi - 0) - 486(2\pi) = -486\pi
\end{aligned}$$

On S_2: The surface is $z = 9$, $x^2 + y^2 \le 9$, with upward orientation and $\mathbf{n} = \mathbf{k}$, so

$$\begin{aligned}
\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} &= \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,d\mathbf{S} = \iint_{S_2} 4z^2\,d\mathbf{S} = \iint_{x^2+y^2\le 9} 4(9)^2\,dA \\
&= 324(\text{area of circle}) = 324\cdot\pi(3)^2 = 2916\pi
\end{aligned}$$

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = -486\pi + 2916\pi = 2430\pi$.

3. $\mathbf{F}(x,y,z) = \langle z, y, x \rangle \Rightarrow \operatorname{div}\mathbf{F} = 0 + 1 + 0 = 1$, so $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = V(E) = \frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi$. S is a sphere of radius 4 centered at the origin which can be parametrized by $\mathbf{r}(\phi,\theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi \rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ (similar to Example 16.6.10). Then

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi \rangle \times \langle -4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0 \rangle \\ &= \left\langle 16\sin^2\phi\cos\theta, 16\sin^2\phi\sin\theta, 16\cos\phi\sin\phi \right\rangle\end{aligned}$$

and $\mathbf{F}(\mathbf{r}(\phi,\theta)) = \langle 4\cos\phi, 4\sin\phi\sin\theta, 4\sin\phi\cos\theta \rangle$. Thus,

$$\begin{aligned}\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) &= 64\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta + 64\cos\phi\sin^2\phi\cos\theta \\ &= 128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta\end{aligned}$$

and

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi} \left(128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta\right) d\phi\,d\theta \\ &= \int_0^{2\pi} \left[\tfrac{128}{3}\sin^3\phi\cos\theta + 64\left(\tfrac{1}{3}\cos^3\phi - \cos\phi\right)\sin^2\theta\right]_{\phi=0}^{\phi=\pi} d\theta \\ &= \int_0^{2\pi} \tfrac{256}{3}\sin^2\theta\,d\theta = \tfrac{256}{3}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{256}{3}\pi\end{aligned}$$

4. $F(x,y,z) = \left\langle x^2, -y, z \right\rangle \Rightarrow \operatorname{div}\mathbf{F} = 2x - 1 + 1 = 2x$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint\limits_{y^2+z^2\le 9} \left[\int_0^2 2x\,dx\right] dA = \iint\limits_{y^2+z^2\le 9} 4\,dA = 4(\text{area of circle}) = 4(\pi \cdot 3^2) = 36\pi$$

Let S_1 be the front of the cylinder (in the plane $x = 2$), S_2 the back (in the yz-plane), and S_3 the lateral surface of the cylinder. S_1 is the disk $x = 2$, $y^2 + z^2 \le 9$. A unit normal vector is $\mathbf{n} = \langle 1,0,0 \rangle$ and $\mathbf{F} = \langle 4, -y, z \rangle$ on S_1, so $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_{S_1} 4\,dS = 4(\text{surface area of } S_1) = 4(\pi \cdot 3^2) = 36\pi$. S_2 is the disk $x = 0$, $y^2 + z^2 \le 9$. Here $\mathbf{n} = \langle -1,0,0 \rangle$ and $\mathbf{F} = \langle 0, -y, z \rangle$, so $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

S_3 can be parametrized by $\mathbf{r}(x,\theta) = \langle x, 3\cos\theta, 3\sin\theta \rangle$, $0 \le x \le 2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 1,0,0 \rangle \times \langle 0, -3\sin\theta, 3\cos\theta \rangle = \langle 0, -3\cos\theta, -3\sin\theta \rangle$. For the outward (positive) orientation we use $-(\mathbf{r}_x \times \mathbf{r}_\theta)$ and $\mathbf{F}(\mathbf{r}(x,\theta)) = \left\langle x^2, -3\cos\theta, 3\sin\theta \right\rangle$, so

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (-(\mathbf{r}_x \times \mathbf{r}_\theta))\;dA = \int_0^2\int_0^{2\pi} \left(0 - 9\cos^2\theta + 9\sin^2\theta\right) d\theta\,dx \\ &= -9\int_0^2 dx\int_0^{2\pi}\cos 2\theta\,d\theta = -9\,(2)\left[\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = 0\end{aligned}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = 36\pi + 0 + 0 = 36\pi$.

5. $\mathbf{F}(x,y,z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k} \Rightarrow$

$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) = ye^z + 2xyz^3 - ye^z = 2xyz^3$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^1 2xyz^3\,dz\,dy\,dx = 2\int_0^3 x\,dx\int_0^2 y\,dy\int_0^1 z^3\,dz \\ &= 2\left[\tfrac{1}{2}x^2\right]_0^3\left[\tfrac{1}{2}y^2\right]_0^2\left[\tfrac{1}{4}z^4\right]_0^1 = 2\left(\tfrac{9}{2}\right)(2)\left(\tfrac{1}{4}\right) = \tfrac{9}{2}\end{aligned}$$

6. $\mathbf{F}(x,y,z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k} \quad \Rightarrow$

$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^a\int_0^b\int_0^c 6xyz\,dz\,dy\,dx = 6\int_0^a x\,dx\int_0^b y\,dy\int_0^c z\,dz \\ &= 6\left[\tfrac{1}{2}x^2\right]_0^a\left[\tfrac{1}{2}y^2\right]_0^b\left[\tfrac{1}{2}z^2\right]_0^c = 6\left(\tfrac{1}{2}a^2\right)\left(\tfrac{1}{2}b^2\right)\left(\tfrac{1}{2}c^2\right) = \tfrac{3}{4}a^2b^2c^2\end{aligned}$$

7. $\mathbf{F}(x,y,z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k} \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have, by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (3y^2+3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta \\ &= 3\int_0^{2\pi}d\theta\int_0^1 r^3\,dr\int_{-1}^2 dx = 3\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1\left[\,x\,\right]_{-1}^2 = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}\end{aligned}$$

8. $\mathbf{F}(x,y,z) = (x^3+y^3)\,\mathbf{i} + (y^3+z^3)\,\mathbf{j} + (z^3+x^3)\,\mathbf{k} \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = 3x^2+3y^2+3z^2$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^\pi\int_0^{2\pi}\int_0^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 3\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^4\,d\rho \\ &= 3\left[-\cos\phi\right]_0^\pi\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_0^2 = 3\,(2)\,(2\pi)\left(\tfrac{32}{5}\right) = \tfrac{384}{5}\pi\end{aligned}$$

9. $\mathbf{F}(x,y,z) = xe^y\,\mathbf{i} + (z-e^y)\,\mathbf{j} - xy\,\mathbf{k} \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = e^y + (-e^y) + 0 = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

10. $\mathbf{F}(x,y,z) = e^y\tan z\,\mathbf{i} + x^2y\,\mathbf{j} + e^x\cos y\,\mathbf{k} \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = x^2$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_{-1}^1\int_{-1}^1\int_0^{2-x-y^3} x^2\,dz\,dx\,dy = \int_{-1}^1\int_{-1}^1 x^2(2-x-y^3)\,dx\,dy \\ &= \int_{-1}^1\int_{-1}^1(2x^2-x^3-x^2y^3)\,dx\,dy = \int_{-1}^1\int_{-1}^1(2x^2-x^2y^3)\,dx\,dy \qquad [x^3 \text{ is odd}] \\ &= \int_{-1}^1\left[\frac{2}{3}x^3 - \frac{x^3}{3}y^3\right]_{x=-1}^{x=1}dy = \int_{-1}^1\left(\frac{4}{3}-\frac{2}{3}y^3\right)dy = 2\int_0^1\frac{4}{3}\,dy \qquad \begin{bmatrix}\frac{4}{3} \text{ is even,} \\ y^3 \text{ is odd}\end{bmatrix} \\ &= \frac{8}{3}\end{aligned}$$

11. $\mathbf{F}(x,y,z) = (2x^3+y^3)\,\mathbf{i} + (y^3+z^3)\,\mathbf{j} + 3y^2z\,\mathbf{k} \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = 6x^2+3y^2+3y^2 = 6x^2+6y^2$, so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 6(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 6r^3(1-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^1(6r^3-6r^5)\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{3}{2}r^4 - r^6\right]_0^1 = 2\pi\left(\tfrac{3}{2}-1\right) = \pi\end{aligned}$$

12. $\mathbf{F}(x,y,z) = (xy+2xz)\,\mathbf{i} + (x^2+y^2)\,\mathbf{j} + (xy-z^2)\,\mathbf{k}$. For $x^2+y^2\le 4$ the plane $z = y-2$ is below the xy-plane, so the solid E bounded by S is $E = \{(x,y,z) \mid x^2+y^2\le 4,\ y-2\le z\le 0\}$. Here $\operatorname{div}\mathbf{F} = y+2z+2y-2z = 3y$, so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3y\,dV = \int_0^{2\pi}\int_0^2\int_{r\sin\theta-2}^0 (3r\sin\theta)\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2(3r^2\sin\theta)(0-r\sin\theta+2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left(-3r^3\sin^2\theta+6r^2\sin\theta\right)dr\,d\theta \\ &= \int_0^{2\pi}\left[-\tfrac{3}{4}r^4\sin^2\theta + 2r^3\sin\theta\right]_{r=0}^{r=2}d\theta = \int_0^{2\pi}\left(-12\sin^2\theta+16\sin\theta\right)d\theta \\ &= \left[-12\left(\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right) - 16\cos\theta\right]_0^{2\pi} = -12\pi - 16 + 16 = -12\pi\end{aligned}$$

13. $\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} + xz^3\,\mathbf{j} + y\ln(x+1)\,\mathbf{k} \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = 2xz$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^4\int_0^3\int_0^{2-(1/2)x} 2xz\,dz\,dy\,dx = \int_0^4\int_0^3 x\left(2-\tfrac{1}{2}x\right)^2\,dy\,dx \\ &= \int_0^4\int_0^3\left(4x - 2x^2 + \tfrac{1}{4}x^3\right)dy\,dx = 3\int_0^4\left(4x-2x^2+\tfrac{1}{4}x^3\right)dx \\ &= 3\left[2x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{16}x^4\right]_0^4 = 3\left(32 - \tfrac{128}{3} + 16\right) = 16\end{aligned}$$

14. $\mathbf{F}(x,y,z) = (xy - z^2)\,\mathbf{i} + x^3\sqrt{z}\,\mathbf{j} + (xy+z^2)\,\mathbf{k} \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = y + 2z$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_{-1}^1\int_{y^2}^1\int_0^{1-x}(y+2z)\,dz\,dx\,dy \\ &= \int_{-1}^1\int_{y^2}^1\left[y(1-x)+(1-x)^2\right]dx\,dz = \int_{-1}^1\int_{y^2}^1 (y - yx + 1 - 2x + x^2)\,dx\,dy \\ &= \int_{-1}^1\left[yx - y\,\frac{x^2}{2} + x - x^2 + \frac{x^3}{3}\right]_{x=y^2}^{x=1} dy = \int_{-1}^1\left(-\frac{y^6}{3} + \frac{y^5}{2} + y^4 - y^3 - y^2 + \frac{y}{2} + \frac{1}{3}\right)dy \\ &= 2\int_0^1\left(-\frac{y^6}{3} + y^4 - y^2 + \frac{1}{3}\right)dy = 2\left[-\frac{y^7}{21} + \frac{y^5}{5} - \frac{y^3}{3} + \frac{y}{3}\right]_0^1 = \frac{32}{105}\end{aligned}$$

15. The tetrahedron has vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and is described by

$$E = \left\{(x,y,z) \mid 0 \le x \le a, 0 \le y \le b\left(1-\tfrac{x}{a}\right), 0 \le z \le c\left(1 - \tfrac{x}{a} - \tfrac{y}{b}\right)\right\}$$

Here we have $\mathbf{F}(x,y,z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k} \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = 0 + 1 + x = x + 1$, so

$$\begin{aligned}\textstyle\iint_S \mathbf{F}\cdot d\mathbf{S} &= \textstyle\iiint_E (x+1)\,dV = \int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})}(x+1)\,dz\,dy\,dx \\ &= \textstyle\int_0^a\int_0^{b(1-\frac{x}{a})}(x+1)\left[c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right]dy\,dx \\ &= \textstyle c\int_0^a(x+1)\left[\left(1-\frac{x}{a}\right)y - \frac{1}{2b}y^2\right]_{y=0}^{y=b(1-\frac{x}{a})}dx \\ &= \textstyle c\int_0^a(x+1)\left[\left(1-\frac{x}{a}\right)\cdot b\left(1-\frac{x}{a}\right) - \frac{1}{2b}\cdot b^2\left(1-\frac{x}{a}\right)^2\right]dx = \frac{1}{2}bc\int_0^a(x+1)\left(1-\frac{x}{a}\right)^2dx \\ &= \textstyle\frac{1}{2}bc\int_0^a\left(\frac{1}{a^2}x^3 + \frac{1}{a^2}x^2 - \frac{2}{a}x^2 + x - \frac{2}{a}x + 1\right)dx \\ &= \textstyle\frac{1}{2}bc\left[\frac{1}{4a^2}x^4 + \frac{1}{3a^2}x^3 - \frac{2}{3a}x^3 + \frac{1}{2}x^2 - \frac{1}{a}x^2 + x\right]_0^a \\ &= \textstyle\frac{1}{2}bc\left(\frac{1}{4}a^2 + \frac{1}{3}a - \frac{2}{3}a^2 + \frac{1}{2}a^2 - a + a\right) = \frac{1}{2}bc\left(\frac{1}{12}a^2 + \frac{1}{3}a\right) = \frac{1}{24}abc(a+4)\end{aligned}$$

16. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}| = \sqrt{x^2+y^2+z^2} \quad\Rightarrow$

$\mathbf{F}(x,y,z) = |\mathbf{r}|^2\,\mathbf{r} = x(x^2+y^2+z^2)\,\mathbf{i} + y(x^2+y^2+z^2)\,\mathbf{j} + z(x^2+y^2+z^2)\,\mathbf{k} \quad\Rightarrow$

$\operatorname{div}\mathbf{F} = x\cdot 2x + (x^2+y^2+z^2) + y\cdot 2y + (x^2+y^2+z^2) + z\cdot 2z + (x^2+y^2+z^2) = 5(x^2+y^2+z^2)$. Then

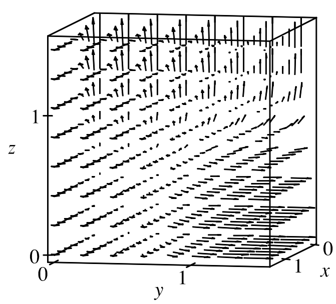
$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 5(x^2+y^2+z^2)\,dV = \int_0^\pi\int_0^{2\pi}\int_0^R 5\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \textstyle 5\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^R\rho^4\,d\rho = 5\left[-\cos\phi\right]_0^\pi\left[\theta\right]_0^{2\pi}\left[\frac{1}{5}\rho^5\right]_0^R = 5\,(2)\,(2\pi)\left(\frac{1}{5}R^5\right) = 4\pi R^5\end{aligned}$$

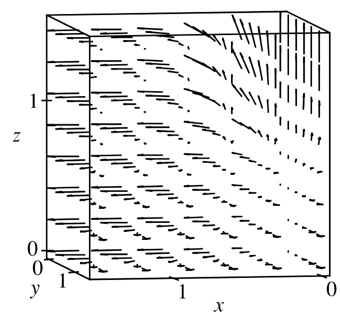
17. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{r}| = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$\mathbf{F}(x,y,z) = |\mathbf{r}|\,\mathbf{r} = x\sqrt{x^2+y^2+z^2}\,\mathbf{i} + y\sqrt{x^2+y^2+z^2}\,\mathbf{j} + z\sqrt{x^2+y^2+z^2}\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\operatorname{div}\mathbf{F} &= x\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) + (x^2+y^2+z^2)^{1/2} \\ &\quad + y\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) + (x^2+y^2+z^2)^{1/2} \\ &\quad + z\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) + (x^2+y^2+z^2)^{1/2} \\ &= (x^2+y^2+z^2)^{-1/2}\left[x^2 + (x^2+y^2+z^2) + y^2 + (x^2+y^2+z^2) + z^2 + (x^2+y^2+z^2)\right] \\ &= \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}\end{aligned}$$

Then $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 4\sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 4\sqrt{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1 4\rho^3\,d\rho = [-\cos\phi]_0^{\pi/2}\,[\theta]_0^{2\pi}\,\left[\rho^4\right]_0^1 = (1)\,(2\pi)\,(1) = 2\pi$$

18. 



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\,\cos^2 y + 3\sin^2 y\,\cos y\,\cos^4 z + 5\sin^4 z\,\cos z\,\cos^6 x\right]dz\,dy\,dx = \tfrac{19}{64}\pi^2$$

19. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D(-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)\,r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan^{-1}z\right) + \frac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally, $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

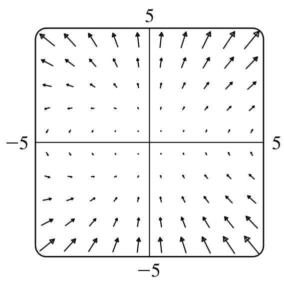
20. As in the hint to Exercise 19, we create a closed surface $S_2 = S\cup S_1$, where S is the part of the paraboloid $x^2+y^2+z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2+y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot(-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}(-1)\,dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E\operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = \int_0^1\int_0^{2\pi}\int_1^{2-r^2} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi}(r - r^3)\,d\theta\,dr = (2\pi)\frac{1}{4} = \frac{\pi}{2}$. Thus the flux of $\mathbf{F}$ across S is $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

22. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

(b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \Rightarrow \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1+2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1+2y$ is negative, and P_2 is a sink.

23.

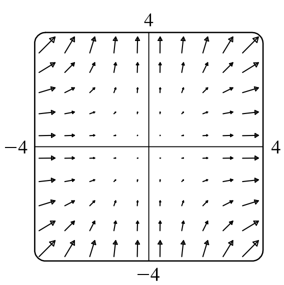


From the graph it appears that for points above the x-axis, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the x-axis, where divergence is negative.

$\mathbf{F}(x,y) = \langle xy, x+y^2\rangle \Rightarrow \operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(x+y^2) = y + 2y = 3y$.

Thus $\operatorname{div}\mathbf{F} > 0$ for $y > 0$, and $\operatorname{div}\mathbf{F} < 0$ for $y < 0$.

24.



From the graph it appears that for points above the line $y = -x$, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the line $y = -x$, where divergence is negative.

$\mathbf{F}(x,y) = \langle x^2, y^2\rangle \Rightarrow \operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) = 2x + 2y$. Then $\operatorname{div}\mathbf{F} > 0$ for $2x + 2y > 0 \Rightarrow y > -x$, and $\operatorname{div}\mathbf{F} < 0$ for $y < -x$.

25. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at $(0,0,0)$ where it is undefined.

26. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (2x+2y+z^2)\,dS$, so $\mathbf{F}\cdot\mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div}\mathbf{F} = 1$.

If $B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$, then, by the Divergence Theorem,

$$\iint_S (2x+2y+z^2)\,dS = \iiint_B dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \tfrac{4}{3}\pi$$

27. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{a}\,dV = 0$ since $\operatorname{div}\mathbf{a} = 0$.

28. $\frac{1}{3}\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\iiint_E \operatorname{div}\mathbf{F}\,dV = \frac{1}{3}\iiint_E 3\,dV = V(E)$

29. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ by Theorem 16.5.11.

30. $\iint_S D_{\mathbf{n}} f\,dS = \iint_S(\nabla f\cdot\mathbf{n})\,dS = \iiint_E \operatorname{div}(\nabla f)\,dV = \iiint_E \nabla^2 f\,dV$

31.
$$\begin{aligned}\iint_S (f\nabla g)\cdot\mathbf{n}\,dS &= \iiint_E \operatorname{div}(f\,\nabla g)\,dV && [\text{by (1) with } \mathbf{F} = f\,\nabla g]\\ &= \iiint_E [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dV && [\text{by Exercise 16.5.27}]\\ &= \iiint_E (f\,\nabla^2 g + \nabla f\cdot\nabla g)\,dV\end{aligned}$$

32. $\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E \left[(f\,\nabla^2 g + \nabla f\cdot\nabla g) - (g\nabla^2 f + \nabla g\cdot\nabla f)\right]dV$ [by Exercise 31].

But $\nabla g\cdot\nabla f = \nabla f\cdot\nabla g$, so that $\iint_S (f\,\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\,dV$.

33. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then

$\operatorname{div}\mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x}\,c_1 + \dfrac{\partial f}{\partial y}\,c_2 + \dfrac{\partial f}{\partial z}\,c_3 = \nabla f\cdot\mathbf{c}$ and the Divergence Theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV \quad\Rightarrow$

$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{c}\,dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{i}\,dV \quad\Rightarrow$

$\displaystyle\iint_S fn_1\,dS = \iiint_E \frac{\partial f}{\partial x}\,dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S fn_2\,dS = \iiint_E \frac{\partial f}{\partial y}\,dV$,

and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S fn_3\,dS = \iiint_E \frac{\partial f}{\partial z}\,dV$. Then

$$\begin{aligned}\iint_S f\mathbf{n}\,dS &= \left(\iint_S fn_1\,dS\right)\mathbf{i} + \left(\iint_S fn_2\,dS\right)\mathbf{j} + \left(\iint_S fn_3\,dS\right)\mathbf{k}\\ &= \left(\iiint_E \frac{\partial f}{\partial x}\,dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\,dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\,dV\right)\mathbf{k} = \iiint_E \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right)dV\\ &= \iiint_E \nabla f\,dV \quad \text{as desired.}\end{aligned}$$

34. By Exercise 33, $\iint_S p\mathbf{n}\,dS = \iiint_E \nabla p\,dV$, so

$\mathbf{F} = -\iint_S p\mathbf{n}\,dS = -\iiint_E \nabla p\,dV = -\iiint_E \nabla(\rho g z)\,dV = -\iiint_E (\rho g\,\mathbf{k})\,dV = -\rho g\left(\iiint_E dV\right)\mathbf{k} = -\rho g V(E)\,\mathbf{k}$.

But the weight of the displaced liquid is volume $\times$ density $\times$ $g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

16 Review

TRUE-FALSE QUIZ

1. False. $\operatorname{div}\mathbf{F}$ is a scalar field.
2. True. See Definition 16.5.1.
3. True. Use Theorem 16.5.3 and the fact that $\operatorname{div}\mathbf{0} = 0$.
4. True. See Theorem 16.3.2.

5. False. See Exercise 16.3.41. (But the assertion is true if D is simply-connected; see Theorem 16.3.6.)

6. False. See the discussion accompanying Figure 8 in Section 16.2.

7. False. For example, $\operatorname{div}(y\,\mathbf{i}) = 0 = \operatorname{div}(x\,\mathbf{j})$ but $y\,\mathbf{i} \neq x\,\mathbf{j}$.

8. True. Line integrals of conservative vector fields are independent of path, and by Theorem 16.3.3, work $= \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C.

9. True. See Exercise 16.5.26.

10. False. $\mathbf{F} \cdot \mathbf{G}$ is a scalar field, so $\operatorname{curl}(\mathbf{F} \cdot \mathbf{G})$ has no meaning.

11. True. Apply the Divergence Theorem and use the fact that $\operatorname{div}\mathbf{F} = 0$ since $\mathbf{F}$ is a constant vector field.

12. False. See Theorem 16.5.11. If the statement were true, then $\operatorname{div}\operatorname{curl}\mathbf{F}$ would equal $1 + 1 + 1 = 3 \neq 0$.

13. False. By Formulas 16.4.5, the area is given by $-\oint_C y\,dx$ or $\oint_C x\,dy$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and $\operatorname{div}\mathbf{F}(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \leq x \leq 1$ so

$$\int_C x\,ds = \int_0^1 x\,\sqrt{1 + (2x)^2}\,dx = \tfrac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\big(5\sqrt{5} - 1\big).$$

3.
$$\begin{aligned}\int_C yz\cos x\,ds &= \int_0^\pi (3\cos t)\,(3\sin t)\cos t\,\sqrt{(1)^2 + (-3\sin t)^2 + (3\cos t)^2}\,dt = \int_0^\pi (9\cos^2 t\,\sin t)\sqrt{10}\,dt \\ &= 9\sqrt{10}\,\big(-\tfrac{1}{3}\cos^3 t\big)\big]_0^\pi = -3\sqrt{10}\,(-2) = 6\sqrt{10}\end{aligned}$$

4. $x = 3\cos t \quad\Rightarrow\quad dx = -3\sin t\,dt$, $y = 2\sin t \quad\Rightarrow\quad dy = 2\cos t\,dt$, $0 \leq t \leq 2\pi$, so

$$\begin{aligned}\int_C y\,dx + \big(x + y^2\big)\,dy &= \int_0^{2\pi} \big[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\big]\,dt \\ &= \int_0^{2\pi} (-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\,dt = \int_0^{2\pi} \big[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\big]\,dt \\ &= \int_0^{2\pi} (6\cos 2t + 8\sin^2 t\,\cos t)\,dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\big]_0^{2\pi} = 0\end{aligned}$$

Or: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}\big(x + y^2\big)$, so $\mathbf{F}(x, y) = \big\langle y, x + y^2 \big\rangle$ is a conservative vector field. Since C is a closed curve, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx + (x + y^2)\,dy = 0$.

5. $\int_C y^3\,dx + x^2\,dy = \int_{-1}^{1}\left[y^3(-2y) + (1-y^2)^2\right]dy = \int_{-1}^{1}(-y^4 - 2y^2 + 1)\,dy$

$$= \left[-\tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 + y\right]_{-1}^{1} = -\tfrac{1}{5} - \tfrac{2}{3} + 1 - \tfrac{1}{5} - \tfrac{2}{3} + 1 = \tfrac{4}{15}$$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1\left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt = \int_0^1(4t^6 + 2te^{t^2} + 3t^9)\,dt$

$$= \left[\tfrac{4}{7}t^7 + e^{t^2} + \tfrac{3}{10}t^{10}\right]_0^1 = e - \tfrac{9}{70}$$

7. C: $x = 1 + 2t \;\;\Rightarrow\;\; dx = 2\,dt,\; y = 4t \;\;\Rightarrow\;\; dy = 4\,dt,\; z = -1 + 3t \;\;\Rightarrow\;\; dz = 3\,dt,\; 0 \le t \le 1.$

$$\int_C xy\,dx + y^2\,dy + yz\,dz = \int_0^1[(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)]\,dt$$

$$= \int_0^1(116t^2 - 4t)\,dt = \left[\tfrac{116}{3}t^3 - 2t^2\right]_0^1 = \tfrac{116}{3} - 2 = \tfrac{110}{3}$$

8. $\mathbf{F}(x,y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1+t)\,\mathbf{j},\; 0 \le t \le \pi \;\;\Rightarrow\;\; \mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j},$

$\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^\pi\left(\tfrac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt$$

$$= \left[\tfrac{1}{2}\left((1+t)\left(-\tfrac{1}{2}\cos 2t\right) + \tfrac{1}{4}\sin 2t\right) + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^\pi = \tfrac{\pi}{4}$$

9. $\mathbf{F}(x,y,z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x+y)\,\mathbf{k}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k},\; 0 \le t \le 1 \;\;\Rightarrow$

$\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2 + t^3)\,\mathbf{k},\; \mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1(2te^{-t} - 3t^5 - (t^2 + t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \tfrac{1}{2}t^6 - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4\right]_0^1 = \tfrac{11}{12} - \tfrac{4}{e}.$

10. (a) $\mathbf{F}(x,y,z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ and C: $x = 3 - 3t,\; y = \tfrac{\pi}{2}t,\; z = 3t,\; 0 \le t \le 1$. Then

$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right]dt = \int_0^1\left[-9t + \tfrac{3\pi}{2}\right]dt = \tfrac{1}{2}(3\pi - 9).$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$= \int_0^{\pi/2}(-9\sin^2 t + 3\cos t + 3t\cos t)\,dt = \left[-9\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2}$

$= -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}$

11. $\mathbf{F}(x,y) = (1+xy)e^{xy}\,\mathbf{i} + (e^y + x^2e^{xy})\,\mathbf{j} \;\;\Rightarrow\;\; \frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \;\;\Rightarrow\;\; g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}(x,y,z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 16.5.4. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then

$f_y(x, y, z) = x\cos y + g_y(y, z)$. But $f_y(x, y, z) = x\cos y$, so $g_y(y, z) = 0 \;\Rightarrow\; g(y, z) = h(z)$. Then $f(x, y, z) = x\sin y + h(z)$ implies $f_z(x, y, z) = h'(z)$. But $f_z(x, y, z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x, y, z) = x\sin y + \cos z + K$.

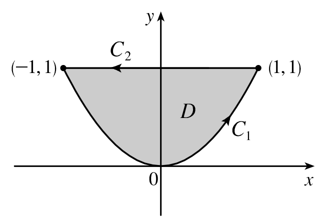
13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j} \;\Rightarrow$

$\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Furthermore, $f(x, y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0, 1)$ and $t = 1$ corresponds to $(1, 1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$.

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k} \;\Rightarrow\; \operatorname{curl}\mathbf{F} = (e^z - e^z)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (e^y - e^y)\,\mathbf{k} = \mathbf{0}$. The domain of $\mathbf{F}$ is $\mathbb{R}^3$ and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore, we can show that $f(x, y, z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$.

15. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.



Then

$$\begin{aligned}\int_C xy^2\,dx - x^2y\,dy &= \int_{-1}^{1}[t(t^2)^2(1) - t^2(t^2)(2t)]\,dt \\ &\qquad + \int_{-1}^{1}[(-t)(1)^2(-1) - (-t)^2(1)(0)]\,dt \\ &= \int_{-1}^{1}(-t^5)\,dt + \int_{-1}^{1} t\,dt \\ &= 0 \qquad [-t^5 \text{ and } t \text{ are odd}]\end{aligned}$$

Using Green's Theorem, we have

$$\begin{aligned}\int_C xy^2\,dx - x^2y\,dy &= \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA = \iint_D (-2xy - 2xy)\,dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx \\ &= \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1}(2x^5 - 2x)\,dx = 0 \qquad [2x^5 - 2x \text{ is odd}]\end{aligned}$$

16. $\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1 + x^3}\right)\right] dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx = \int_0^1 9x^2\,dx = \left[3x^3\right]_0^1 = 3$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint_{x^2+y^2\le 4}\left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint_{x^2+y^2\le 4}(-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$

18. $\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k} \;\Rightarrow$

$\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$, and $\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$.

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \operatorname{div}(2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}) = 2 + 3z - 2xz$. Since $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$ [by (16.5.11)], such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} = &\left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\\ &- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right.\\ &\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

and

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} = &\left[\left(P_2\,\frac{\partial P_1}{\partial x} + Q_2\,\frac{\partial P_1}{\partial y} + R_2\,\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\,\frac{\partial Q_1}{\partial x} + Q_2\,\frac{\partial Q_1}{\partial y} + R_2\,\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right.\\ &\left. + \left(P_2\,\frac{\partial R_1}{\partial x} + Q_2\,\frac{\partial R_1}{\partial y} + R_2\,\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\\ &- \left[\left(P_1\,\frac{\partial P_2}{\partial x} + Q_1\,\frac{\partial P_2}{\partial y} + R_1\,\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\,\frac{\partial Q_2}{\partial x} + Q_1\,\frac{\partial Q_2}{\partial y} + R_1\,\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right.\\ &\left. + \left(P_1\,\frac{\partial R_2}{\partial x} + Q_1\,\frac{\partial R_2}{\partial y} + R_1\,\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G}\\ &\quad= \left[\left(P_1\,\frac{\partial Q_2}{\partial y} + Q_2\,\frac{\partial P_1}{\partial x}\right) - \left(P_2\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial P_2}{\partial y}\right)\right.\\ &\qquad\left. - \left(P_2\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial P_2}{\partial z}\right) + \left(P_1\,\frac{\partial R_2}{\partial z} + R_2\,\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}\\ &\qquad + \left[\left(Q_1\,\frac{\partial R_2}{\partial z} + R_2\,\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial Q_2}{\partial z}\right)\right.\\ &\qquad\left. - \left(P_1\,\frac{\partial Q_2}{\partial x} + Q_2\,\frac{\partial P_1}{\partial x}\right) + \left(P_2\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}\\ &\qquad + \left[\left(P_2\,\frac{\partial R_1}{\partial x} + R_1\,\frac{\partial P_2}{\partial x}\right) - \left(P_1\,\frac{\partial R_2}{\partial x} + R_2\,\frac{\partial P_1}{\partial x}\right)\right.\\ &\qquad\left. - \left(Q_1\,\frac{\partial R_2}{\partial y} + R_2\,\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}\\ &\quad= \left[\frac{\partial}{\partial y}\,(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}\,(P_2R_1 - P_1R_2)\right]\mathbf{i}\\ &\qquad + \left[\frac{\partial}{\partial z}\,(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}\,(P_1Q_2 - P_2Q_1)\right]\mathbf{j}\\ &\qquad + \left[\frac{\partial}{\partial x}\,(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}\,(Q_1R_2 - Q_2R_1)\right]\mathbf{k}\\ &\quad= \operatorname{curl}(\mathbf{F}\times\mathbf{G})\end{aligned}$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x}\,g(y) - \frac{\partial}{\partial y}\,f(x)\right] dA = \iint_D 0\,dA = 0.$

22. $\nabla^2(fg) = \dfrac{\partial^2(fg)}{\partial x^2} + \dfrac{\partial^2(fg)}{\partial y^2} + \dfrac{\partial^2(fg)}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\,g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\,g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\,g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\,g + 2\,\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,g + 2\,\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,g + 2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 14.6.43(b), 16.5.25, and 16.5.27 [with div $\mathbf{F} = \nabla \cdot \mathbf{F}$], we have

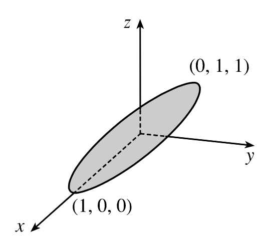
$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (f\,\nabla g + g\,\nabla f) = f\nabla^2 g + \nabla g \cdot \nabla f + g\nabla^2 f + \nabla f \cdot \nabla g$$
$$= f\,\nabla^2 g + g\,\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA$$
$$= -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 16.3.3.

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is contained in the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$; with $0 \le t \le 2\pi$ we get the entire intersection (an ellipse).



(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

25. $z = f(x, y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

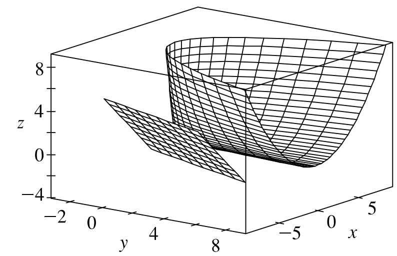
$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\,dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5 + 4x^2}\,dx = \frac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \frac{1}{6}\left(27 - 5\sqrt{5}\right).$

26. (a) $\mathbf{r}(u,v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k}$, $0 \le u \le 3$, $-3 \le v \le 3 \quad \Rightarrow$

$\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and

$\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4,-2,1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4,-2,1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 16.6.6, the area of S is given by

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\,dA = \int_0^3\int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3\int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$$

(d) $\mathbf{F}(x,y,z) = \dfrac{z^2}{1+x^2}\,\mathbf{i} + \dfrac{x^2}{1+y^2}\,\mathbf{j} + \dfrac{y^2}{1+z^2}\,\mathbf{k}$. By Equation 16.7.9, the surface integral is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\,\mathbf{r}_u \times \mathbf{r}_v\,)\,dA = \int_0^3\int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \left\langle 2u^2, 4uv, 2v^2 \right\rangle\,dv\,du$$

$$= \int_0^3\int_{-3}^3 \left(\frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv\,du \approx 1524.0190$$

27. $z = g(x,y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$. By Formula 16.7.4,

$$\iint_S z\,dS = \iint_{x^2+y^2\le 4} (x^2+y^2)\sqrt{(2x)^2 + (2y)^2 + 1}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{17} \left(\tfrac{u-1}{4}\right)\sqrt{u}\,\left(\tfrac{1}{8}\,du\right)d\theta \quad \begin{bmatrix} u = 1 + 4r^2, \\ du = 8r\,dr \end{bmatrix} \quad = \tfrac{1}{32}(2\pi)\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{17}$$

$$= \tfrac{\pi}{16}\cdot\tfrac{2}{15}\left[(3u^2 - 5u)\sqrt{u}\right]_1^{17} = \tfrac{\pi}{120}\left[782\sqrt{17} + 2\right] = \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

28. $z = g(x,y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$. By Formula 16.7.4,

$$\iint_S (x^2z + y^2z)\,dS = \iint_{x^2+y^2\le 4} (x^2+y^2)(4+x+y)\sqrt{1^2+1^2+1}\,dA$$

$$= \int_0^2\int_0^{2\pi} \sqrt{3}\,r^3(4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

29. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (z-2)\,dV = \iiint_E z\,dV - 2\iiint_E dV$$

$$= 0 \begin{bmatrix} \text{odd function in } z \\ \text{and } E \text{ is symmetric} \end{bmatrix} - 2\cdot V(E) = -2\cdot\tfrac{4}{3}\pi(2)^3 = -\tfrac{64}{3}\pi$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\,\cos\theta\,\cos\phi\,\mathbf{i} - 4\sin\phi\,\sin\theta\,\mathbf{j} + 6\sin\phi\,\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\,\cos\theta\,\mathbf{i} + 4\sin^2\phi\,\sin\theta\,\mathbf{j} + 4\sin\phi\,\cos\phi\,\mathbf{k}$, and

$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\,\cos^2\theta\,\cos\phi - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} (16\sin^3\phi\,\cos\phi\,\cos^2\theta - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi$$

30. $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$, $z = f(x,y) = x^2 + y^2$, $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then, by Formula 16.7.9,

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+y^2\le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA \\ &= \int_0^1\int_0^{2\pi}(-2r^3\cos^3\theta - 2r^3\cos\theta\,\sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2} \end{aligned}$$

31. $F(x,y,z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S(\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ and

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-\cos^2 t\,\sin t + \sin^2 t\,\cos t)\,dt = \left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0.$

32. By Stokes' Theorem, $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$. So $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\,\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t\sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t$. Thus,

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}(-16\cos^2 t\,\sin^2 t + 4\sin t\,\cos t)\,dt \\ &= \left[-16\left(-\tfrac{1}{4}\sin t\,\cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi \end{aligned}$$

33. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$. $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (1-x-y)\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then, by Formula 16.7.9,

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D(-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dA \\ &= \iint_D(-1)\,dA = -(\text{area of } D) = -\tfrac{1}{2}. \end{aligned}$$

34. By the Divergence Theorem,

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^2(3r^2+3z^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^1(6r^3+8r)\,dr = 11\pi.$

35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_{x^2+y^2+z^2\le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\,\cos^2\theta + \sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi = \sin\phi$ and

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi.$

36. Here we must use Equation 16.9.7 since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1 = \mathbf{r}/|\mathbf{r}|$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,dS = -\iint_{S_1}\mathbf{F}\cdot(-\mathbf{n}_1)\,dS + \iiint_E \operatorname{div}\mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$ with $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$, so

$\operatorname{div}\mathbf{F} = \nabla\cdot\left(|\mathbf{r}|^{-3}\,\mathbf{r}\right) = |\mathbf{r}|^{-3}(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\left(\nabla\,|\mathbf{r}|^{-3}\right) = |\mathbf{r}|^{-3}\,(3) + \mathbf{r}\cdot\left(-3\,|\mathbf{r}|^{-4}\right)\left(\mathbf{r}\,|\mathbf{r}|^{-1}\right) = 0$

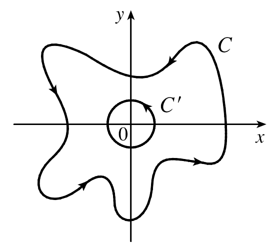
[Exercises 16.5.32 and 16.5.33]. And $\mathbf{F}\cdot\mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1.

Thus $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,dS = \iint_{S_1}dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi.$

37. Because $\operatorname{curl} \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = 3x^2yz - 3y$ implies $f(x, y, z) = x^3yz - 3xy + g(y, z) \quad \Rightarrow \quad f_y(x, y, z) = x^3z - 3x + g_y(y, z)$. But $f_y(x, y, z) = x^3z - 3x$, so $g(y, z) = h(z)$ and $f(x, y, z) = x^3yz - 3xy + h(z)$. Then $f_z(x, y, z) = x^3y + h'(z)$ but $f_z(x, y, z) = x^3y + 2z$, so $h(z) = z^2 + K$ and a potential function for $\mathbf{F}$ is $f(x, y, z) = x^3yz - 3xy + z^2$. Hence

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, 3, 0) - f(0, 0, 2) = 0 - 4 = -4$.

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0,$$

so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \quad [\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}]$$

$$= \int_0^{2\pi} \left[\frac{2a^3\cos^3 t + 2a^3\cos t\,\sin^2 t - 2a\sin t}{a^2} (-a\sin t) + \frac{2a^3\sin^3 t + 2a^3\cos^2 t\,\sin t + 2a\cos t}{a^2} (a\cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$

39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3(\text{volume of } E) = 3(8 - 1) = 21$.

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

41. Let $\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle = \langle a_2z - a_3y, a_3x - a_1z, a_1y - a_2x \rangle$. Then $\operatorname{curl} \mathbf{F} = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{a}$, and $\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$ by Stokes' Theorem.

PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\displaystyle\iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV$.

But
$$\begin{aligned}\nabla\cdot\frac{\mathbf{r}}{r^3} &= \left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}},\frac{y}{(x^2+y^2+z^2)^{3/2}},\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle\\ &= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}} = 0\end{aligned}$$

$\Rightarrow\ \displaystyle\iint_{\partial S_1}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} 0\,dV = 0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n}=0\ \Rightarrow$

$\displaystyle 0=\iint_{\partial S_1}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S_L}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS\quad\Rightarrow$

$\displaystyle\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=-\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r=a\ \Rightarrow\ \mathbf{n}=-\dfrac{\mathbf{r}}{r}=-\dfrac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r}=r^2=a^2$, so

that $\displaystyle-\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS=\iint_{S(a)}\frac{a^2}{a^4}\,dS=\frac{1}{a^2}\iint_{S(a)}dS=\frac{\text{area of }S\,(a)}{a^2}=|\Omega(S)|$.

Therefore $\displaystyle|\Omega(S)|=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

2. By Green's Theorem

$$\int_C (y^3-y)\,dx-2x^3\,dy=\iint_D\left[\frac{\partial(-2x^3)}{\partial x}-\frac{\partial(y^3-y)}{\partial y}\right]dA=\iint_D(1-6x^2-3y^2)\,dA$$

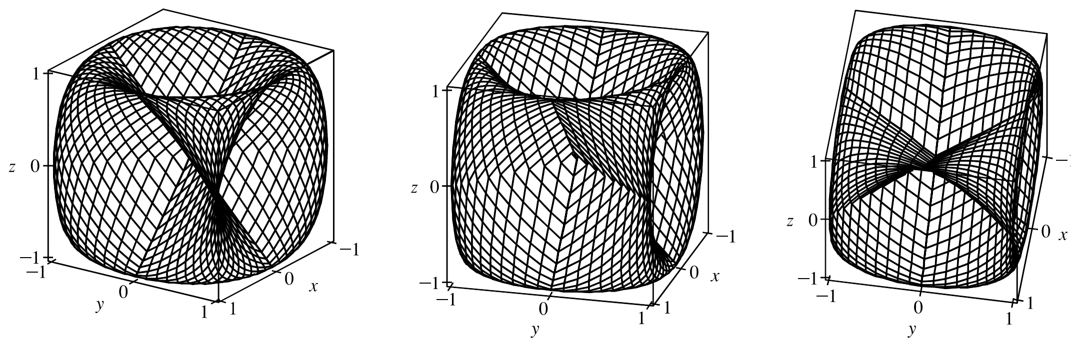
Notice that for $6x^2+3y^2>1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2+3y^2=1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2+3y^2=1$.

3. The given line integral $\frac{1}{2}\int_C(bz-cy)\,dx+(cx-az)\,dy+(ay-bx)\,dz$ can be expressed as $\int_C\mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z)=P\,\mathbf{i}+Q\,\mathbf{j}+R\,\mathbf{k}=\frac{1}{2}(bz-cy)\,\mathbf{i}+\frac{1}{2}(cx-az)\,\mathbf{j}+\frac{1}{2}(ay-bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$. Now

$$\begin{aligned}\operatorname{curl}\mathbf{F}&=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\\&=\left(\tfrac{1}{2}a+\tfrac{1}{2}a\right)\mathbf{i}+\left(\tfrac{1}{2}b+\tfrac{1}{2}b\right)\mathbf{j}+\left(\tfrac{1}{2}c+\tfrac{1}{2}c\right)\mathbf{k}=a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}=\mathbf{n}\end{aligned}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\mathbf{n}\cdot\mathbf{n}=|\mathbf{n}|^2=1$, hence $\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_S dS$ which is simply the surface area of S. Thus, $\int_C\mathbf{F}\cdot d\mathbf{r}=\frac{1}{2}\int_C(bz-cy)\,dx+(cx-az)\,dy+(ay-bx)\,dz$ is the plane area enclosed by C.

4. The surface given by $x = \sin u$, $y = \sin v$, $z = \sin(u + v)$ is difficult to visualize, so we first graph the surface from three different points of view.

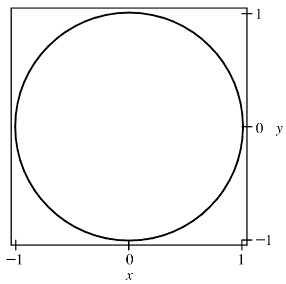


The trace in the horizontal plane $z = 0$ is given by $z = \sin(u + v) = 0 \;\Rightarrow\; u + v = k\pi$ [k an integer]. Then we can write $v = k\pi - u$, and the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin(k\pi - u) = \sin k\pi\,\cos u - \cos k\pi\,\sin u = \pm\sin u$, and since $\sin u = x$, the trace consists of the two lines $y = \pm x$.

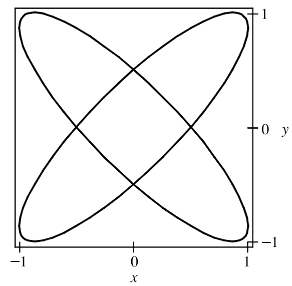
If $z = 1$, $z = \sin(u + v) = 1 \;\Rightarrow\; u + v = \frac{\pi}{2} + 2k\pi$. So $v = \left(\frac{\pi}{2} + 2k\pi\right) - u$ and the trace in $z = 1$ is given by the parametric equations $x = \sin u$, $y = \sin v = \sin\!\left(\left(\frac{\pi}{2} + 2k\pi\right) - u\right) = \sin\!\left(\frac{\pi}{2} + 2k\pi\right)\cos u - \cos\!\left(\frac{\pi}{2} + 2k\pi\right)\sin u = \cos u$. This curve is equivalent to $x^2 + y^2 = 1$, $z = 1$, a circle of radius 1. Similarly, in $z = -1$ we have $z = \sin(u + v) = -1 \;\Rightarrow\; u + v = \frac{3\pi}{2} + 2k\pi \;\Rightarrow\; v = \left(\frac{3\pi}{2} + 2k\pi\right) - u$, so the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin\!\left(\left(\frac{3\pi}{2} + 2k\pi\right) - u\right) = \sin\!\left(\frac{3\pi}{2} + 2k\pi\right)\cos u - \cos\!\left(\frac{3\pi}{2} + 2k\pi\right)\sin u = -\cos u$, which again is a circle, $x^2 + y^2 = 1$, $z = -1$.

If $z = \frac{1}{2}$, $z = \sin(u + v) = \frac{1}{2} \;\Rightarrow\; u + v = \alpha + 2k\pi$ where $\alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Then $v = (\alpha + 2k\pi) - u$ and the trace in $z = \frac{1}{2}$ is given by the parametric equations $x = \sin u$, $y = \sin v = \sin[(\alpha + 2k\pi) - u] = \sin(\alpha + 2k\pi)\,\cos u - \cos(\alpha + 2k\pi)\,\sin u = \frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}\sin u$. In rectangular coordinates, $x = \sin u$ so $y = \frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}x \;\Rightarrow\; y \pm \frac{\sqrt{3}}{2}x = \frac{1}{2}\cos u \;\Rightarrow\; 2y \pm \sqrt{3}\,x = \cos u$. But then $x^2 + \left(2y \pm \sqrt{3}\,x\right)^2 = \sin^2 u + \cos^2 u = 1 \;\Rightarrow\; x^2 + 4y^2 \pm 4\sqrt{3}\,xy + 3x^2 = 1 \;\Rightarrow\; 4x^2 \pm 4\sqrt{3}\,xy + 4y^2 = 1$, which may be recognized as a conic section. In particular, each equation is an ellipse rotated $\pm 45^\circ$ from the standard orientation (see the following graph). The trace in $z = -\frac{1}{2}$ is similar: $z = \sin(u + v) = -\frac{1}{2} \;\Rightarrow\; u + v = \beta + 2k\pi$ where $\beta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Then $v = (\beta + 2k\pi) - u$ and the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin[(\beta + 2k\pi) - u] = \sin(\beta + 2k\pi)\,\cos u - \cos(\beta + 2k\pi)\,\sin u = -\frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}\sin u$. If we convert to rectangular coordinates, we arrive at the same pair of equations, $4x^2 \pm 4\sqrt{3}\,xy + 4y^2 = 1$, so the trace is identical to the trace in $z = \frac{1}{2}$.

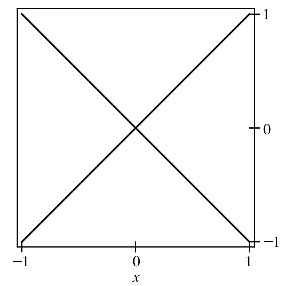
Graphing each of these, we have the following 5 traces.



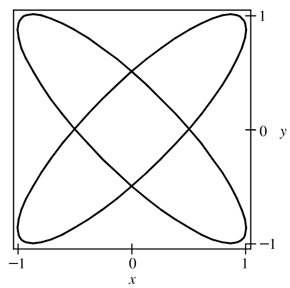
$z = -1$



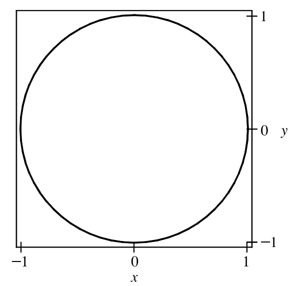
$z = -\frac{1}{2}$



$z = 0$



$z = \frac{1}{2}$



$z = 1$

Visualizing these traces on the surface reveals that horizontal cross sections are pairs of intersecting ellipses whose major axes are perpendicular to each other. At the bottom of the surface, $z = -1$, the ellipses coincide as circles of radius 1. As we move up the surface, the ellipses become narrower until at $z = 0$ they collapse into line segments, after which the process is reversed, and the ellipses widen to again coincide as circles at $z = 1$.

5. $$(\mathbf{F}\cdot\nabla)\,\mathbf{G} = \left(P_1\frac{\partial}{\partial x} + Q_1\frac{\partial}{\partial y} + R_1\frac{\partial}{\partial z}\right)(P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k})$$

$$= \left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}$$
$$+ \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

$$= (\mathbf{F}\cdot\nabla P_2)\,\mathbf{i} + (\mathbf{F}\cdot\nabla Q_2)\,\mathbf{j} + (\mathbf{F}\cdot\nabla R_2)\,\mathbf{k}.$$

Similarly, $(\mathbf{G}\cdot\nabla)\,\mathbf{F} = (\mathbf{G}\cdot\nabla P_1)\,\mathbf{i} + (\mathbf{G}\cdot\nabla Q_1)\,\mathbf{j} + (\mathbf{G}\cdot\nabla R_1)\,\mathbf{k}$. Then

$$\mathbf{F}\times\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_1 & Q_1 & R_1 \\ \partial R_2/\partial y - \partial Q_2/\partial z & \partial P_2/\partial z - \partial R_2/\partial x & \partial Q_2/\partial x - \partial P_2/\partial y \end{vmatrix}$$

$$= \left(Q_1\frac{\partial Q_2}{\partial x} - Q_1\frac{\partial P_2}{\partial y} - R_1\frac{\partial P_2}{\partial z} + R_1\frac{\partial R_2}{\partial x}\right)\mathbf{i} + \left(R_1\frac{\partial R_2}{\partial y} - R_1\frac{\partial Q_2}{\partial z} - P_1\frac{\partial Q_2}{\partial x} + P_1\frac{\partial P_2}{\partial y}\right)\mathbf{j}$$
$$+ \left(P_1\frac{\partial P_2}{\partial z} - P_1\frac{\partial R_2}{\partial x} - Q_1\frac{\partial R_2}{\partial y} + Q_1\frac{\partial Q_2}{\partial z}\right)\mathbf{k}$$

[continued]

and

$$\mathbf{G}\times\operatorname{curl}\mathbf{F}=\left(Q_2\frac{\partial Q_1}{\partial x}-Q_2\frac{\partial P_1}{\partial y}-R_2\frac{\partial P_1}{\partial z}+R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i}+\left(R_2\frac{\partial R_1}{\partial y}-R_2\frac{\partial Q_1}{\partial z}-P_2\frac{\partial Q_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}\right)\mathbf{j}$$
$$+\left(P_2\frac{\partial P_1}{\partial z}-P_2\frac{\partial R_1}{\partial x}-Q_2\frac{\partial R_1}{\partial y}+Q_2\frac{\partial Q_1}{\partial z}\right)\mathbf{k}.$$

Then

$$(\mathbf{F}\cdot\nabla)\,\mathbf{G}+\mathbf{F}\times\operatorname{curl}\mathbf{G}=\left(P_1\frac{\partial P_2}{\partial x}+Q_1\frac{\partial Q_2}{\partial x}+R_1\frac{\partial R_2}{\partial x}\right)\mathbf{i}+\left(P_1\frac{\partial P_2}{\partial y}+Q_1\frac{\partial Q_2}{\partial y}+R_1\frac{\partial R_2}{\partial y}\right)\mathbf{j}$$
$$+\left(P_1\frac{\partial P_2}{\partial z}+Q_1\frac{\partial Q_2}{\partial z}+R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

and

$$(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{G}\times\operatorname{curl}\mathbf{F}=\left(P_2\frac{\partial P_1}{\partial x}+Q_2\frac{\partial Q_1}{\partial x}+R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i}+\left(P_2\frac{\partial P_1}{\partial y}+Q_2\frac{\partial Q_1}{\partial y}+R_2\frac{\partial R_1}{\partial y}\right)\mathbf{j}$$
$$+\left(P_2\frac{\partial P_1}{\partial z}+Q_2\frac{\partial Q_1}{\partial z}+R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}.$$

Hence

$$(\mathbf{F}\cdot\nabla)\,\mathbf{G}+\mathbf{F}\times\operatorname{curl}\mathbf{G}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{G}\times\operatorname{curl}\mathbf{F}$$
$$=\left[\left(P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_1}{\partial x}\right)+\left(Q_1\frac{\partial Q_2}{\partial x}+Q_2\frac{\partial Q_1}{\partial y}\right)+\left(R_1\frac{\partial R_2}{\partial x}+R_2\frac{\partial R_1}{\partial x}\right)\right]\mathbf{i}$$
$$+\left[\left(P_1\frac{\partial P_2}{\partial y}+P_2\frac{\partial P_1}{\partial y}\right)+\left(Q_1\frac{\partial Q_2}{\partial y}+Q_2\frac{\partial Q_1}{\partial y}\right)+\left(R_1\frac{\partial R_2}{\partial y}+R_2\frac{\partial R_1}{\partial y}\right)\right]\mathbf{j}$$
$$+\left[\left(P_1\frac{\partial P_2}{\partial z}+P_2\frac{\partial P_1}{\partial z}\right)+\left(Q_1\frac{\partial Q_2}{\partial z}+Q_2\frac{\partial Q_1}{\partial z}\right)+\left(R_1\frac{\partial R_2}{\partial z}+R_2\frac{\partial R_1}{\partial z}\right)\right]\mathbf{k}$$
$$=\nabla(P_1P_2+Q_1Q_2+R_1R_2)=\nabla(\mathbf{F}\cdot\mathbf{G}).$$

6. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t)\geq 0$ is the distance from the top of the cylinder to the piston at time t. Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t)=x(t)\,\mathbf{i}$, $a\leq t\leq b$. (Thus, the curve lies on the positive x-axis and reverses direction several times.) The force on the piston is $AP(t)\,\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t. As in Section 16.2, the work done on the piston is $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_a^b AP(t)\,\mathbf{i}\cdot x'(t)\,\mathbf{i}\,dt=\int_a^b AP(t)\,x'(t)\;dt$. Here, the volume of the cylinder at time t is $V(t)=Ax(t)\quad\Rightarrow\quad V'(t)=Ax'(t)\quad\Rightarrow\quad\int_a^b AP(t)\,x'(t)\;dt=\int_a^b P(t)\,V'(t)\;dt$. Since the curve C in the PV-plane corresponds to the values of P and V at time t, $a\leq t\leq b$, we have

$$W=\int_a^b AP(t)\,x'(t)\;dt=\int_a^b P(t)\,V'(t)\;dt=\int_C P\,dV$$

Another method: If we divide the time interval $[a,b]$ into n subintervals of equal length Δt, the amount of work done on the piston in the ith time interval is approximately $AP(t_i)[x(t_i)-x(t_{i-1})]$. Thus we estimate the total work done during

one cycle to be $\sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})]$. If we allow $n \to \infty$, we have

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})] = \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[Ax(t_i) - Ax(t_{i-1})]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[V(t_i) - V(t_{i-1})] = \int_C P\,dV$$

(b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 16.4.5 we know the area of the lower loop in the PV-plane is given by $-\oint_{C_L} P\,dV$. C_U is negatively oriented, so the area of the upper loop is given by $-\left(-\oint_{C_U} P\,dV\right) = \oint_{C_U} P\,dV$. From part (a),

$$W = \int_C P\,dV = \int_{C_L \cup C_U} P\,dV = \oint_{C_L} P\,dV + \oint_{C_U} P\,dV$$
$$= \oint_{C_U} P\,dV - \left(-\oint_{C_L} P\,dV\right),$$

the difference of the areas enclosed by the two loops of C.

7. (a) For each value of $u = u_0$, $\mathbf{X}(u_0, v) = \mathbf{r}(u_0) + q\cos v\,\mathbf{N}(u_0) + q\sin v\,\mathbf{B}(u_0)$ is a circle of radius q that is perpendicular to the tangent vector, $\mathbf{r}'(u_0)$. Thus, the union of all circles, $a \le u \le b$ gives a Tube(C, q) around the curve C.

(b) $\mathbf{X}(u, v) = \mathbf{r}(u) + q\cos v\,\mathbf{N}(u) + q\sin v\,\mathbf{B}(u)$, $a \le u \le b$, $0 \le v \le 2\pi$.

First, we find $\mathbf{X}_u(u, v)$ and $\mathbf{X}_v(u, v)$. Note that, as $\mathbf{r}(u)$ is parametrized with respect to arc length, $|\mathbf{r}'(u)| = 1$ and thus, $\mathbf{r}'(u) = \mathbf{T}(u)$. With the Frenet-Serret Formulas, this gives

$$\begin{aligned}
\mathbf{X}_u(u, v) &= \mathbf{r}'(u) + q\cos v\,\mathbf{N}'(u) + q\sin v\,\mathbf{B}'(u) \\
&= \mathbf{T}(u) + q\cos v\,\left(-\kappa\mathbf{T}(u) + \tau\mathbf{B}(u)\right) + q\sin v\,\left(-\tau\mathbf{N}(u)\right) \\
&= \mathbf{T}(u) - q\kappa\cos v\,\mathbf{T}(u) + q\tau\cos v\,\mathbf{B}(u) - q\tau\sin v\,\mathbf{N}(u) \\
&= (1 - q\kappa\cos v)\mathbf{T}(u) + q\tau\cos v\,\mathbf{B}(u) - q\tau\sin v\,\mathbf{N}(u)
\end{aligned}$$

$$\begin{aligned}
\mathbf{X}_v(u, v) &= \mathbf{0} - q\sin v\,\mathbf{N}(u) + q\cos v\,\mathbf{B}(u) \\
&= -q\sin v\,\mathbf{N}(u) + q\cos v\,\mathbf{B}(u)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v) &= [(1 - q\kappa\cos v)\mathbf{T}(u) + q\tau\cos v\,\mathbf{B}(u) - q\tau\sin v\,\mathbf{N}(u)] \\
&\qquad \times\ [-q\sin v\,\mathbf{N}(u) + q\cos v\,\mathbf{B}(u)] \\
&= (1 - q\kappa\cos v)(-q\sin v)\mathbf{T}(u) \times \mathbf{N}(u) + (1 - q\kappa\cos v)(q\cos v)\mathbf{T}(u) \times \mathbf{B}(u) \\
&\qquad - q^2\tau\cos v\sin v\,\mathbf{B}(u) \times \mathbf{N}(u) + q^2\tau\cos^2 v\,\mathbf{B}(u) \times \mathbf{B}(u) \\
&\qquad + q^2\tau\sin^2 v\,\mathbf{N}(u) \times \mathbf{N}(u) - q^2\tau\sin v\cos v\,\mathbf{N}(u) \times \mathbf{B}(u)
\end{aligned}$$

The last four terms drop out since $\mathbf{N}(u) \times \mathbf{N}(u) = \mathbf{B}(u) \times \mathbf{B}(u) = \mathbf{0}$ and $\mathbf{N}(u) \times \mathbf{B}(u) = -(\mathbf{B}(u) \times \mathbf{N}(u))$, so

$$\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v) = q(1 - q\kappa\cos v)[-\sin v\,\mathbf{B}(u) - \cos v\,\mathbf{N}(u)] \qquad [\mathbf{B} = \mathbf{T} \times \mathbf{N}, \mathbf{N} = -\mathbf{T} \times \mathbf{B}]$$

[continued]

Next,

$$\begin{aligned}|\mathbf{X}_u(u,v)\times\mathbf{X}_v(u,v)|^2 &= |q(1-q\kappa\cos v)[-\sin v\,\mathbf{B}(u)-\cos v\,\mathbf{N}(u)]|^2\\ &= q^2(1-q\kappa\cos v)^2[(-\sin v)^2\,|\mathbf{B}(u)|^2+(-\cos v)^2\,|\mathbf{N}(u)|^2]\\ &\qquad\qquad\text{[by the Pythagorean Theorem for vectors]}\\ &= q^2(1-q\kappa\cos v)^2\left[\sin^2 v\,(1)+\cos^2 v\,(1)\right] = q^2(1-q\kappa\cos v)^2\end{aligned}$$

Thus, $|\mathbf{X}_u(u,v)\times\mathbf{X}_v(u,v)| = q(1-q\kappa\cos v)$. Therefore, the surface area of $\text{Tube}(C,q)$ is

$$\begin{aligned}S(q) &= \int_a^b\int_0^{2\pi}|\mathbf{X}_u(u,v)\times\mathbf{X}_v(u,v)|\,dv\,du = \int_a^b\int_0^{2\pi}q(1-q\kappa\cos v)\,dv\,du\\ &= q\int_a^b\left[v-q\kappa\sin v\right]_{v=0}^{v=2\pi}\,du = q\int_a^b 2\pi\,du = 2\pi q(b-a) = 2\pi qL \quad [\text{as } b-a=L]\end{aligned}$$

(c) Volume V of $\text{Tube}(C,r) = \int_0^r S(q)\,dq = \int_0^r 2\pi qL\,dq = \left[\pi q^2L\right]_0^r = \pi r^2L$

(d) First find the length L of the helix: $\mathbf{r}(t)=\langle\cos t,\sin t,t\rangle \;\Rightarrow\; \mathbf{r}'(t)=\langle-\sin t,\cos t,1\rangle \;\Rightarrow\; |\mathbf{r}'(t)|=\sqrt{2} \;\Rightarrow$

$L=\int_a^b|\mathbf{r}'(t)|\,dt=\int_0^{4\pi}\sqrt{2}\,dt=4\sqrt{2}\pi$. Then by part (c), with $r=0.2$ and $L=4\sqrt{2}\pi$, we have

$V=\pi r^2L=\pi(0.2)^2\left(4\sqrt{2}\pi\right)=\frac{4}{25}\sqrt{2}\pi^2$.

(e) We create the torus by forming a tube of radius r around the circle of radius R. The length of the curve is $L=2\pi R$ and by part (c), $V=\pi r^2L=\pi r^2(2\pi R)=2\pi^2r^2R$.